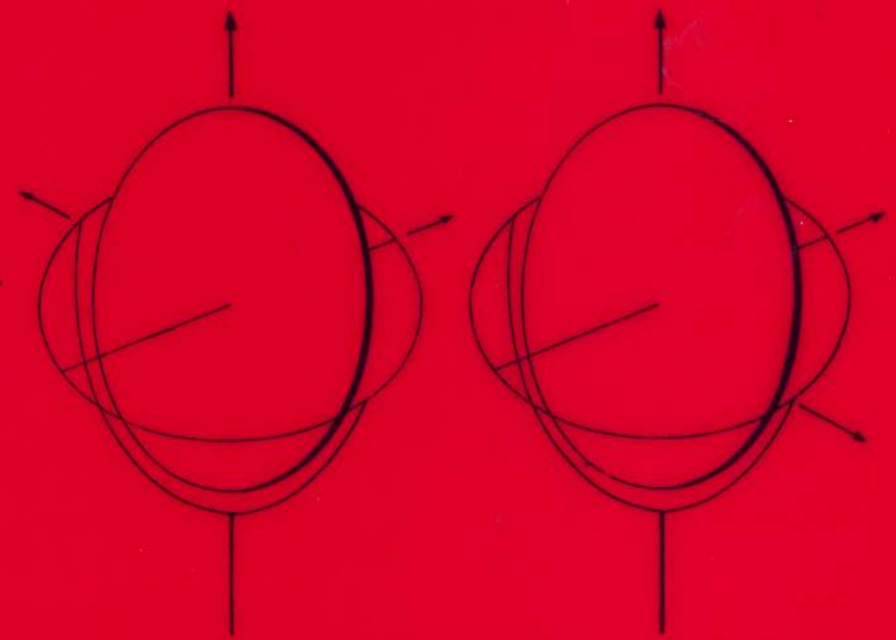
APPLIED MATHEMATICS FOR ENGINEERS AND SCIENTISTS



C. G. LAMBE

APPLIED MATHEMATICS
FOR ENGINEERS
AND SCIENTISTS
C. G. LAMBE

This book gives a thorough account of those parts of Dynamics, Statics, Hydromechanics and Wave Motion which are of fundamental importance to Engineers and Scientists. The author has been guided by the syllabus in Mathematics for all parts of the London B.Sc. Engineering degree examination. Most of the Applied Mathematics required for Part II of the London B.Sc. General degree examination is also covered.

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APPLIED MATHEMATICS
for
ENGINEERS AND SCIENTISTS

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GENERAL EDITOR'S FOREWORD

General Editor:

SIR GRAHAM SUTTON, C.B.E., D.Sc., LL.D., F.R.S.

THE present volume is one of a number planned to extend the Physical Science Texts beyond the Advanced or Scholarship levels of the General Certificate of Education. The earlier volumes in this series were prepared as texts for class teaching or self study in the upper forms at school, or in the first year at the university or technical college. In this next stage, the treatment necessarily assumes a greater degree of maturity in the student than did the earlier volumes, but the emphasis is still on a strongly realistic approach aimed at giving the sincere reader technical proficiency in his chosen subject. The material has been carefully selected on a broad and reasonably comprehensive basis, with the object of ensuring that the student acquires a proper grasp of the essentials before he begins to read more specialized texts. At the same time due regard has been paid to modern developments, and each volume is designed to give the reader an integrated account of a subject up to the level of an honours degree of any British or Commonwealth university, or the graduate membership of a professional institution.

A course of study in science may take one of two shapes. It may spread horizontally rather than vertically, with greater attention to the security of the foundations than to the level attained, or it may be deliberately designed to reach the heights by the quickest possible route. The tradition of scientific education in this country has been in favour of the former method, and despite the need to produce technologists quickly, I am convinced that the traditional policy is still the sounder. Experience shows that the student who has received a thorough unhurried training in the fundamentals reaches the stage of productive or original work very little, if at all, behind the man who has been persuaded to specialize at a much earlier stage, and in later life there is little doubt who is the better educated man. It is my hope that in these texts we have provided materials for a sound general education in the physical sciences, and that the student who works conscientiously through these books will face more specialized studies with complete confidence.

O. G. SUTTON

PREFACE

THIS book includes under the heading of Applied Mathematics the subjects of Dynamics, Statics, Hydromechanics and Wave Motion. The author has been guided by the new syllabus in Mathematics for all parts of the London B.Sc. Engineering degree examination, and most of the Applied Mathematics required for Part II of the London B.Sc. General degree examination is also covered.

The book is in some sense a sequel to the author's *Advanced Level Applied Mathematics*, and although there is inevitably some repetition of the earlier parts of Dynamics and Statics the scope of the present volume is much wider. A practical approach is maintained and there is a large number of examples and exercises the working of which is so essential for the understanding of the fundamental principles.

Many topics are included which are not usually found in books whose chief concern is with Dynamics and Statics. Thus under the heading of the Oscillation of Particles there is a treatment of Servomechanisms and Electric Circuits; Fourier Series are included in the chapter on Vibrations, and the Free Gyroscope in the chapter on Lagrange's equations. In Statics, which follows Dynamics without prejudice to the order in which the subjects may be taught, there is a section on Structural Statics dealing with Deflexions of Beams and the Theory of Struts. Stability of Flotation is dealt with in the chapter on the Motion of a Fluid, which also contains an account of the theory of Stream Functions. In the final chapter Legendre and Bessel functions are derived from their differential equations and applied to the solution of boundary value problems for Laplace's Equation and the Heat Flow Equation.

I wish to express my thanks to Sir Graham Sutton for his help and encouragement in the preparation of this book and to many of my colleagues at the Royal Military College of Science for their constructive criticisms. I am also indebted to the following who have read part of the manuscript and offered valuable advice:

Professors A. N. Black and E. T. Davies of the University of Southampton, Professor D. G. Christopherson of Imperial College, Professor T. A. A. Broadbent of the Royal Naval College, Greenwich, Dr. S. G. Hooker, Chief Engineer of the Bristol Aeroplane Company, Group Captain J. M. R. Morgan, Deputy Director of Education Services, R.A.F., and the Staff of the Royal Naval Engineering College, Plymouth.

For permission to use questions set in various examinations I wish to thank the Head of the Department of Engineering, University of

Cambridge, the Syndics of the Cambridge University Press and the Senate of the University of London.

C. G. LAMBE

Shrivenham

The source of examples and exercises is indicated by the following symbols:

Q.E.	Mechanical Sciences Tripos Qualifying Examination.
C.U.	Cambridge Intercollegiate Examination in Mechanical Sciences.
L.U., Pt. I L.U., Pt. II	Parts I and II of the London B.Sc. Engineering Examination.
L.U.	Other London University Examinations.

CONTENTS

CHAPTER 1

MOTION OF A PARTICLE IN A PLANE

1.1 Vector Addition of Displacements, Velocities and Accelerations

A displacement is described by giving its length and its direction. It can therefore be represented graphically by a straight line whose magnitude represents the length of the displacement to a certain scale and whose direction represents the direction of the displacement.

This straight line is called a *vector* and may be specified by its end points. Thus AB denotes a displacement from A to B (Fig. 1). The magnitude of a vector is called its *modulus* and the angle which it makes with some fixed direction its *amplitude*. Thus in a given plane a vector is determined by its amplitude and its modulus. A vector representing a displacement is not considered as indicating the starting-point or the finishing-point of the displacement, but merely its length and direction. With this convention all equal parallel vectors are equivalent. The sum of two displacements represented by straight lines AB and BC (Fig. 1) is represented by the straight line AC which is the third side of the triangle formed by the lines AB and BC placed end to end. This does not represent the numerical sum of the magnitudes of the displacements but the resulting displacement which follows from making the two displacements AB and BC in any order or simultaneously.

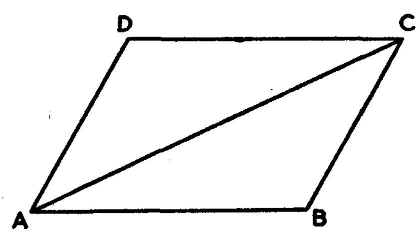

Fig. 1

The vector AC is called the *vector sum* of the vectors AB and BC and may be thought of as the third side of the triangle ABC, or as the diagonal of the parallelogram $ABCD$ of which AB and BC (or AD) are adjacent sides.

It follows, therefore, that the resulting displacement of two displacements is given by the vector sum of the two displacement vectors. The term *vector* is applied to straight lines which represent quantities in magnitude and direction only when the vector sum represents the result of combining two such quantities. Thus we shall see that velocities, accelerations, etc., are vectors, but there are certain quantities, for example, finite rotations of a rigid body about an axis, which can

be represented by straight lines but for which the vector sum does not give the resultant.

If a particle has a velocity u in a given direction its displacement in the infinitesmal time δt is $u\delta t$ in this direction. If it has at the same time a velocity v in another direction its displacement is $v\delta t$ in this other direction. These two displacements can be added as vectors, giving resultant displacement which we shall call $w\delta t$ in time δt (Fig. 2). Hence w is the resultant velocity of u and v and, since δt only modifies

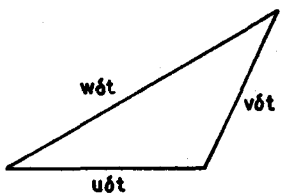

Fig. 2

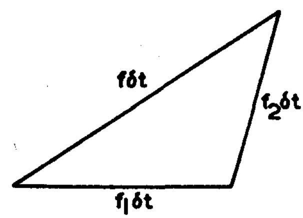

Fig. 3

the scale of the vectors, w is the vector sum of u and v. Therefore velocity is a vector.

Similarly, if a particle has accelerations f_1 and f_2 in two directions the velocities gained in time δt are $f_1\delta t$ and $f_2\delta t$. These velocities can be added as vectors (Fig. 3) giving a resultant velocity which we may call $f\delta t$. Then f is the resultant acceleration and is the vector sum of f_1 and f_2. Therefore acceleration is a vector.

Now by Newton's second law the force on a particle is proportional to its acceleration and is in the same direction. Therefore, a diagram showing two accelerations and the resultant acceleration also shows the forces causing the accelerations on a different scale. Therefore, forces are added vectorially and force is a vector.

It is clear that any number of vectors may be added by drawing successive triangles, adding the sum of the first two vectors to the third, and so on, and that these vectors need not be all in the same plane.

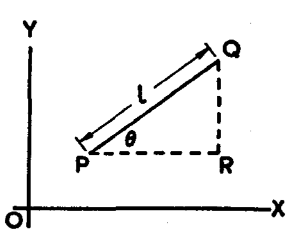

Fig. 4

1.2 Resolution of Vectors

If a vector PQ whose modulus is l makes an angle θ with a straight line OX, and OY is perpendicular to OX, it is evident that PQ is the vector sum of two vectors PR and RQ parallel to OX and OY whose moduli are $l \cos \theta$ and $l \sin \theta$ respectively (Fig. 4).

The vectors PR and RQ are called the *components* of the vector PQ in the directions OX and OY. By drawing a triangle of which one side

is PQ and whose other sides have two given directions, components of a vector in any two directions may be found.

A vector is said to be *resolved* into components in two directions. If the two directions are at right angles, by Pythagoras' Theorem the sum of the squares of the moduli of the components is equal to the square of the modulus of the vector.

Thus if the components of a vector parallel to OX and OY have moduli x and y respectively the vector has modulus $(x^2 + y^2)^{1/2}$ and its amplitude measured from OX is $\tan^{-1}(y/x)$.

1.3 Differentiation of a Vector

We now consider the rate of change of a vector with respect to time. When a vector changes, its modulus may change or its amplitude or both. Let a vector OP have modulus l and amplitude θ at time t, and at time $t + \delta t$ let the vector be OQ with modulus $l + \delta l$ and amplitude $\theta + \delta\theta$ so that δl and $\delta\theta$ are the changes in modulus and amplitude in time δt (Fig. 5). The components of OQ along OP and perpendicular to OP are $(l + \delta l) \cos \delta\theta$ and $(l + \delta l) \sin \delta\theta$. To the first order of small quantities these components are

$$l + \delta l \text{ and } l\delta\theta.$$

Hence the first order increment of the vector has components δl in the direction of OP and $l\delta\theta$ in a perpendicular direction and the total

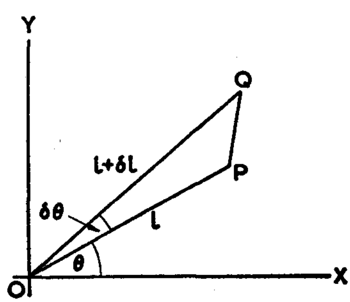

Fig. 5

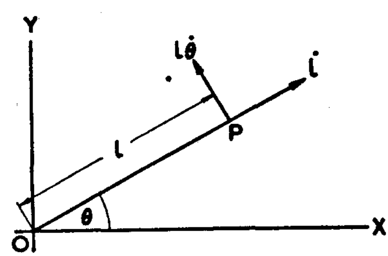

Fig. 6

first order increment is the resultant of these two vectors. These increments occur in time δt and in the limit we have rates of change (Fig. 6)

$$\frac{dl}{dt} = \dot{l}, \text{ in the direction } OP,$$

$$l\frac{d\theta}{dt} = l\dot{\theta}, \text{ perpendicular to } OP.$$

Hence the differential coefficient of a vector is itself a vector whose component along the line of the vector represents the rate of change of

the modulus and whose component in the perpendicular direction is proportional to the rate of change of the amplitude.

If the amplitude is constant we have $\dot{\theta} = 0$ and the differential coefficient of the vector is $\dot{l}$ in the direction of its length. If the length is constant we have $\dot{l} = 0$ and the differential coefficient is $l\dot{\theta}$ perpendicular to the vector in the direction of θ increasing.

The differential coefficient of a vector with respect to the time is called its derivative. We shall show that the derivative of a vector is the vector sum of the derivatives of its components.

Thus the vector OP (Fig. 6) has components $l \cos \theta$ and $l \sin \theta$ in the fixed directions OX and OY. These components have constant amplitude as θ changes and their derivatives are therefore the derivatives of their moduli. We have

$$\frac{d}{dt}(l \cos \theta) = \dot{l} \cos \theta - l\dot{\theta} \sin \theta,$$

$$\frac{d}{dt}(l \sin \theta) = \dot{l} \sin \theta + l\dot{\theta} \cos \theta.$$

These quantities are the sums of the components of $\dot{l}$ and $l\dot{\theta}$ in the directions OX and OY and are, therefore, the components of the derivative of the vector OP.

1.4 Velocity of a Particle in a Plane

The position of a particle may be described by giving its coordinates (x, y) with respect to fixed axes OX and OY. Then OP (Fig. 7) is the displacement vector of the particle from O and its components parallel to the axes are x and y. Now, velocity is rate of change of position, that is, it is represented by the derivative of the vector OP. Since the components x and y have fixed amplitudes their derivatives are $\dot{x}$ and $\dot{y}$ and, therefore, the velocity is the vector sum of components $\dot{x}$ and $\dot{y}$ parallel to OX and OY respectively.

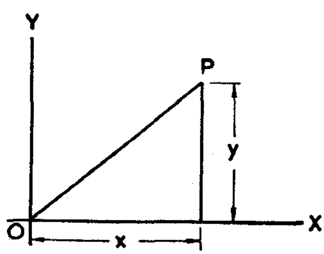

Fig. 7

Hence the direction of the velocity is inclined to OX at an angle

$$\tan^{-1}\frac{\dot{y}}{\dot{x}} = \tan^{-1}\frac{dy}{dx} = \psi,$$

where ψ is the angle made by the tangent to the path of the particle with OX.

The magnitude of the velocity (the speed) is given by

$$v = (\dot{x}^2 + \dot{y}^2)^{1/2}.$$

Since $$v \cos\psi = \frac{dx}{dt},$$

$$\cos\psi = \frac{dx}{ds},$$

$$v = \frac{ds}{dx}\cdot\frac{dx}{dt} = \frac{ds}{dt}.$$

Therefore the velocity has magnitude $\frac{ds}{dt}$ and is directed along the tangent to the path.

In polar coordinates the displacement of the particle from O is a vector whose modulus is r and whose amplitude is θ. The velocity which is the differential coefficient of this vector has therefore components

$\dot{r}$, parallel to OP,

$r\dot{\theta}$, perpendicular to OP, in the direction of θ increasing.

The quantity $\dot{\theta}$ is the rate at which the line OP is changing direction and $\dot{\theta}$ is called the angular velocity of the point P about O.

1.5 Acceleration of a Particle in a Plane

The acceleration is the derivative of the velocity which is a vector with modulus $v(= \dot{s})$ and amplitude ψ.

Therefore the acceleration vector f has components $\dot{v}$ along the tangent, $v\dot{\psi}$ in a perpendicular direction. Now the radius of curvature ϱ of the path is given by

$$\varrho = \frac{ds}{d\psi},$$

$$\text{and } \frac{d\psi}{dt} = \frac{d\psi}{ds}\cdot\frac{ds}{dt} = \frac{v}{\varrho},$$

so the component of acceleration perpendicular to the tangent is usually written as v^2/ϱ.

The velocity vector has components $\dot{x}$ and $\dot{y}$ in fixed directions parallel to OX and OY respectively. Hence the derivative of the velocity vector which is the acceleration vector f is the vector sum of the derivatives of $\dot{x}$ and $\dot{y}$. That is, the acceleration has components $\ddot{x}$ and $\ddot{y}$ parallel to OX and OY respectively.

In polar coordinates the velocity has components $\dot{r}$ and $r\dot{\theta}$ along and perpendicular to OP.

The derivative of $\dot{r}$ has components $\ddot{r}$ along OP and $\dot{r}\dot{\theta}$ perpendicular to OP (Fig. 8).

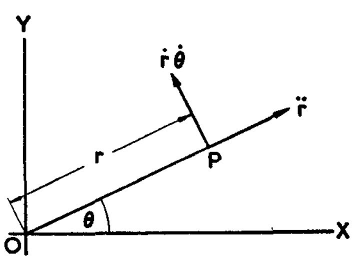

Fig. 8

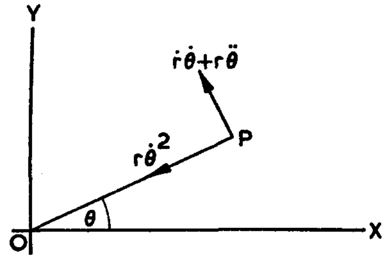

Fig. 9

The derivative of $r\dot{\theta}$ has components $\frac{d}{dt}(r\dot{\theta})$ perpendicular to OP and $r\dot{\theta}(\dot{\theta})$ in the perpendicular direction which is the direction PO (Fig. 9).

Now $$\frac{d}{dt}(r\dot{\theta}) = \dot{r}\dot{\theta} + r\ddot{\theta}$$

and the totals of the components of acceleration are

$$\ddot{r} - r\dot{\theta}^2 \text{ in the direction } OP,$$
$$r\ddot{\theta} + 2\dot{r}\dot{\theta} \text{ in the perpendicular direction.}$$

The component perpendicular to OP is more easily remembered in the form

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}).$$

We thus have three different sets of components for the acceleration of a particle which are suitable for use in various problems. In general the

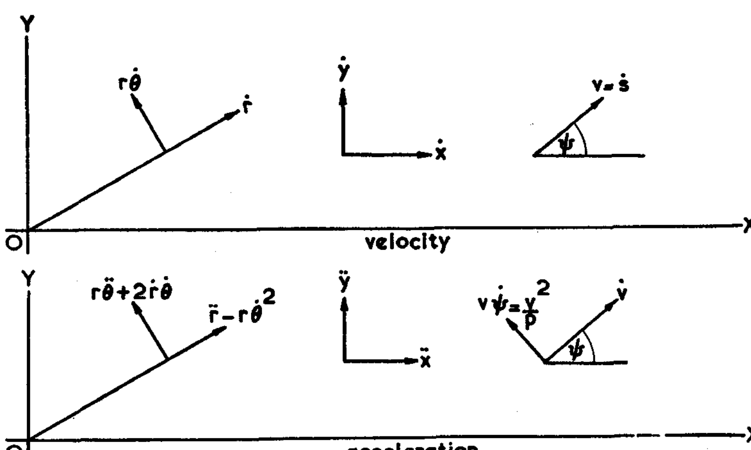

Fig. 10

Cartesian components are most useful, but if the force is given as a head-on resistance to a particle moving along a curve the components $\dot{v}$ and v^2/ϱ are more suitable, while if the force is a central one directed towards a fixed point the components of acceleration in polar coordinates lead to a simpler solution. The components of velocity and acceleration for the different types of coordinates are shown in Fig. 10.

1.6 Linear Momentum and Angular Momentum

The linear momentum of a particle moving in a plane is the product of its mass and velocity mv, and since velocity is a vector, momentum is also a vector and can be resolved into components $m\dot{x}$, $m\dot{y}$ parallel to fixed axes.

If P be the force acting on the particle and f its acceleration the directions of P and f are the same and we have $P = mf$. If P has components X and Y parallel to fixed axes of coordinates OX and OY we have

$$m\ddot{x} = X,$$
$$m\ddot{y} = Y.$$

Integrating these equations with respect to the time from $t = t_1$ to $t = t_2$ we have

$$\Big[m\dot{x}\Big]_{t_1}^{t_2} = \int_{t_1}^{t_2} X\,dt,$$

$$\Big[m\dot{y}\Big]_{t_1}^{t_2} = \int_{t_1}^{t_2} Y\,dt.$$

Hence the change in momentum in each direction is the time integral of the component of force in that direction, that is, the component of impulse in that direction.

When two bodies collide, because of their elasticity they are in contact for a short period. During this period there is force, which may vary during the period, normal to the common tangent plane of the bodies. By Newton's third law the forces exerted by each body on the other are equal and opposite; therefore the impulses, which are the time integrals of these forces over the period of contact, are equal and opposite. Now the impulse is equal to the change of momentum, therefore the amount of momentum which is lost by one body is imparted to the other and the total momentum of the bodies is unaltered.

When we are concerned only with their motion of translation we may speak of the bodies as particles and we have the *Principle of Conservation of Linear Momentum* that the total momentum of a system of particles is unaltered by internal forces between the particles.

A simple illustration of the conservation of momentum is to be found in the recoil of guns, where the forward momentum given to the shell is equal to the backward momentum given to the barrel of the gun, if the

forward momentum given to the gases formed by the explosion be neglected. The barrel recoils along its axis against a buffer fixed to the gun carriage which remains stationary. In some large guns the barrel is fixed to the carriage, which is allowed to recoil along a horizontal or inclined plane, and this recoil is due to the component of the backward momentum in this direction.

The *angular momentum* of a particle about a point is defined as the moment of its linear momentum about the point, that is pmv where p is the perpendicular from the point on to the line of the velocity.

If a point O is the origin of polar coordinates and the position of the particle is (r, θ), its linear momentum has components $m\dot{r}$ and $mr\dot{\theta}$ along and perpendicular to the radius r and its angular momentum about the origin is $mr^2\dot{\theta}$.

In Cartesian coordinates if the position of the particle be (x, y) its linear momentum has components $m\dot{x}$ and $m\dot{y}$ and the angular momentum about the origin is $m(x\dot{y} - y\dot{x})$. In each case the angular momentum is taken as positive if the movement about the point is in a counter-clockwise direction.

The concept of angular momentum is necessary when dealing with the motion of a rigid body.

We may note that its derivative is equal to the moment of the force about the point, that is,

$$\begin{aligned}\frac{d}{dt}m(x\dot{y} - y\dot{x}) &= m(x\ddot{y} - y\ddot{x}),\\ &= (xY - yX),\\ &= \text{the moment of the force about } O.\end{aligned}$$

Hence if the force has no moment about a point O the angular momentum about this point is constant.

Since the sum of the moments of the internal forces of a system of particles about any point is zero, if the moment of the external forces acting on the system about some point is zero the total angular momentum about the point is constant and we have *Conservation of Angular Momentum.*

1.7 Energy and Work

The kinetic energy of a particle is

$$\frac{1}{2}mv^2 = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2).$$

Since
$$\frac{d}{dx}\left(\frac{1}{2}\dot{x}^2\right) = \dot{x}\frac{d\dot{x}}{dt}\frac{dt}{dx} = \ddot{x},$$

$$\frac{d}{dy}\left(\frac{1}{2}\dot{y}^2\right) = \dot{y}\frac{d\dot{y}}{dt}\frac{dt}{dy} = \ddot{y},$$

the equations $m\ddot{x} = X$, $m\ddot{y} = Y$ integrated between the positions (x_1, y_1) and (x_2, y_2) give

$$\left[\frac{1}{2}m\dot{x}^2\right]_{x_1}^{x_2} = \int_{x_1}^{x_2} X dx,$$

$$\left[\frac{1}{2}m\dot{y}^2\right]_{y_1}^{y_2} = \int_{y_1}^{y_2} Y dy.$$

Hence, the change in kinetic energy is the sum of the space integrals of the components of force, that is, the work done. If the force P makes at any instant an angle ϕ with OX we have

$$X = P\cos\phi, \quad Y = P\sin\phi,$$
$$dx = ds\cos\psi, \quad dy = ds\sin\psi,$$

$$\int_{x_1}^{x_2} X dx + \int_{y_1}^{y_2} Y dy = \int_{s_1}^{s_2} P(\cos\phi\cos\psi + \sin\phi\sin\psi)ds,$$
$$= \int_{s_1}^{s_2} P\cos(\phi - \psi)ds,$$

where s_1 and s_2 are the corresponding positions on the path. Thus the work done by the components of force is the same as the space integral of the force along the path of the particle.

1.8 Units of Force

From Newton's second law of motion we have that the force P required to give a particle of mass m an acceleration f is proportional to the product mf. By choosing appropriate units we are able to write $P = mf$. Thus if the mass is in pounds and the acceleration in feet per second per second the force P given by this equation is in poundals. A poundal is therefore the force required to give a mass of 1 lb. an acceleration 1 ft./sec.^{2}. In metric units a dyne is the force required to give a mass of 1 gramme an acceleration 1 cm./sec.^{2}.

Now a particle of mass m lb., falling freely under gravity has an acceleration g ft./sec.^{2}, therefore the force acting on it, namely its weight, is mg poundals. That is, m lb.wt. $= mg$ poundals.

We thus have two kinds of units of force: absolute units such as poundals and dynes, and gravitational units such as pounds weight and grammes weight.

When the force P is expressed in absolute units we have

$$P = mf,$$

when P is expressed in gravitational units we have

$$Pg = mf.$$

Gravitational units are commonly used in Engineering and in theoretical work it is usual to denote the mass of a body by the symbol

w as a reminder that these units are being used. The fundamental formula connecting force and acceleration is then written

$$P = \frac{w}{g}f.$$

An alternative unit of mass called the *slug* is commonly used, particularly in aircraft engineering, and one slug is g pounds, that is, a mass of approximately 32 pounds. Thus if m be the mass of a particle in pounds and $\bar{m}$ its mass in slugs we have $\bar{m}g = m$ and hence with the force in lb.wt. and the acceleration in ft./sec.^{2} we have

$$P = \bar{m}f.$$

An advantage of using slugs is that the local variability of the unit of force is balanced by the corresponding variability of the unit of mass.

The quantity mf (or $\frac{w}{g}f$ or $\bar{m}f$) is called the *effective* force on the particle and has, of course, the same units as the force.

The units of momentum are those of impulse which is the time integral of force. Thus the momentum of a particle whose velocity is v ft./sec. is mv pdl. sec., or $\bar{m}v = \frac{w}{g}v$ lb.sec.

The units of kinetic energy are those of work which is the space integral of force. Thus the kinetic energy is $\frac{1}{2}mv^2$ ft. pdl., or $\frac{1}{2}\bar{m}v^2 = \frac{w}{2g}v^2$ ft. lb.

Power is rate of doing wórk and is, therefore, the rate of change of kinetic energy.

Now
$$\frac{d}{dt}\left(\frac{w}{2g}v^2\right) = \frac{w}{g}v\frac{dv}{dt},$$
$$= Pv, \text{ where } P \text{ is the force.}$$

The units of power are therefore ft. pdl./sec. or ft. lb./sec. depending on the units of force. The standard unit of one horse-power is a rate of 550 ft. lb./sec. and thus if the force is in lb.wt. the power is $Pv/550$ horse-power.

In the metric system the kilowatt is a power of 10^{10} C.G.S. units, that is 10^{10} absolute units of power. Therefore, if P is in dynes and v in cm./sec., the power is $10^{-10}\,Pv$ kilowatts.

1.9 Impulsive Forces

The change of momentum of a particle, or of a system of particles, may be used to measure the impulse of the external force causing this change. In the case of a short sharp impact it is rarely possible to measure the force itself which may vary considerably in magnitude

during the period of impact, but the time integral of this force may be found from the change of momentum.

It should be noted, however, that the time integral of the force gives the change of momentum over long as well as short periods.

If a jet of water is played on a fixed surface at right-angles to the jet, the momentum of a certain mass of water is destroyed in each second by the impulse of the surface on the water. This is equal to the impulse of the water on the surface and since its duration is one second the average thrust on the surface during that time may be found.

The same method may be used to measure the thrust of an air current on a surface.

Example 1. *A jet of water 2 in. in diameter moving horizontally at 40 ft./sec. strikes at right-angles the vertical face of a block of mass 10 lb. which rests on a rough horizontal plane, the coefficient of friction between the block and the plane being 0·5. Find the initial acceleration of the block and the greatest speed with which it can be driven by the jet.*

The mass of water issuing from the jet in one second is

$$\frac{\pi}{144} \times 40 \times 62{\cdot}5 = 54{\cdot}5 \text{ lb.}$$

The momentum destroyed in one second is

$$\frac{54{\cdot}5 \times 40}{32} = 68{\cdot}1 \text{ lb.sec.}$$

Since this momentum is destroyed in one second the average force is 68·1 lb.wt. and since the frictional resistance is 5 lb.wt.

$$68{\cdot}1 = \frac{10}{32} \times f,$$

$$f = 202 \text{ ft./sec.}^2$$

The acceleration of the block is zero when the thrust of the water on it is 5 lb.wt. Let v ft./sec. be its velocity at that instant.

The mass of water reaching the block in each second is now $\pi(40 - v)62{\cdot}5/144$ and its velocity is reduced from 40 to v ft./sec.

Therefore
$$5 = \frac{\pi \times 62{\cdot}5}{144 \times 32}(40 - v)^2,$$

$$40 - v = 10{\cdot}8,$$

$$v = 29{\cdot}2 \text{ ft./sec.}$$

Example 2. *A water turbine has flat radial blades and a stream of water delivered horizontally at the rate of 240 gallons per minute from a jet 2 in. in diameter strikes the blades normally at 2 ft. from the centre of the wheel. If the wheel is turning at 60 revolutions per minute find the thrust on the blades, the horse-power developed by the wheel and the theoretical efficiency of the arrangement.*

Since a gallon of water weighs 10 lb., 40 lb. of water is delivered each second with velocity given by

$$\frac{\pi}{144} \times 62{\cdot}5 \times v = 40,$$

$$v = 29{\cdot}3 \text{ ft./sec.}$$

The angular velocity of the wheel being 2π radians/sec., the velocity of the blade where the water strikes is 4π ft./sec., and hence the impulse is

$$\frac{40}{32}(29\cdot3 - 4\pi) = 20\cdot92 \text{ lb.sec.}$$

The average thrust is therefore 20·92 lb.wt.
The horse-power developed is

$$\frac{20\cdot92 \times 4\pi}{550} = 0\cdot48.$$

The work of which the jet is capable in each second is the kinetic energy available, that is,

$$\frac{40}{2g}(29\cdot3)^2 = 537 \text{ ft.lb.}$$

Therefore the efficiency is

$$\frac{20\cdot92 \times 4\pi}{537} = 0\cdot49, \text{ or } 49 \text{ per cent.}$$

1.10 Conservation of Energy

We have seen that the change in kinetic energy of a particle is the space integral of the force moving it taken over the distance moved and that this is the work done by the force in the displacement.

Let us consider the work done in stretching a spring (or an elastic string) of stiffness s. Let x_1 and x_2 be the initial and final extensions respectively.

The force in the spring is the product of the stiffness and the extension, so that when the extension is x the force is sx.

The space integral of this force from x_1 to x_2

$$= \int_{x_1}^{x_2} sx dx,$$

$$= \frac{1}{2}s(x_2^2 - x_1^2),$$

$$= \frac{1}{2}(sx_2 + sx_1)(x_2 - x_1).$$

This is easily seen to be the product of the mean of the initial and final tensions and the extension.

If a particle is moving against the force in the spring the work done in stretching the spring is equal to the kinetic energy lost by the particle. Hence we may regard this work as energy stored in the spring and this energy is converted back into kinetic energy as the spring unstretches. Thus if we call the energy stored in the spring its potential energy we have that the total of kinetic and potential energy remains constant throughout the motion.

In the same way a particle of weight w rising vertically against

gravity through a distance h does work which is $\int w\,dx = wh$. This is equal to the loss of kinetic energy and may be regarded as an increase in potential energy and hence again the total of kinetic and potential energy remains constant. The term *potential energy* is used to denote the work done in a displacement to some standard position by so-called *conservative forces*. These are forces such that the work they do as the particle changes its position depends only on the initial and final positions. Thus friction which tends to reduce kinetic energy whichever way the particle is moving is excluded, and, in general, non-conservative forces cause an apparent loss of energy by converting mechanical energy into other forms such as heat and sound.

We have, therefore, the *Principle of Conservation of Energy*, that for conservative forces the sum of kinetic and potential energies is constant.

It should be noted that the standard position from which potential energy is measured is of little importance since it is the change of potential energy that is required to find the change in kinetic energy.

It should also be noted that the energy equation, which equates the change in kinetic energy to the space integral of the force, holds for all forces conservative or not.

Example 3. *A railway truck of weight W tons impinges on a truck of weight W' tons which is at rest and the total energy which can be absorbed by the buffer springs is E ft.tons. Prove that, neglecting road resistance, the buffers will not be driven home if the speed of the truck is less than*

$$2gE\left(\frac{1}{W} + \frac{1}{W'}\right)^{1/2}.$$

In any case, show that this expression gives the final relative speed of the trucks provided that E is the energy restored by the springs during the separation of the trucks. (L.U., Pt. I)

Let U ft./sec. be the velocity of approach of the truck W and let u ft./sec. be the common velocity of the trucks at the instant of maximum compression.

Then
$$WU = (W + W')u. \qquad (1)$$

The loss of kinetic energy at this instant is

$$\frac{W}{2g}U^2 - \frac{(W + W')}{2g}u^2$$
$$= \frac{WW'}{W + W'}\frac{U^2}{2g}, \text{ from (1).}$$

If this quantity is less than E the buffers have not been driven home, that is if

$$U < \left\{2gE\left(\frac{1}{W} + \frac{1}{W'}\right)\right\}^{1/2}.$$

Let v and v' respectively be the velocities after separation in the same direction as U.

Then
$$Wv + W'v' = WU. \qquad (2)$$

Now the energy gained since maximum compression being E, we have

$$\frac{W}{2g}v^2 + \frac{W'}{2g}v'^2 = E + \frac{W + W'}{2g}u^2$$

$$= E + \frac{W^2}{W + W'}\frac{U^2}{2g}. \qquad (3)$$

Eliminating U between (2) and (3) we find

$$WW'(v^2 - 2vv' + v'^2) = 2gE(W + W'),$$

$$v - v' = \left\{2gE\left(\frac{1}{W} + \frac{1}{W'}\right)\right\}^{1/2}.$$

EXERCISES 1 (*a*)

1. A train of trucks is being started from rest, and just before the last coupling becomes taut, the front part has acquired a velocity of 15 m.p.h. If the front portion of the train weighs 72 tons and the last truck weighs 6 tons, find the jerk in the coupling in ft.-lb.-sec. units. (L.U.)

2. A ship A of 500 tons tows a ship B of 2000 tons by means of a cable fixed to A and passing round a bollard on B in such a way that a pull of 4 tons on the cable causes it to slip round the bollard. Initially A is moving forward at 5 ft./sec., B is stationary and the cable is just taut. Neglecting resistance to motion find the velocity of the ships and the length of cable paid out when slipping ceases, (*a*) when A's engines are not started, (*b*) when A's engines exert a steady forward thrust of 3 tons.

3. Waves are striking against a vertical sea-wall with a speed of 50 ft./sec. Taking a cubic foot of sea-water to weigh 64 lb., show that the pressure on the wall, due to the destruction of the momentum of the waves is very approximately 34·7 lb. wt./sq. in.

4. A land yacht, running on wheels, has one sail of area 50 sq. ft. With the sail set at right angles to the centre line of the vessel and a following wind of 30 m.p.h., it is found that the yacht moves forward at 22·5 m.p.h. Assuming that the air after striking the sail is reduced to the speed of the sail, determine the h.p. of a motor that would maintain the speed of the yacht if the sail were removed. The density of the air may be taken as 0·08 lb. per cu. ft. (Q.E.)

5. A motor-driven pump raises 40 cu. ft. of water per minute through a height of 35 ft., and delivers it at the upper level in a jet with a velocity of 40 ft./sec.

   (*a*) If 40 per cent of the energy output from the motor is transferred to the water in the jet as kinetic and potential energy, find the power output of the motor.

   (*b*) If the jet is directed normal to a flat surface, what force will it apply to that surface, assuming the water does not rebound? (Q.E.)

6. A tug is propelled by pumps which suck 6000 gallons of water per minute through an intake in the bottom and discharge this water

straight astern with a velocity of 20 ft./sec., relative to the tug. What tow-rope pull can the tug exert when stationary? If the resistance to motion is $1{\cdot}75v^2$ lb., where v is the speed in ft./sec., what is the greatest speed which the tug can attain? (A gallon of water weighs 10 lb.) (Q.E.)

7. A Pelton wheel carries semicircular-shaped buckets on its rim so that water directed against the wheel flows round a bucket and leaves it with the same relative speed with which it entered but in the opposite direction.

   Such a wheel is 4 ft. in diameter and is supplied with 560 cu. ft. of water per minute through a pipe of 14 sq. in. cross-section. When the wheel is making 400 revolutions per minute find the horse-power developed.

8. A vehicle of mass 3000 lb. has an engine which can exert a maximum propulsive power of 60 h.p. Friction and wind resistance to the motion of the vehicle may be taken as a force of $(100 + 0{\cdot}05V^2)$ lb., where V is its velocity in ft./sec. Show that its maximum speed on a level road is a little less than 80 ft./sec., assuming the propulsive power to be at its greatest.

   Find also the acceleration of the vehicle when the engine is exerting 36 h.p. and the vehicle is travelling at 40 ft./sec. up an incline of 1 in 15. (Q.E.)

9. An electric tram-car of mass 18 tons, driven by two motors, is travelling on a horizontal road at 15 m.p.h. and accelerating at 0·45 m.p.h. per sec. The total forces resisting motion amount to 370 lb. If the power drawn from the electric wires is 60 h.p., calculate the efficiency of the motors at this speed.

10. A tug of 500 tons tows a ship of 4000 tons by means of a cable fixed in both ships. Initially the tug is moving forward at 5 ft./sec. and the other ship is stationary. Find the common velocity of the ships when the tow rope has its greatest extension and the loss of kinetic energy at this instant. If a pull of 10 tons stretches the cable 1 foot find the maximum tension in the cable while taking the strain.

11. A train consists of an engine weighing 30 tons and three wagons each weighing 10 tons, and initially the buffers are in contact and all the couplings, which are inelastic, slack to the extent of 1 ft. The locomotive starts from rest and the driving wheels exert a propulsive force of 1 ton. Neglecting frictional and wind resistances calculate the velocity with which each truck starts to move and the loss of energy due to the jerks.

12. A rifle, weighing 7 lb., is suspended by cords with its bore horizontal. It fires a $\frac{3}{4}$-oz. bullet, and in recoiling rises 1 ft. Find the muzzle velocity of the bullet. If the accelerations of the rifle and bullet are taken as constant, and the bore is 2 ft. long, show that the time of the bullet in the bore is about 1/300 sec., and that in this time the rifle will only have moved about 0·16 in.

13. The pump of a fire-fighting launch throws a jet of water 2 in. in diameter, at an angle of elevation of 45°. The velocity of the water is such that the highest point of its trajectory is 50 ft. above the nozzle, which is itself 10 ft. above the water line. Neglecting air resistance and all losses, find the power needed to drive the pump. Find also the horizontal force on the launch. (Q.E.)

14. A ship is propelled by the reaction of a jet of water produced by a centrifugal pump. The pump takes in water from inlets in the skin and ejects it in a backward direction with a velocity relative to the ship of 20 ft. per sec. If the speed of the ship is 10 ft. per sec., and the cross-sectional area of the jet is 3 sq. ft., calculate the propulsive thrust on the ship due to the jet. If the efficiency of the pump is 50 per cent and there are no further losses other than that of the kinetic energy of the jet, find the overall efficiency of the system and the power required to drive the pump. (C.U.)

1.11 Variable Acceleration

The acceleration of a particle (or the force causing the acceleration) may be a constant or may be given as a function of the time, the distance or the velocity. When the acceleration is constant the elementary formulae

$$v = u + ft,$$
$$s = ut + \tfrac{1}{2}ft^2,$$
$$v^2 = u^2 + 2fs,$$

may be used to find the velocity and the displacement at any time or at any distance of a particle moving in a straight line.

When the acceleration is variable these formulae cannot be used. Instead, the velocity and displacement are found by equating one of the general expressions for acceleration, such as $\frac{dv}{dt}$, $\frac{d^2x}{dt^2}$ or $\frac{vdv}{dx}$, to the particular expression given for the acceleration, thus forming a differential equation. This is in general a second order differential equation and the two arbitrary constants in its solution are determined by the initial values of the velocity and displacement.

1.12 Acceleration as a Function of the Time

When the acceleration is a function of the time we equate it to the expression for $\frac{dv}{dt}$ and integrate to find v. Since $v = \frac{dx}{dt}$, a second integration gives the distance.

Example 4. *A train starts with acceleration* 1·1 *ft./sec.*^{2} *which decreases uniformly to zero in* 2 *minutes. The train then travels with uniform speed for* 3 *minutes, after which it is brought to rest by the brakes with a constant retardation of* 3 *ft./sec.*^{2}. *Find the greatest speed of the train and the length of the journey.*

The acceleration is $1{\cdot}1\left(1 - \frac{t}{120}\right)$ ft./sec.^{2} and we have

$$\frac{dv}{dt} = \frac{11}{1200}(120 - t),$$

$$v = \frac{11}{1200}\left(120t - \frac{1}{2}t^2\right) + a,$$

$$x = \frac{11}{1200}\left(60t^2 - \frac{1}{6}t^3\right) + at + b.$$

Since the initial velocity and displacement are zero, the constants of integration a and b are both zero.

When $t = 120$ we have

$$v = 66 \text{ ft./sec.}$$
$$x = 5280 \text{ ft.}$$

The distance covered in 3 minutes at the constant speed of 66 ft./sec. is 11,880 ft. For the final period with constant retardation the formula $v^2 = u^2 + 2fs$, with $v = 0$ and $u = 66$ ft./sec. gives $s = 726$ ft.

Thus the total distance is 17,886 ft. $= 3{\cdot}3875$ miles and the greatest speed is 66 ft./sec. or 45 m.p.h.

1.13 Acceleration as a Function of the Distance

In this case the acceleration is taken as $v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$, and integration leads to an expression for the velocity in terms of the distance; putting $v = \frac{dx}{dt}$ a second integration gives the relation between time and distance.

An important case is that of simple harmonic motion which will be considered separately.

Example 5. *A particle is projected vertically upwards from the earth's surface. Neglecting air resistance find the initial velocity required for the particle to escape from the earth's attraction. If the initial velocity is 10,000 ft./sec. find the greatest height reached and the time taken to reach this height.*

If x be the distance of the particle from the earth's centre at time t and R the earth's radius the acceleration downwards is $g\frac{R^2}{x^2}$.

Hence
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{gR^2}{x^2},$$

$$\frac{1}{2}v^2 = \frac{gR^2}{x} + c \text{ (a constant).}$$

Now if $v = V$ initially when $x = R$ we have

$$\frac{1}{2}V^2 = gR + c,$$

and
$$v^2 = \frac{2gR^2}{x} + (V^2 - 2gR). \tag{1}$$

The particle will escape from the earth's attraction if the velocity v becomes zero only when x is infinite and for this equation (1) gives us that $V^2 = 2gR$. Taking $R = 4000$ miles $= 21{,}120{,}000$ ft. this gives

$$\begin{aligned} V &= 36{,}760 \text{ ft./sec.} \\ &= 6\cdot 95 \text{ miles per second.} \end{aligned}$$

When $V = 10^4$ ft./sec. from (1) we have $v = 0$ when

$$\begin{aligned} x &= \frac{2gR^2}{2gR - V^2} \\ &= 4320 \text{ miles.} \end{aligned}$$

Hence the particle comes to rest 320 miles above the surface of the earth.

To find the time for this distance we have

$$v = \frac{dx}{dt} = \left\{\frac{2gR^2 - (2gR - V^2)x}{x}\right\}^{1/2},$$

and
$$dt = \frac{x^{1/2}dx}{\{2gR^2 - (2gR - V^2)x\}^{1/2}}.$$

We may avoid adding constants and find the time by taking the limits of integration for x as $x = R$ to $x = \dfrac{2gR^2}{2gR - V^2}$.

Substituting $x = \dfrac{2gR^2}{2gR - V^2}\sin^2\theta$, the limits of integration for θ are $\theta = \alpha$ to $\theta = \dfrac{\pi}{2}$, where $\sin\alpha = \left(\dfrac{2gR - V^2}{2gR}\right)^{1/2}$.

$$\begin{aligned} \alpha &= 1\cdot 295 \text{ radians.} \\ t &= \frac{4gR^2}{(2gR - V^2)^{3/2}}\int_\alpha^{\pi/2}\sin^2\theta d\theta, \\ &= \frac{2gR^2}{(2gR - V^2)^{3/2}}\Big[(\theta - \sin\theta\cos\theta)\Big]_\alpha^{\pi/2}, \\ &= 644\cdot 7\left(\frac{\pi}{2} - \alpha + \sin\alpha\cos\alpha\right), \\ &= 346 \text{ seconds approximately.} \end{aligned}$$

EXERCISES 1 (*b*)

1. A train of mass M lb. running at 45 m.p.h. is brought to rest on an upgrade of 1 vertically in 128 along the track by the brakes which exert a force of $M\left(\dfrac{1}{10} + \dfrac{t}{110}\right)$ pdl. where t (seconds) is measured from the instant when the brakes are applied. Show that the train is brought to rest in 1 min. 28 sec., and find the distance in miles that it travels in this time. (L.U.)

2. A particle of mass 3 lb. is acted upon by a force which diminishes uniformly from $\frac{3}{5}$ lb.wt. to $\frac{1}{10}$ lb.wt. in $\frac{1}{2}$ min. If it starts from rest find its greatest velocity in this half minute and the distance traversed. Find, also, the velocity of the particle when the force on it becomes zero. (L.U., Pt. I)

3. The acceleration of a particle moving in a straight line when at a distance x from a fixed point O is k^2x away from O, k being a constant. If the particle starts from rest at a distance a from O, show that its velocity at any instant is $v = k(x^2 - a^2)^{1/2}$, after time t given by $t = \frac{1}{k}\log_e \frac{x + (x^2 - a^2)^{1/2}}{a}$.

4. A particle of mass m lb. is projected on a smooth horizontal plane from a fixed point A with velocity u ft./sec. in the direction OA, O being a fixed origin and OA equal to a ft. It is subject to a resistance which varies inversely as the cube of its distance from O and is equal to m lb.wt. at A. Obtain the equation of motion of the particle and find its velocity when it is x ft. from O.

   Show also, that the time t secs., that the particle has been in motion is given by $(u^2 - ga)t = \{(u^2 - ga)x^2 + ga^3\}^{1/2} - au$. (LU., Pt. I)

5. A 25-lb. shell 3 in. in diameter has a travel of 10 ft. in the bore of the gun. The pressure on the base of the shell is initially 18 tons/in.^{2} and varies inversely as the volume behind the shell being 2 tons/in.^{2} as the shell is ejected. Find the muzzle velocity of the shell.

6. A particle of mass m moves from a fixed point O in a straight line with initial speed V, under a force which produces an acceleration k^2x directed away from O, when x is the displacement from O. Determine (i) the time taken to attain a speed U, (ii) the work done by the accelerating force in time t. (L.U., Pt. I)

7. A particle is let fall from a great height h above the earth. Neglecting air resistance, find the velocity with which it reaches the earth.

8. A particle is projected vertically upwards from the earth with a velocity of 1 mile per second. Neglecting air resistance find the height to which it rises and the total time in the air.

1.14 Acceleration as a Function of Velocity

A body moving through the air experiences a resistance opposing its motion and of different magnitude at different speeds. This resistance varies also with the shape of the body and with the density of the air. For velocities up to about 80 ft./sec. the resistance is found to be proportional to the velocity (Stoke's Law), while for velocities above this and below 1000 ft./sec., it is found to be proportional to the square of the velocity (Newton's Law).

It is usual to take the resistance at all speeds as being of the form

$$\tfrac{1}{2}\varrho v^2 A f_D$$

where v is the velocity, ϱ the density of the air, A the area presented and f_D a non-dimensional coefficient. For the range of velocities within which Newton's Law holds f_D is a constant. For higher velocities its value is found by accurate timing of a shell over successive distances or by wind tunnel measurements. Values of f_D are plotted against the Mach number, that is the ratio of the velocity to the speed of sound

(about 1120 ft./sec.), and the graph (Fig. 11) shows a sudden rise near the velocity of sound followed by a falling off for higher speeds. The resistance of course continues to increase at higher speeds since the coefficient f_D is multiplied by v^2.

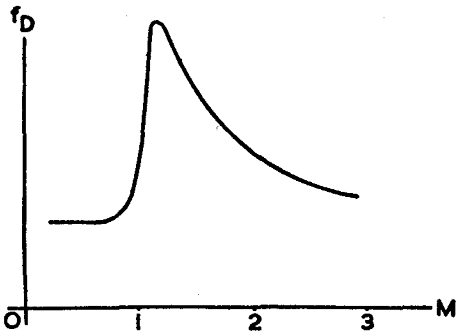

Fig. 11

We shall consider in detail the mathematical treatment of the cases of motion when the retardation due to air resistance is kv and when it is kv^2.

In either case we can form differential equations by equating the retardation to $\frac{dv}{dt}$ or to $v\frac{dv}{dx}$. One gives the connection between velocity and time, the other gives the connection between velocity and distance.

1.15 Resistance Proportional to Velocity—Horizontal Motion

Let the air resistance to a particle of mass m moving horizontally with velocity v be mkv.

We have
$$\frac{dv}{dt} = -kv,$$
$$\int\frac{dv}{v} = -k\int dt,$$
$$\log v = -kt + c \text{ (a constant).}$$

If U be the initial velocity, when $t = 0$, $c = \log U$ and
$$v = Ue^{-kt}.$$

Therefore
$$\frac{dx}{dt} = Ue^{-kt},$$
$$x = -\frac{U}{k}e^{-kt} + d \text{ (a constant).}$$

Taking $x = 0$ when $t = 0$ we have $d = U/k$, and

$$x = \frac{U}{k}(1 - e^{-kt}).$$

Thus the velocity decreases exponentially with the time and the distance x tends asymptotically to the value U/k as t tends to infinity.

We have also
$$v\frac{dv}{dx} = -kv,$$
$$\int dv = -k\int dx,$$
$$v = -kx + U.$$

Thus the velocity decreases uniformly with the distance.

1.16 Resistance Proportional to Velocity—Vertical Motion

In this case if the distance y be measured vertically upwards, the acceleration is $g + kv$ downwards. Therefore we have the two differential equations

$$\frac{dv}{dt} = -g - kv, \quad v\frac{dv}{dy} = -g - kv.$$

It should be noticed that these equations hold for both the up and the down motion since the change of sign of the velocity takes care of the change in direction of the retardation.

Let the initial velocity be V, then from the first equation

$$\int\frac{dv}{g + kv} = -\int dt,$$
$$\log(g + kv) = -kt + \text{constant},$$
$$= -kt + \log(g + kV),$$

and hence
$$g + kv = (g + kV)e^{-kt},$$
$$v = -\frac{g}{k} + \frac{g + kV}{k}e^{-kt}. \tag{1}$$

At the highest point $v = 0$ and hence if t_1 be the time to this point

$$t_1 = \frac{1}{k}\log\left(1 + \frac{kV}{g}\right).$$

We have also
$$\frac{dy}{dt} = -\frac{g}{k} + \frac{g + kV}{k}e^{-kt},$$
$$y = -\frac{gt}{k} - \frac{g + kV}{k^2}e^{-kt} + \text{constant}.$$

Since $y = 0$ when $t = 0$

$$y = -\frac{gt}{k} + \frac{g + kV}{k^2}(1 - e^{-kt}). \tag{2}$$

If y_1 be the highest point reached we have on putting $t = t_1$

$$y_1 = \frac{V}{k} - \frac{g}{k^2}\log\left(1 + \frac{kV}{g}\right).$$

Addition of (1) to (2) multiplied by k shows that throughout the motion

$$v = -gt - ky + V. \tag{3}$$

This equation may be obtained by direct integration of the differential equation with respect to the time.

Taking the second differential equation

$$v\frac{dv}{dy} = -g - kv,$$

$$\int\left(1 - \frac{g}{g + kv}\right)dv = -k\int dy,$$

$$v - \frac{g}{k}\log(g + kv) = -ky + V - \frac{g}{k}\log(g + kV). \tag{4}$$

This equation results also from the elimination of t between (1) and (2). The total time in the air may be found by putting $y = 0$ in (2) and solving numerically or graphically.

There comes a point, when the body is falling downwards, when the resistance of the air just balances the force of gravity and the acceleration is zero. The velocity at which this occurs is called the *terminal velocity*. In this case it occurs when $g = kv$ and hence the terminal velocity is g/k. A body falling in the air will never acquire a velocity greater than this and equation (1) shows that this is a limiting velocity which will not be acquired in finite time.

1.17 Resistance Proportional to Velocity—Trajectory Equation

Let the air resistance to a particle of mass m moving in a vertical plane with velocity v be mkv directly opposing the velocity. If the velocity v has components $\dot{x}$ and $\dot{y}$ parallel to a horizontal x-axis and a vertical y-axis respectively, the resistance has components $-mk\dot{x}$ and $-mk\dot{y}$ in the same directions.

The components of acceleration parallel to the axes are $\ddot{x}$ and $\ddot{y}$ and we have the equations of motion

$$m\ddot{x} = -mk\dot{x},$$
$$m\ddot{y} = -mk\dot{y} - mg.$$

These are the equations of motion which were solved in § 1.15 and § 1.16, and hence if initially $x = y = 0$, $\dot{x} = U$, $\dot{y} = V$, we have

$$x = \frac{U}{k}(1 - e^{-kt}), \tag{1}$$

$$= -\frac{gt}{k} + \left(\frac{V}{k} + \frac{g}{k^2}\right)(1 - e^{-kt}). \tag{2}$$

These equations give the position of the particle at any time and the components of velocity parallel to the axes are the derivatives of the expressions for x and y. Elimination of t between the equations (1) and (2) gives the trajectory equation

$$y = \left(V + \frac{g}{k}\right)\frac{x}{U} + \frac{g}{k^2}\log\left(1 - \frac{kx}{U}\right).$$

The highest point is reached when $\dot{y} = 0$, that is

$$-\frac{g}{k} + \left(V + \frac{g}{k}\right)e^{-kt} = 0,$$

$$t = \frac{1}{k}\log\left(1 + \frac{kV}{g}\right).$$

Substituting this value of t in (1) and (2) we have the coordinates of the highest point, namely

$$x = \frac{UV}{g + kV},$$

$$y = \frac{V}{k} - \frac{g}{k^2}\log\left(1 + \frac{kV}{g}\right).$$

The time for the whole trajectory is found by putting $y = 0$ in (2), giving

$$gkt = (g + kV)(1 - e^{-kt}),$$

$$t = \frac{g + kV}{gk}(kt - \tfrac{1}{2}k^2t^2 + \ldots),$$

$$1 = \left(1 + \frac{kV}{g}\right)(1 - \tfrac{1}{2}kt + \ldots).$$

A first approximation to the time when k is small is $t = 2V/g$, as in parabolic motion. A second approximation is

$$t = \frac{2V}{g}\left(1 - \frac{kV}{3g}\right),$$

and the corresponding horizontal displacement is

$$x = \frac{2UV}{g}\left(1 - \frac{4kV}{3g}\right).$$

1.18 Resistance Proportional to Velocity Squared—Horizontal Motion

Let the air resistance to a particle of mass m moving horizontally with velocity v be the mkv^2.

We have
$$\frac{dv}{dt} = -kv^2,$$
$$\int \frac{dv}{v^2} = -k\int dt,$$
$$\frac{1}{v} = kt + c.$$

If U be the initial velocity
$$c = 1/U,$$
$$v = \frac{U}{1 + kUt}.$$

Integrating again
$$x = \frac{1}{k}\log(1 + kUt),$$
the additive constant being zero.

Also since
$$v\frac{dv}{dx} = -kv^2,$$
$$\int \frac{dv}{v} = -k\int dx,$$
$$\log v = -kx + d \text{ (a constant)},$$
$$= -kx + \log U,$$
and
$$v = Ue^{-kx}.$$

1.19 Resistance Proportional to Velocity Squared—Vertical Motion

When the body is falling it reaches its terminal velocity when $g = kv^2$. Hence, denoting this terminal velocity by v_0 we have
$$v_0 = (g/k)^{1/2}.$$
The retardation when the body is ascending is thus $k(v_0^2 + v^2)$ and its acceleration when descending is $k(v_0^2 - v^2)$.

For motion vertically *upwards* with initial velocity V we have
$$\frac{dv}{dt} = -k(v_0^2 + v^2); \quad v\frac{dv}{dy} = -k(v_0^2 + v^2).$$

The first equation gives
$$\int \frac{dv}{v_0^2 + v^2} = -k\int dt,$$
$$\tan^{-1}\frac{v}{v_0} = -kv_0t + a \text{ (a constant)},$$
$$v = v_0 \tan(a - kv_0t), \tag{1}$$
where
$$a = \tan^{-1}\frac{V}{v_0}.$$

Also
$$y = v_0\int \tan(\alpha - kv_0t)dt,$$
$$= \frac{1}{k}\log\cos(\alpha - kv_0t) + c \text{ (a constant)},$$
$$= \frac{1}{k}\log\frac{\cos(\alpha - kv_0t)}{\cos\alpha},$$
$$= \frac{1}{k}\log\left(\cos v_0kt + \frac{V}{v_0}\sin v_0kt\right). \tag{2}$$

At the highest point
$$v = 0,$$
$$t_1 = \frac{1}{kv_0}\tan^{-1}\frac{V}{v_0}, \text{ from (1)},$$
$$y_1 = \frac{1}{2k}\log\left(1 + \frac{V^2}{v_0^2}\right), \text{ from (2)}.$$

From the second equation
$$\int\frac{vdv}{v_0^2 + v^2} = -k\int dy,$$
$$\log(v_0^2 + v^2) = -2ky + \log(v_0^2 + V^2),$$
$$v^2 = V^2e^{-2ky} - v_0^2(1 - e^{-2ky}). \tag{3}$$

Now consider the motion downwards, y being measured downwards and the initial velocity being zero.

We have
$$\frac{dv}{dt} = k(v_0^2 - v^2); \quad \frac{vdv}{dy} = k(v_0^2 - v^2).$$

From the first equation
$$\int\frac{dv}{v_0^2 - v^2} = k\int dt,$$
$$\log\frac{v_0 + v}{v_0 - v} = 2kv_0t + c \text{ (a constant which is zero)},$$
$$\frac{v_0 + v}{v_0 - v} = e^{-2kv_0t},$$
$$v = v_0\frac{1 - e^{-2kv_0t}}{1 + e^{-2kv_0t}},$$
$$= v_0\tanh kv_0t. \tag{4}$$
$$y = \int v_0\tanh kv_0t\,dt,$$
$$= \frac{1}{k}\log\cosh kv_0t + c \text{ (a constant which is zero)}. \tag{5}$$

The second equation gives

$$\int \frac{v\,dv}{v_0^2 - v^2} = k \int dy,$$

whence

$$\log (v_0^2 - v^2) = -2ky + \log v_0^2,$$
$$v^2 = v_0^2(1 - e^{-2ky}).$$

If the body falls the height y_1 in time t_2 we have from (5)

$$\frac{1}{2k} \log \left(1 + \frac{V^2}{v_0^2}\right) = \frac{1}{k} \log \cosh (kv_0 t_2),$$

and

$$t_2 = \frac{1}{kv_0} \sinh^{-1} \left(\frac{V}{v_0}\right).$$

The velocity v_2 acquired in this time is given by (4) as

$$v_2 = v_0 \tanh \left(\sinh^{-1} \frac{V}{v_0}\right),$$
$$= v_0\left(1 + \frac{v_0^2}{V^2}\right)^{-1/2}.$$

Hence for the up and down motion,

the time of flight $= \dfrac{1}{kv_0}\left\{\tan^{-1} \dfrac{V}{v_0} + \sinh^{-1} \dfrac{V}{v_0}\right\},$

the greatest height $= \dfrac{1}{2k} \log \left(1 + \dfrac{V^2}{v_0^2}\right),$

the final velocity $= v_0\left(1 + \dfrac{v_0^2}{V^2}\right)^{-1/2}.$

1.20 Trajectory with Resistance Proportional to Velocity Squared

Let the air resistance to a particle of mass m moving with velocity v be mkv^2 along the tangent to the trajectory, that is, inclined at an angle ψ to the horizontal. It is convenient in this case to use the components of acceleration $\dot{v}$ and v^2/ϱ or $v\dot{\psi}$ along the tangent and perpendicular to the tangent in the direction of ψ increasing.

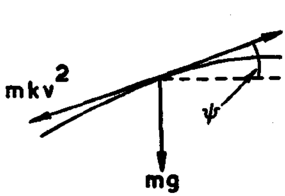

Fig. 12

The forces being mkv^2 and mg we have (Fig. 12) the equations of motion

$$m\dot{v} = -mkv^2 - mg \sin \psi,$$
$$mv\dot{\psi} = -mg \cos \psi.$$

Hence
$$\frac{\dot v}{\dot\psi} = \frac{dv}{d\psi} = \frac{kv^3}{g\cos\psi} + v\tan\psi,$$
$$\cos\psi\frac{dv}{d\psi} - v\sin\psi = \frac{k}{g}v^3,$$
$$\frac{d}{d\psi}(v\cos\psi) = \frac{k}{g}v^3,$$
$$\frac{-2}{(v\cos\psi)^3}\frac{d}{d\psi}(v\cos\psi) = -\frac{2k}{g}\sec^3\psi,$$
$$\frac{1}{(v\cos\psi)^2} = -\frac{2k}{g}\int\sec^3\psi d\psi,$$
$$= -\frac{k}{g}\{\sec\psi\tan\psi + \log(\sec\psi + \tan\psi)\} + c.$$
With the initial conditions $v = V$ when $\psi = \alpha$, we have
$$\frac{1}{(v\cos\psi)^2} = \frac{1}{(V\cos\alpha)^2} - \frac{k}{g}\{f(\psi) - f(\alpha)\},$$
where
$$f(\psi) = \sec\psi\tan\psi + \log(\sec\psi + \tan\psi).$$
Hence
$$v^2 = \frac{V^2\cos^2\alpha\sec^2\psi}{1 - \dfrac{kV^2\cos^2\alpha}{g}\{f(\psi) - f(\alpha)\}}.$$
From the second equation of motion we have
$$\dot\psi = -\frac{g\cos\psi}{v},$$
$$\frac{ds}{d\psi} = \frac{ds}{dt}\frac{dt}{d\psi} = \frac{v}{\dot\psi},$$
$$= -\frac{v^2}{g\cos\psi},$$
$$= -\frac{1}{g}\frac{V^2\cos^2\alpha\sec^3\psi}{1 - \dfrac{kV^2\cos^2\alpha}{g}\{f(\psi) - f(\alpha)\}}.$$
Now
$$\frac{d}{d\psi}f(\psi) = 2\sec^3\psi,$$
therefore
$$ds = -\frac{1}{2k}\frac{\dfrac{kV^2\cos^2\alpha}{g}df(\psi)}{1 - \dfrac{kV^2\cos^2\alpha}{g}\{f(\psi) - f(\alpha)\}},$$
$$s = \frac{1}{2k}\log\left[1 - \frac{kV^2\cos^2\alpha}{g}\{f(\psi) - f(\alpha)\}\right].$$

There is no added constant since $s = 0$ when $\psi = \alpha$ and we have

$$e^{2ks} = 1 - \frac{kV^2 \cos^2 \alpha}{g}\{f(\psi) - f(\alpha)\}.$$

This is the s, ψ equation of the trajectory. Equations connecting x, y and t with ψ are obtained from the equations

$$\frac{dx}{d\psi} = \cos \psi \frac{ds}{d\psi} = -\frac{v^2}{g},$$

$$\frac{dy}{d\psi} = \sin \psi \frac{ds}{d\psi} = -\frac{v^2}{g} \tan \psi,$$

$$\frac{dt}{d\psi} = -\frac{v}{g} \sec \psi.$$

The integration necessary to obtain x, y and t explicitly in terms of ψ can only be carried out numerically.

EXERCISES 1 (*c*)

1. A particle is projected vertically upwards under gravity with initial speed V, there being a resistance to the motion which at every instant is proportional to the velocity at that instant. The particle returns to the starting point after time T. Show that its speed is then $gT - V$. (L.U., Pt. I)

2. A sphere of mass m is allowed to fall freely in a medium. If the resistance of the medium to the motion is k times the velocity of the sphere, show that this velocity will approach a limiting value $V = mg/k$. Show also that the distance fallen from rest in time t is

$$Vt - \frac{mV}{k}(1 - e^{-kt/m}).$$ (L.U., Pt. I)

3. An engine which works at the constant rate of 396 h.p., draws a train of 275 tons along a level track. The resistance to motion is partly constant and partly varies as the velocity. When the velocity is 45 m.p.h., the resistance is 12 lb.wt. per ton, and when the velocity is 15 m.p.h., it is $4\frac{4}{5}$ lb.wt. per ton; prove that the maximum velocity of the train is 45 m.p.h.

   Show further that an increase of 1 per cent in the above given h.p. produces approximately an increase of 0·5 per cent in the maximum velocity. (L.U., Pt. II)

4. A small body falls from rest in a medium whose resistance per unit mass of the body is k times the square of the velocity. Show that the distance it must fall to acquire a velocity v is

$$\frac{1}{2k} \log \frac{g}{g - kv^2},$$

   and that the velocity approaches a limiting value $(g/k)^{1/2}$.

If the body is projected upward with this limiting velocity, show that its speed on again reaching the point of projection will be $(g/2k)^{1/2}$. (L.U., Pt. I)

5. A body of mass 1 lb. falls from rest under gravity in a medium exerting a resistance $0{\cdot}01v^2$ lb.wt. when the speed is v ft. per sec. State the terminal velocity.

Taking the value of g to be 32 ft./sec.^{2} show that the speed attained after falling 5 ft. is 9·8 ft. per sec. approximately and find the time taken. (L.U., Pt. I)

6. A toboggan of mass 200 lb. descends from rest down a slope 50 yd. long at 5 in 13. The coefficient of sliding friction is 0·1 and air resistance varies as the square of the speed, being 3 lb.wt. at a speed of 10 ft./sec. Show that it reaches the bottom with a speed of 38·6 ft./sec., and that if the slope were longer, it would never attain a speed of 44·2 ft./sec. (L.U., Pt. I)

7. A train is drawn by an engine which exerts a constant pull at all speeds, and the total resistance to motion varies as the square of the speed. The combined mass of engine and train is 300 tons, the maximum speed on the level is 60 m.p.h., and the horse-power then developed is 1500. Prove that, when climbing a slope of 1 in 100, the maximum speed is nearly 32 m.p.h.

Write down the equation of motion on the level, and find the distance travelled from rest in acquiring a speed of 45 m.p.h. (L.U., Pt. I)

8. A train of weight W tons moves on the level under the action of a pull P tons wt. against a resistance R tons wt., and the speed at any instant is v ft./sec. Show that the distance travelled while the speed varies from v_0 to v_1 is

$$\frac{W}{g}\int_{v_0}^{v_1}\frac{vdv}{P-R}.$$

If $W = 300$ and $R = 0{\cdot}9 + 0{\cdot}007v^2$, show that the distance travelled in slowing down from 45 to 30 m.p.h. with power cut off is about 520 ft. (L.U., Pt. I)

9. If the retardation caused by the resistance to the motion of a train is $a + bv^2$, where v is the velocity, show that it will come to rest from velocity V in a distance $\frac{1}{2b}\log_e\left(1+\frac{bV^2}{a}\right)$.

The resistances to motion of a train of 300 tons wt. are $(900 + 7v^2)/100$ tons wt. when the speed is v ft./sec. Find the distance travelledinsl owing from 45 m.p.h. to 30 m.p.h. with the steam cut off.

10. A particle of mass m moves in a straight line in a medium whose resistance is $mkv(v^2 + a^2)$, where v is the speed and a and k are constants. Show that whatever the initial speed the total distance moved will be less than $\pi/2ak$ and that the particle will not come to rest in a finite time. (L.U., Pt. I)

11. A particle of unit mass moves under gravity in a medium whose resistance is kv where v is the velocity and k is constant. If the particle is projected vertically upwards with velocity v_0 find the time to reach the highest point.

If it is projected with a velocity whose horizontal and vertical components are u_0 and v_0 respectively, prove that the horizontal distance of the highest point of the path from the starting point is

$$u_0v_0/(g + kv_0). \qquad \text{(L.U., Pt. I)}$$

12. Assuming that the resistance to the motion of a projectile varies as the velocity, so that the retardation is kv when the velocity is v, determine the position at time t of a particle projected with velocity V at an angle α above the horizontal.

Show also that asymptotically the particle falls vertically with constant velocity. (L.U., Pt. II)

13. A bomb is released from an aeroplane travelling horizontally at a speed of U ft. per sec. at an altitude of H feet. If the air resistance on the bomb per unit mass is k times its velocity, show that the path of the bomb, t seconds after release, will be inclined to the horizontal at an angle

$$\tan^{-1} g(e^{kt}-1)/kU.$$

Show also that, if the altitude H is great, the bomb-sights, directed on to the target at the instant of release, will be inclined to the horizontal at an angle $\tan^{-1} kH/U$. (L.U., Pt. I)

14. An aeroplane flying horizontally with speed V releases a bomb. If the air resistance is kv when the speed is v, show that the horizontal and vertical distances travelled by the bomb in time t are $V(1 - e^{-kt})/k$ and $g(e^{-kt} - 1 + kt)/k^2$ respectively.

If the bomb is released at a height of 19,600 ft. when the speed of the aeroplane is 200 m.p.h. and if the squares and higher powers of k are negligible show that it reaches the ground in $(210 + 1225k)/6$ sec., after having covered a horizontal distance $880(105 - 1225k)/9$ ft. (L.U., Pt. II)

15. A particle of mass m, moving in the x, y plane, is attracted towards the axis of x by a force $\mu m/y^3$ when at a point (x, y). Show that, if μ is constant and the particle is projected from the point $(0, k)$ with component velocities U, V parallel to the axes of x and y, it will reach a highest point if $\mu > k^2V^2$. Find the coordinates of the highest point if this condition is satisfied. (L.U., Pt. II)

1.21 Motion with Varying Mass

Newton's second law of motion tells us that the force on a particle is equal to the rate of change of momentum, that is

$$P = \frac{d}{dt}(mv),$$

$$= m\frac{dv}{dt} + v\frac{dm}{dt}.$$

When the mass is varying this equation is only true in certain circumstances. For the full solution of the problem we need to know the relative velocity of the mass which is being added or lost.

Suppose that a particle of mass m is moving in a straight line with velocity v under the action of a force P and that in time δt it receives an additional amount of mass δm which is moving in the opposite direction with velocity V. Then the relative velocity of δm with respect to m is $(V + v)$ in the direction of V and we write for the relative velocity

$$u = V + v.$$

To the first order of small quantities the momentum of δm is altered from $V\delta m$ to $v\delta m$ in the opposite direction, and its loss of momentum is $\delta m(V + v) = u\delta m$.

Therefore the force R between m and δm which destroys this momentum is given by

$$R\delta t = u\delta m,$$

that is, in the limit,
$$R = u\frac{dm}{dt}.$$

Therefore, the equation of motion of the mass m is

$$P - R = m\frac{dv}{dt},$$

$$P = m\frac{dv}{dt} + u\frac{dm}{dt}.$$

In this equation $\frac{dm}{dt}$ is positive if mass is being added and negative if it is being lost. Also, u is the relative velocity backwards of the mass which is being added or lost.

If the added mass is at rest before being picked up or the lost mass is brought to rest as it drops off, $u = v$ and we have the equation in Newton's form.

1.22 The Rocket Equation

When a rocket is fired vertically upwards mass is projected backwards with relative velocity u and, if air resistance be neglected, P is the gravitational pull downwards. We have, therefore, for the motion of the rocket while matter is being ejected

$$-mg = m\frac{dv}{dt} + u\frac{dm}{dt}.$$

Example 6. *A uniform chain of length l and mass m lies coiled on the edge of a smooth table, and a mass equal to the weight of the chain is attached to one end and hangs over the edge. If the table is $2l$ above the floor find the velocity of the mass as the chain leaves the table and the thrust on the floor when half of the chain has reached the floor.*

When a length x of the chain is over the edge, the moving mass is $m(1 + x/l)$, and its velocity is $\dot{x}$. Also, the added mass is brought into motion from rest and we have

$$m\left(1 + \frac{x}{l}\right)g = \frac{d}{dt}\left\{m\left(1 + \frac{x}{l}\right)\dot{x}\right\},$$

$$(x + l)g = \frac{d}{dt}(x + l)\dot{x},$$

$$= \dot{x}\frac{d}{dx}(x + l)\dot{x},$$

$$(x + l)^2 g = (x + l)\dot{x}\frac{d}{dx}(x + l)\dot{x},$$

$$= \frac{1}{2}\frac{d}{dx}(x + l)^2\dot{x}^2.$$

Then on integration

$$\frac{1}{3}(x + l)^3 g = \frac{1}{2}(x + l)^2\dot{x}^2 + c \text{ (a constant)}.$$

Since $\dot{x} = x = 0$ initially, $c = \frac{1}{3}gl^3$ and we have

$$\dot{x}^2 = \frac{2}{3}g\frac{x^3 + 3x^2 l + 3xl^2}{(x + l)^2}.$$

Hence the velocity as the chain leaves the table is $\left(\frac{7}{6}gl\right)^{1/2}$.

When the top of the chain is at a height $l - y$ above the floor it has fallen freely a further distance $l + y$ and its velocity is given by

$$v^2 = \frac{7}{6}gl + 2g(l + y).$$

At this instant a length of chain $v\delta t$ is brought to rest by the reaction of the floor from a velocity v in time δt. Hence the reaction is given by

$$R\delta t = \left(\frac{m}{l}v\right)v\delta t,$$

$$R = m\frac{v^2}{l},$$

$$= \frac{mg}{l}\left(\frac{7}{6}l + 2l + 2y\right).$$

In addition, there is the mass $m + my/l$ already on the floor so that the total reaction is

$$R_1 = \frac{mg}{l}\left(\frac{25}{6}l + 3y\right),$$

and when $y = \frac{1}{2}l$ this is $\frac{17}{3}mg$.

Example 7. *A rocket has mass a when full, it burns for time T and after time t its mass is $a - bt$. The relative backward velocity of the gases is u. If the rocket is ignited and moves vertically upwards from rest, find the velocity and height reached after time t ($< T$), (i) when the air resistance is neglected, (ii) when the air resistance is proportional to the velocity.*

(i) We have $\dfrac{dm}{dt} = -b$, and

$$-(a - bt)g = (a - bt)\frac{dv}{dt} - bu,$$

$$\frac{dv}{dt} = -g + \frac{bu}{a - bt},$$

$$v = -gt - u \log (a - bt) + \text{constant},$$

$$= -gt - u \log \left(1 - \frac{b}{a}t\right).$$

Integrating once more we have

$$x = -\frac{1}{2}gt^2 - u\int \log \left(1 - \frac{b}{a}t\right)dt,$$

$$= -\frac{1}{2}gt^2 - u\left(t - \frac{a}{b}\right) \log \left(1 - \frac{b}{a}t\right) + ut.$$

(ii) Let the air resistance be kbv when the velocity is v, k being constant. The equation of motion is

$$-(a - bt)g - kbv = (a - bt)\frac{dv}{dt} - bu,$$

$$\frac{dv}{dt} + \frac{kbv}{a - bt} = -g + \frac{bu}{a - bt},$$

$$\frac{d}{dt}(a - bt)^{-k}\, v = -g(a - bt)^{-k} + bu(a - bt)^{-k-1},$$

$$(a - bt)^{-k}\, v = \frac{g}{b(1 - k)}(a - bt)^{1-k} + \frac{u}{k}(a - bt)^{-k} + c.$$

Since $v = 0$ when $t = 0$

$$-c = \left\{\frac{ga}{b(1 - k)} + \frac{u}{k}\right\}a^{-k},$$

$$v = \frac{u}{k} + \frac{g(a - bt)}{b(1 - k)} - \left\{\frac{ga}{b(1 - k)} + \frac{u}{k}\right\}\left(1 - \frac{b}{a}t\right)^k.$$

Integrating once more we have

$$x = \frac{ut}{k} + \frac{g}{b(1 - k)}\left(at - \frac{1}{2}bt^2\right)$$
$$- \left\{\frac{ga^2}{b^2(1 - k^2)} + \frac{au}{bk(1 + k)}\right\}\left\{1 - \left(1 - \frac{bt}{a}\right)^{k+1}\right\}.$$

If the air resistance is taken as proportional to the velocity squared the differential equation obtained is not soluble by elementary methods, and numerical integration must be used.

EXERCISES 1 (*d*)

1. A uniform chain is coiled on a table. One end is raised vertically from the coil to pass over a light smooth pulley 1 ft. in diameter whose centre is 2 ft. above the table and 3 ft. of the chain hangs vertically on the other side of the pulley. If the system is released from rest in this position find the velocity of the chain when a further 10 ft. has passed over the pulley.

2. A uniform chain 3 ft. long weighing $\frac{1}{2}$ lb./ft. is hung vertically with its lower end 4 ft. above the ground. If the chain is released in this position find the thrust on the ground when 2 ft. of the chain has reached the ground.

3. Two masses, P and $(P + Q)$ are attached to the ends of a light inextensible string which passes over a smooth fixed pulley. If Q loses mass at a constant rate, becoming zero in time T, and the system starts from rest, find the velocity when Q vanishes. (Neglect the inertia of the pulley.) (L.U., Pt. I)

4. A uniform chain of length l and weight W lies on the ground. A rope is attached to one end and hauled vertically at a steady speed v until the whole chain is clear of the ground. Find the pull in the rope at this instant.

5. A rocket moves vertically upwards with initial speed V. During the ascent matter is continuously ejected vertically downwards with constant velocity u relative to the rocket. If m is the mass of the rocket at time t, and if v is the velocity at this instant, prove that the equation of motion is
$$m\left(g + \frac{dv}{dt}\right) = -u\frac{dm}{dt}.$$
If m_0 is the mass of the rocket initially (when its speed is V and when $t = 0$), prove that
$$v = V - gt + u \log \frac{m_0}{m}. \qquad \text{(L.U., Pt. II)}$$

6. A rocket has initial mass 40 lb. and the charge burns at the rate of 3 lb./sec. being all burnt after 10 sec. The relative backward velocity of the gases is 640 ft./sec. If the rocket is ignited and begins to move vertically upwards find, neglecting air resistance, the velocity and the height reached when the charge is all burnt.

7. If the rocket in the preceding example is subject to a resistance $1{\cdot}5v$ pdl. when moving at v ft./sec. and moves vertically upwards, find the velocity and the height reached when the charge is all burnt.

8. A rocket of initial mass m_0 and rate of burning r ejects matter backwards with relative velocity u. If it moves horizontally against an air resistance proportional to the square of its velocity such that its terminal velocity in the horizontal motion is v_0 and the effect of gravity is neglected find an expression for its velocity when it has burnt for time t.

1.23 Motion in a Circle

When a particle moves in a circle of radius r its position at any instant is determined by one coordinate θ, the angle which the radius to the particle makes with a fixed direction, that is, its position is given by polar coordinates (r, θ) where r is constant. Hence its velocity is $r\dot{\theta}$

perpendicular to the radius in the direction of θ increasing. Also $\dot{\theta}$ is the angular velocity with which the circle is being described.

The components of accelerations are readily found in the polar form to be $r\ddot{\theta}$ along the tangent in the direction of θ increasing, and $r\dot{\theta}^2$ towards the centre of the circle (Fig. 13).

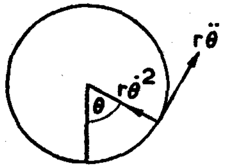

Fig. 13

1.24 Conical Pendulum

A conical pendulum consists of a particle suspended by a light inextensible string from a fixed point and projected so as to describe a horizontal circle with uniform velocity.

Let l be the length of the string, h the depth of the circle below the fixed point and m the mass of the particle.

The forces acting on the particle are its weight and the tension of the string T and there is no component of force along the tangent to the horizontal circle. Hence, if $\dot{\theta}$ be the angular velocity, $\ddot{\theta} = 0$ and $\dot{\theta}$ is constant.

If $h = l \cos a$, a is the semi-vertical angle of the cone swept out by the string, the radius of the horizontal circle is $l \sin a$ and the only acceleration is $l \sin a \, \dot{\theta}^2$ towards the centre of the circle (Fig. 14).

Hence
$$T \cos a = mg,$$
$$T \sin a = ml \sin a \, \dot{\theta}^2.$$

Therefore
$$\dot{\theta}^2 = \frac{g}{l \cos a} = \frac{g}{h},$$
$$T = \frac{mgl}{h} = ml\dot{\theta}^2,$$

and the time of a revolution is $2\pi\sqrt{(h/g)}$.

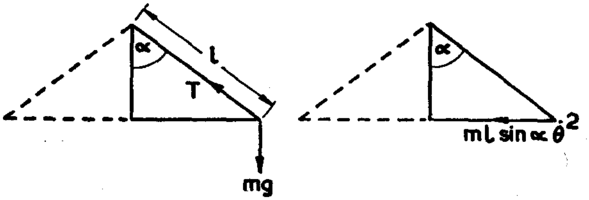

Fig. 14

Thus the speed of rotation is related to the depth of the circle below the fixed point and the tension is proportional to the square of the speed. This principle is used in the governor of a steam engine where an increase of speed causes an increase in the tension of the framework which opens a valve and reduces the speed.

1.25 Tension in a Belt running over a Pulley

Let a be the radius of the pulley, m the mass per unit length of the belt and consider the forces on a small element $ma\delta\psi$ of the belt (Fig. 15). If T and $T + \delta T$ be the tensions at the ends of the element, their difference has components,

$$(T + \delta T) \cos \delta\psi - T, \text{ tangentially,}$$
$$(T + \delta T) \sin \delta\psi, \text{ towards the centre.}$$

That is, to the first order of small quantities,

$$\delta T, \text{ tangentially,}$$
$$T\delta\psi, \text{ towards the centre.}$$

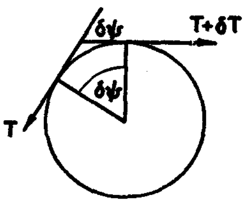

Fig. 15

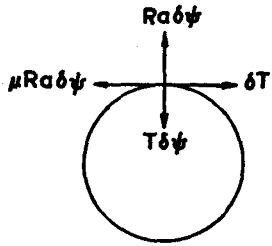

Fig. 16

Let R be the reaction per unit length and μ the coefficient of friction between the belt and the pulley. Assuming the friction to be limiting we have therefore forces

$$Ra\delta\psi, \text{ normal,}$$
$$\mu Ra\delta\psi, \text{ tangentially (Fig. 16).}$$

Hence if the belt is running at uniform speed v the central acceleration of the element is v^2/a and we have

$$\delta T = \mu Ra\delta\psi,$$
$$ma\delta\psi.\frac{v^2}{a} = T\delta\psi - Ra\delta\psi.$$

Eliminating R we have $$\delta T = \mu(T - mv^2)\delta\psi,$$

and in the limit $$\frac{dT}{d\psi} = \mu(T - mv^2).$$

Hence $$\log (T - mv^2) = \mu\psi + \text{constant},$$

that is $$T - mv^2 = Ce^{\mu\psi}.$$

Therefore, if T_1 and T_2 be the tensions at the points where the belt clears the pulley, πa apart,

$$T_1 - mv^2 = (T_2 - mv^2)e^{\mu\pi}.$$

1.26 Deviation of a Plumb Line

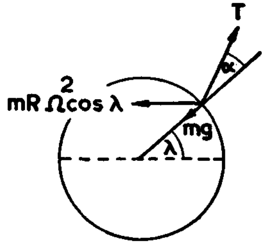

Fig. 17

A point on the surface of the earth in latitude λ is rotating in a circle of radius $R\cos\lambda$, where R is the earth's radius, with angular velocity Ω and has an acceleration $R\Omega^2\cos\lambda$ towards the axis of rotation (Fig. 17), where $\Omega = 10^{-5}\times 7{\cdot}29$ radians/sec., approximately. Hence a plumb line of mass m held in this latitude will deviate from the vertical towards the equator by an angle α so that its tension and the force of gravity may combine to provide this acceleration.

That is
$$T\sin\alpha = mR\Omega^2\cos\lambda.\sin\lambda,$$
$$T\cos\alpha = mg - mR\Omega^2\cos\lambda.\cos\lambda.$$

Therefore
$$\tan\alpha = \frac{R\Omega^2\cos\lambda\sin\lambda}{g - R\Omega^2\cos^2\lambda},$$
$$= \frac{R\Omega^2\sin 2\lambda}{2g}, \text{ approximately.}$$

This deviation has its greatest value of approximately 6 minutes in latitude 45°.

The decrease in g due to the rotation is $R\Omega^2\cos^2\lambda$. This is greatest at the equator where its value is approximately 0·112 ft./sec.²

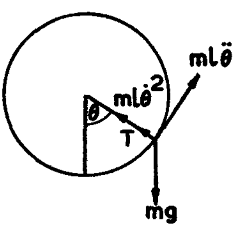

Fig. 18

1.27 The Simple Pendulum

If a particle is moving in a vertical circle at the end of a light inextensible string whose other end is fixed, the forces acting on it are its weight and the tension in the string (Fig. 18). Hence if m be the mass of the particle, l the length of the string and θ the angle which it makes with the vertical at any instant we have the equations of motion

$$ml\ddot\theta = -mg\sin\theta,$$
$$ml\dot\theta^2 = T - mg\cos\theta.$$

The first equation may be integrated with respect to θ. We have

$$l\frac{d}{d\theta}\left(\frac{1}{2}\dot\theta^2\right) = -g\sin\theta,$$
$$l\dot\theta^2 = 2g\cos\theta + \text{constant}.$$

If θ remains small, so that $\sin\theta$ is approximately equal to θ, the equation of motion gives

$$l\ddot\theta = -g\theta.$$

This is the equation of a simple harmonic oscillation whose period is $2\pi\sqrt{(l/g)}$ and the motion is said to be that of a simple pendulum. If the amplitude of the oscillation is α we have

$$\theta = \alpha \cos \sqrt{(g/l)}t.$$

Since $\dot{\theta} = 0$ when $\theta = \alpha$ we have

$$\begin{aligned} l\dot{\theta}^2 &= 2g(\cos\theta - \cos\alpha), \\ &= 4g(\sin^2 \tfrac{1}{2}\alpha - \sin^2 \tfrac{1}{2}\theta), \end{aligned}$$

$$2\left(\frac{g}{l}\right)^{\frac{1}{2}} dt = \frac{d\theta}{(\sin^2 \frac{1}{2}\alpha - \sin^2 \frac{1}{2}\theta)^{1/2}}.$$

If T be the period, the quarter period is the time between $\theta = 0$ and $\theta = \alpha$ and we have

$$\frac{1}{2}\left(\frac{g}{l}\right)^{\frac{1}{2}} T = \int_0^\alpha \frac{d\theta}{(\sin^2 \frac{1}{2}\alpha - \sin^2 \frac{1}{2}\theta)^{1/2}}.$$

Substituting $\sin \frac{1}{2}\theta = k \sin\phi$, where $k = \sin \frac{1}{2}\alpha$, we have

$$\begin{aligned} \frac{1}{2}\left(\frac{g}{l}\right)^{\frac{1}{2}} T &= 2\int_0^{\pi/2} \frac{d\phi}{(1 - k^2 \sin^2\phi)^{1/2}}, \\ &= 2\int_0^{\pi/2} d\phi \left\{1 + \frac{1}{2}k^2 \sin^2\phi + \frac{1.3}{2.4}k^4 \sin^4\phi \right. \\ &\qquad \left. + \frac{1.3.5}{2.4.6}k^6 \sin^6\phi + \dots \right\}. \end{aligned}$$

Now
$$2\int_0^{\pi/2} d\phi = \pi,$$

$$2\int_0^{\pi/2} \sin^2\phi\, d\phi = \frac{1}{2}\pi,$$

$$2\int_0^{\pi/2} \sin^4\phi\, d\phi = \frac{1.3}{2.4}\pi, \text{ and so on.}$$

Therefore

$$T = 2\pi\left(\frac{l}{g}\right)^{\frac{1}{2}} \left\{1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1.3}{2.4}\right)^2 k^4 + \left(\frac{1.3.5}{2.4.6}\right)^2 k^6 \dots \right\}.$$

Since $k = \sin \frac{1}{2}\alpha$, a second approximation to the period is found by putting $k = \frac{\alpha}{2}$ and ignoring α^4 and higher powers of α, giving

$$T = 2\pi\left(\frac{l}{g}\right)^{\frac{1}{2}}\left(1 + \frac{\alpha^2}{16}\right).$$

EXERCISES 1 (*e*)

1. A mass of 10 lb. rests on a rough horizontal table with coefficient of friction $\frac{1}{2}$. It is attached to one end of a light inextensible string which passes through a smooth hole in the table and carries a mass of 4 lb. at its free end. If the mass of 4 lb. describes a horizontal circle with a uniform velocity of 8 ft./sec. and the mass on the table is on the point of slipping, find the radius of the circle and the length of string below the table.

2. A heavy bob B is attached to a pivot P by a light rod 20 in. long. P is on an arm rigidly attached to a vertical shaft and is 6 in. from the axis of the shaft. The shaft rotates at 4 rad./sec. Find graphically or otherwise the angle θ which PB makes with the vertical to the nearest degree. (L.U., Pt. I)

3. The smooth inside surface of a bowl is a segment of a sphere of radius 8 in., the height of the segment being 4 in. The bowl has its axis fixed and vertical, and it is rotated about the axis. Find the greatest permissible angular velocity of the bowl, in revolutions per minute, if a particle placed in it can remain at rest relative to the bowl just within the rim. (L.U.)

4. A steam governor consists of four equal light rods, AB, BC, CD, DA, of length 10 in., freely hinged at their ends and rotating about a vertical axis to which it is hinged at the fixed point A and on which it slides without friction on a light collar attached to the rods at C below A. It carries masses of 6 lb. at B and D, and is kept in position by a spring at C. Find the number of revolutions being made per minute if the force exerted by the spring is 2 lb.wt. downwards and AC equals 12 in. (L.U.)

5. In a simple governor four equal, light smoothly jointed links form a rhombus $ABCD$ of side 1 ft. Two balls each of mass 6 lb. are attached at B and D. The point A is fixed on a vertical rotating shaft, and at C there is a collar of mass 40 lb. which can slide freely on the shaft. Find the speed of the shaft at which the balls will rotate in a circle of radius 6 in. If the speed is increased by 10 r.p.m. find the amount by which the collar rises.

6. Two pulley wheels each 3 ft. in diameter are coupled together by a leather belt the angle of contact on each wheel being 180°. The belt weighs 1·5 lb. per ft. run and runs at a speed of 50 ft./sec. If the tension in the belt is to be limited to 800 lb. and the coefficient of friction between the belt and the wheels is 0·3, calculate the maximum torque and horse-power which can be transmitted at this speed. (C.U.)

7. An elastic band 1 ft. in length weighs 0·5 lb. and its modulus of elasticity is 1 lb. The band is fitted to the outside of the rim of a wheel 2 ft. in diameter. Show that the reaction between the band and the wheel is 5·28 lb. per ft. run and find the speed of rotation at which the band will become loose on the wheel.

8. Show that if a seconds pendulum is allowed to swing through an angle of 30° either side of the vertical its half-period will be 1·017 sec., and that a clock governed by this pendulum will lose just over 24 minutes a day.
9. A particle is suspended by a light inextensible string of length l from a fixed point and given a horizontal velocity $(ngl)^{1/2}$. Show that if it describes a circle without the string becoming slack $n > 5$. Show that the time to complete the circle is the same as the period of a simple pendulum of length l/n oscillating through an angle $2 \sin^{-1} (2/n^{1/2})$ either side of the vertical.

1.28 Motion on a Curve

If a particle moves along a smooth curve the forces acting on it at any point will be the reaction normal to the tangent and its weight (Fig. 19).

Hence the equations of motion are

$$m\dot{v} = -mg \sin \psi,$$
$$mv\dot{\psi} = R - mg \cos \psi.$$

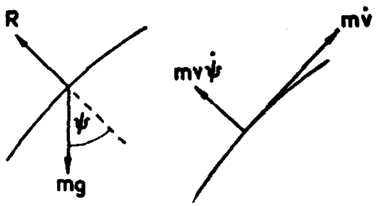

Fig. 19

Since $\dot{v} = v\dfrac{dv}{ds}$,

$$\frac{1}{2}v^2 = -g\int \sin \psi ds,$$
$$= -g\int \frac{dy}{ds}ds,$$
$$= -gy + \text{constant}.$$

This equation follows immediately from the energy equation

$$\frac{1}{2}mv^2 + mgy = \text{constant}.$$

If the equation of the curve is given in the intrinsic form $s = f(\psi)$, $\int \sin \psi ds$ may be evaluated and hence v obtained in terms of s, and since $v = \dfrac{ds}{dt}$ the relation between distance and time is obtained by a second integration.

If there is in addition a tangential component of force T (Fig. 20) the first equation of motion becomes

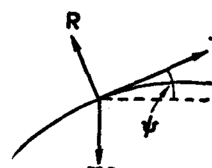

Fig. 20

$$m\dot{v} = T - mg \sin \psi$$

and $$\frac{1}{2}mv^2 = \int Tds - mgy + \text{constant},$$

that is the energy is increased by the work done by the force T as the particle moves along the curve.

1.29 Trajectories of Shells

The trajectory of a shell fired with a given velocity at a given elevation is determined by the force of gravity and the air resistance to the motion which acts backwards in the direction of the tangent to the trajectory. The air resistance to a shell of mass m when the velocity is v is taken as $mcv^2P(v)$, where c is a constant and $P(v)$ is found experimentally and tabulated for different values of v.

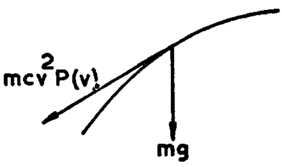

Fig. 21

The equations of motion are then (Fig. 21)

$$m\dot{v} = -mg \sin\psi - mcv^2P(v), \tag{1}$$
$$mv\dot{\psi} = -mg\cos\psi. \tag{2}$$

Hence $\quad \dot{v}\cos\psi + g\cos\psi\sin\psi = -cv^2P(v)\cos\psi,$

and substituting for $\cos\psi$ from (2)

$$\dot{v}\cos\psi - v\dot{\psi}\sin\psi = \frac{c}{g}v^3P(v)\dot{\psi},$$

that is
$$\frac{d}{d\psi}(v\cos\psi) = \frac{c}{g}\,v^3P(v). \tag{3}$$

Here $v\cos\psi$ is the horizontal component of velocity and the vertical component is $(v\cos\psi)\tan\psi$. The equation (3) is integrated numerically by what is called the *small arc process*. Values $\psi_1, \psi_2, \ldots, \psi_r \ldots$ of the slope of the trajectory are taken differing by a constant amount of one degree or less. The corresponding values of v are given by equations

$$v_{r+1}\cos\psi_{r+1} - v_r\cos\psi_r = \frac{c}{g}\int_{\psi_r}^{\psi_{r+1}} v^3P(v)dv.$$

A value v_r having been found, a forward estimate of v_{r+1} is made by the methods of finite differences and the integral is evaluated to verify this estimate, the process being repeated if there is disagreement. Thus a relationship between v and ψ is built up over the whole trajectory. A relationship between $\dot{\psi}$ and ψ follows from (2) whence a relationship between ψ and t is built up over the same small arcs.

We have also

$$\frac{dx}{d\psi} = \frac{1}{\dot{\psi}}\frac{dx}{dt} = \frac{v\cos\psi}{\dot{\psi}} = -\frac{v^2}{g}, \text{ from (2)},$$
$$\frac{dy}{d\psi} = \frac{1}{\dot{\psi}}\frac{dy}{dt} = \frac{v\sin\psi}{\dot{\psi}} = -\frac{v^2}{g}\tan\psi.$$

These equations integrated over the same arcs provide the complete solution of the trajectory.

Alternatively, the trajectory may be divided up according to equal intervals of time and all integration carried out with respect to t.

This laborious computation is performed in a few seconds on an electronic computer.

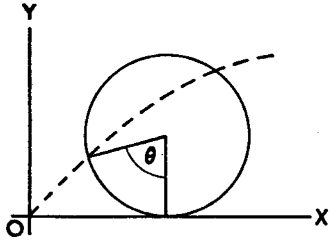

Fig. 22

1.30 Cycloidal Pendulum

A cycloid is the curve traced out by a point on the circumference of a circle as the circle rolls along a horizontal straight line. When the circle of radius a has turned through an angle θ the point of contact (Fig. 22) has moved a distance $a\theta$ and the horizontal and vertical displacements of the point are

$$x = a\theta - a\sin\theta, \quad y = a - a\cos\theta.$$

Hence,
$$\frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{\sin\theta}{1-\cos\theta} = \cot\frac{1}{2}\theta.$$

The inclination of the tangent to the horizontal is therefore

$$\psi = \left(\frac{\pi}{2} - \frac{1}{2}\theta\right).$$

Now
$$\begin{aligned} ds &= (dx^2 + dy^2)^{1/2}, \\ &= \{a^2(1-\cos\theta)^2 + a^2\sin^2\theta\}^{1/2}d\theta, \\ &= 2a\sin\frac{1}{2}\theta d\theta, \\ &= -4a\cos\psi d\psi. \end{aligned}$$

Therefore
$$s = 4a\,(1 - \sin\psi).$$

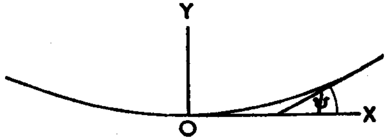

Fig. 23

Here the distance s is measured from the point O. The intrinsic equation of an inverted cycloid (Fig. 23) when the distance is measured from the lowest point of the curve is therefore

$$s = 4a\sin\psi,$$

If a particle is free to move on a smooth inverted cycloid the tangential component of force is $mg\sin\psi$ and we have

$$m\dot{v} = -mg\sin\psi,$$

that is
$$\begin{aligned} \frac{d^2s}{dt^2} &= -g\sin\psi, \\ &= -\frac{g}{4a}s. \end{aligned}$$

This is an equation of simple harmonic motion for s and the period is

$2\pi(4a/g)^{1/2}$. The period is independent of the amplitude of the oscillation, provided the particle remains on the curve.

If a particle is suspended by a light string between guides which are arcs of a cycloid (Fig. 24) the path of the particle is an involute of the cycloid. If the guides are generated by a circle of radius a and the string is of length $4a$ the involute is a cycloid of the same dimensions as the guides and the particle oscillates with simple harmonic motion of period $2\pi(4a/g)^{1/2}$.

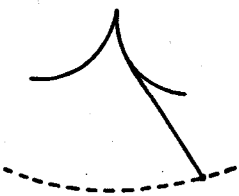

Fig. 24

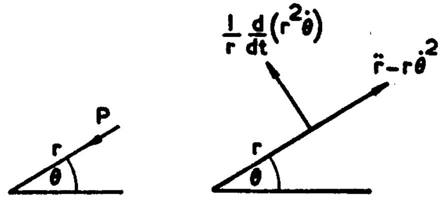

Fig. 25

1.31. Use of Polar Components of Acceleration

The polar components of acceleration are chiefly used when

(a) the force on a particle depends only on its distance from a fixed point,

(b) the force is always directed towards or away from a fixed point,

(c) the force is always perpendicular to the radius from a fixed point to the particle.

If the force P is always directed towards a fixed point, taking this point as the origin of polar coordinates the components of force and acceleration are as shown in Fig. 25.

Therefore
$$m(\ddot{r} - r\dot{\theta}^2) = -P,$$
$$\frac{m}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0.$$

From the second equation it follows that $r^2\dot{\theta}$ is constant and may be determined from the initial conditions. Writing $r^2\dot{\theta} = h$ we have
$$r\dot{\theta}^2 = \frac{h^2}{r^3},$$
and substituting in the first equation
$$m\left(\ddot{r} - \frac{h^2}{r^3}\right) = -P.$$

If P is a function of r this equation may be integrated with respect to r giving
$$\frac{1}{2}m\left(\dot{r}^2 + \frac{h^2}{r^2}\right) = -\int P\,dr + \text{constant}.$$

This equation may indeed be written down directly as the energy equation since the kinetic energy is $\frac{1}{2}m\left(\dot{r}^2 + \frac{h^2}{r^2}\right)$ and $-\int P dr$ is the work done.

Example 8. *A particle P is attached to one end of an inextensible string which passes through a small hole in a smooth horizontal table, and a particle Q of equal mass is attached to the other end of the string. Initially the system is at rest with P held on the table at a distance c from the hole, and with Q hanging vertically. If P is projected on the table in a direction perpendicular to the string with velocity* $(\frac{1}{3}gc)^{1/2}$ *find the distance of P from the hole when it is next instantaneously moving in a direction perpendicular to the string.* (L.U., Pt. I)

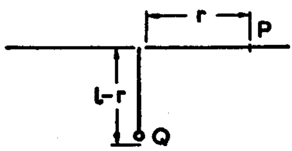

Fig. 26

Let the position of P be given by polar coordinates (r, θ) so that r is its distance from the hole (Fig. 26). Then if l be the length of the string, Q is at a depth $l - r$ below the hole and its acceleration is $\frac{d^2}{dt^2}(l - r) = -\ddot{r}$ downwards.

Then if T be the tension in the string we have for Q

$$T - mg = m\ddot{r}.$$

For P we have the equations

$$m(\ddot{r} - r\dot{\theta}^2) = -T,$$
$$r^2\dot{\theta} = \text{constant}.$$

Initially $r = c$ and $r\dot{\theta} = (gc/3)^{1/2}$, and therefore

$$(r^2\dot{\theta})^2 = \frac{1}{3}gc^3,$$

$$r\dot{\theta}^2 = \frac{1}{3}g\frac{c^3}{r^3}.$$

Therefore $$m\left(\ddot{r} - \frac{1}{3}g\frac{c^3}{r^3}\right) = -T = -mg - m\ddot{r},$$

$$2\ddot{r} - \frac{1}{3}g\frac{c^3}{r^3} = -g,$$

$$\dot{r}^2 + \frac{1}{6}g\frac{c^3}{r^2} = -gr + \text{constant},$$

and since $\dot{r} = 0$ when $r = c$,

$$\dot{r}^2 = \frac{1}{6}gc\left(1 - \frac{c^2}{r^2}\right) - g(r - c),$$

$$= -\frac{1}{6}\frac{g}{r^2}(r - c)(2r - c)(3r + c).$$

P is moving in a direction perpendicular to the string when $\dot{r} = 0$, and this next occurs when $r = \frac{1}{2}c$.

Example 9. *A smooth rod turns in a horizontal plane about a vertical axis through one end. A small ring slides on the rod whose angular velocity is maintained constant by a suitably varied torque. If the ring is projected along the rod from the axis with a speed V find its distance from the axis and the value of the torque after a time t.* (L.U., Pt. II)

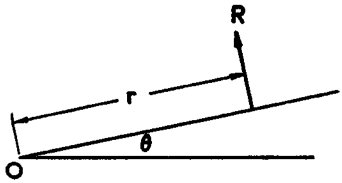

Fig. 27

If θ be the inclination of the rod at any instant and r the displacement of the ring from the axis (Fig. 27), (r, θ) are polar coordinates of the ring. We have $\dot{\theta}$ = constant = ω (say) and initially $r = 0$, $\dot{r} = V$.
The only force acting on the ring is the reaction of the rod normal to the rod. We have therefore

$$\ddot{r} - r\dot{\theta}^2 = 0,$$
$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = R.$$

That is, since $\dot{\theta} = \omega$ and $\ddot{\theta} = 0$

$$\ddot{r} - r\omega^2 = 0,$$
$$R = 2m\omega\dot{r}.$$

We have the solution of the equation for r

$$r = Ae^{\omega t} + Be^{-\omega t},$$

and when $t = 0$,

$$0 = A + B,$$
$$V = (A - B)\omega,$$

therefore

$$r = \frac{V}{2\omega}(e^{\omega t} - e^{-\omega t}),$$
$$= \frac{V}{\omega}\sinh \omega t.$$

Also

$$R = 2m\omega\dot{r},$$
$$= 2m\omega V\cosh \omega t.$$
$$rR = 2mV^2\cosh \omega t \sinh \omega t,$$
$$= mV^2 \sinh 2\omega t.$$

This is the torque exerted by the ring on the rod and is therefore the torque required to maintain the angular velocity.

1.32 Equation of an Ellipse

The usual form of the equation of an ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

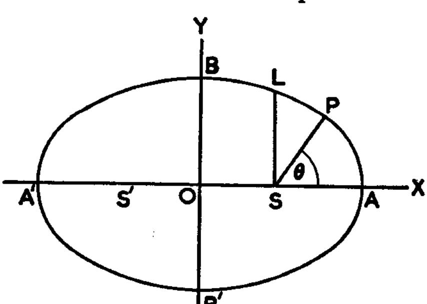

Fig. 28

where a and b are the major and minor semi-axes respectively. The eccentricity e is given by the equation $b^2 = a^2(1 - e^2)$; the semi latus rectum SL (Fig. 28) is denoted by l, and $l = a(1 - e^2)$.

Hence

$$a = \frac{l}{1 - e^2}, \quad b = \frac{l}{(1 - e^2)^{1/2}}.$$

The distance OS of the focus from the centre of the ellipse is ae and hence if (r, θ) be the polar coordinates of a point P with reference to S we have

$$x = \frac{le}{1 - e^2} + r\cos\theta, \quad y = r\sin\theta.$$

Substitution for x and y in the equation of the ellipse leads to the polar equation

$$\frac{l}{r} = 1 + e\cos\theta.$$

This is the polar equation of any conic and represents a circle if $e = 0$, an ellipse if $e < 1$, a parabola if $e = 1$ and a hyperbola if $e > 1$.

Differentiating this equation we have

$$-\frac{l}{r^2}\frac{dr}{d\theta} = -e\sin\theta,$$

$$= -\left\{e^2 - \left(\frac{l}{r} - 1\right)^2\right\}^{1/2}.$$

$$\frac{l^2}{r^4}\left(\frac{dr}{d\theta}\right)^2 = -\frac{l^2}{r^2} + \frac{2l}{r} - (1 - e^2),$$

$$\left(\frac{dr}{d\theta}\right)^2 = -r^2 + \frac{2}{l}r^3 - \frac{1 - e^2}{l^2}r^4.$$

This is the differential equation satisfied by the polar equation of the ellipse. Its solution would involve an arbitrary constant and would be

$$\frac{l}{r} = 1 + e\cos(\theta + a).$$

1.33 Newton's Law of Universal Gravitation

If two bodies have masses m_1 lb. and m_2 lb. and are at a distance d ft. apart they attract one another with a force

$$\frac{\gamma m_1 m_2}{d^2} \text{ pdl.}$$

where $\gamma = 10^{-9} \times 1{\cdot}0685$, approximately. This is Newton's Law of Gravitation on the basis of which the motion of the earth and other planets around the sun is explained.

If M be the mass of the sun and m the mass of a planet their mutual attraction when at a distance r apart is $\dfrac{\gamma Mm}{r^2}$. Therefore the sun has an acceleration $\dfrac{\gamma m}{r^2}$ towards the planet and the planet has an acceleration

$\dfrac{\gamma M}{r^2}$ towards the sun. Thus the *relative* acceleration of the planet towards the sun is $\dfrac{\gamma(M+m)}{r^2}$.

To have an idea of the magnitudes of the forces involved, let M be the mass of the sun, m the mass of the earth and d its mean distance from the sun.

$$M = 4\cdot392 \times 10^{30} \text{ lb.}$$
$$m = 1\cdot315 \times 10^{25} \text{ lb.}$$
$$d = 4\cdot911 \times 10^{11} \text{ ft.}$$
$$\frac{\gamma M m}{d^2} = 2\cdot56 \times 10^{23} \text{ pdl.}$$
$$\frac{\gamma(M+m)}{d^2} = 0\cdot0195 \text{ ft./sec.}^2$$

1.34 Orbit of a Planet

Let the position of a planet relative to the sun be given by polar coordinates (r, θ) (Fig. 29). The acceleration is $\dfrac{\gamma(M+m)}{r^2}$ directed towards the sun.

We have therefore

$$\ddot{r} - r\dot{\theta}^2 = -\frac{\gamma(M+m)}{r^2}, \qquad (1)$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0,$$

that is $\qquad r^2\dot{\theta} = h$ (a constant). $\qquad$ (2)

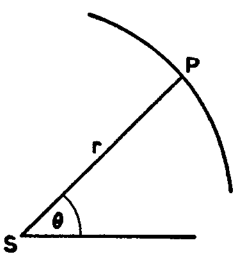

Fig. 29

Substituting for $\dot{\theta}$ in (1) we have

$$\ddot{r} - \frac{h^2}{r^3} = -\frac{\gamma(M+m)}{r^2}.$$

Integrating with respect to r we have

$$\dot{r}^2 + \frac{h^2}{r^2} = \frac{2\gamma(M+m)}{r} + C \text{ (a constant).}$$

We shall take $\quad C = -\dfrac{\gamma(M+m)}{a}$, where a is constant,

and hence $\quad \dot{r}^2 + \dfrac{h^2}{r^2} = \gamma(M+m)\left(\dfrac{2}{r} - \dfrac{1}{a}\right).$ $\qquad$ (3)

It should be noticed that since $h^2/r^2 = r^2\dot{\theta}^2$ equation (3) gives the square of the velocity for any value of r and that the velocity increases as r diminishes, that is as the planet moves nearer to the sun.

We have

$$\dot{r}^2 = -\frac{h^2}{r^2} + \frac{2\gamma(M+m)}{r} - \frac{\gamma(M+m)}{a},$$

$$\dot{\theta}^2 = \frac{h^2}{r^4},$$

therefore $$\left(\frac{dr}{d\theta}\right)^2 = -r^2 + \frac{2\gamma(M+m)}{h^2}r^3 - \frac{\gamma(M+m)}{h^2 a}r^4.$$

This is the differential equation which was obtained for an ellipse and its solution is

$$\frac{l}{r} = 1 + e\cos(\theta + \alpha),$$

where $$l = \frac{h^2}{\gamma(M+m)},$$

$$\frac{1-e^2}{l^2} = \frac{\gamma(M+m)}{h^2 a} = \frac{1}{la},$$

that is $$\frac{l}{1-e^2} = a.$$

It follows that the constant a which was introduced is the major semi-axis of the ellipse. The minor semi-axis is $(la)^{1/2}$.

Since $mr^2\dot{\theta}$ is the angular momentum about S, the constant h is the angular momentum per unit mass. We shall show that $\frac{1}{2}h$ is the rate at which area is swept out by the radius from the sun to the planet.

If (r, θ) and $(r + \delta r, \theta + \delta\theta)$ be the coordinates of P at time t and time $t + \delta t$ respectively, the area enclosed between the radii is to the first order of small quantities $\frac{1}{2}r^2\delta\theta$. This is the area swept by the radius in time δt and hence in the limit the rate at which area is swept is

$$\frac{1}{2}r^2\dot{\theta} = \frac{1}{2}h.$$

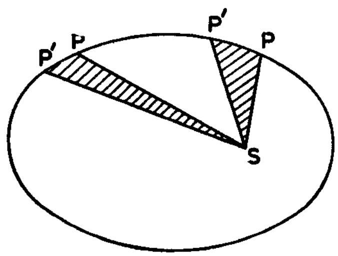

Fig. 30

Thus the area is swept at a constant rate, that is, areas SPP' (Fig. 30) swept in equal times are equal.

It follows that the time T in which the planet makes a complete circuit of the sun is the area of the ellipse divided by $\frac{1}{2}h$.

That is $$T = \frac{\pi ab}{\frac{1}{2}h},$$

$$T^2 = \frac{4\pi^2 a^2 b^2}{h^2}.$$

Now
$$h^2 = \gamma(M + m)l,$$
$$b^2 = la,$$
therefore
$$T^2 = \frac{4\pi^2 a^3}{\gamma(M + m)},$$
$$T = \frac{2\pi a^{3/2}}{\{\gamma(M + m)\}^{1/2}}.$$

Since m is small compared with M the periodic time of a planet is seen to be proportional to $a^{3/2}$, where a is its major semi-axis. This fact was obtained from observations by Kepler and is known as Kepler's Third Law.

For the motion of the earth about the sun the following are approximate values of the constants involved:

$$\begin{aligned} a &= 4{\cdot}9106 \times 10^{11} \text{ ft.} \\ b &= 4{\cdot}9099 \times 10^{11} \text{ ft.} \\ l &= 4{\cdot}9092 \times 10^{11} \text{ ft.} \\ e &= 0{\cdot}01675. \\ h &= 4{\cdot}800 \times 10^{16} \text{ ft.}^2/\text{sec.} \\ T &= 3{\cdot}156 \times 10^{7} \text{ sec.} \\ \gamma M &= 4{\cdot}6928 \times 10^{21} \text{ ft.}^3/\text{sec.}^2 \end{aligned}$$

There is not agreement on some of these figures to this degree of accuracy, but the figures given above are a consistent set.

EXERCISES 1 (*f*)

1. A heavy cable hangs in a catenary whose equation is $s = c \tan \psi$. A ring can slide freely on the cable. If the ring is given a velocity v_0 along the cable from its lowest point find how far it will move before coming to rest.

2. A ring can slide smoothly along a heavy cable in the form of a catenary. Show that if the displacement of the ring is small compared with the parameter c of the catenary, the periodic time of an oscillation about the position of equilibrium is $2\pi(c/g)^{1/2}$.

3. A body attached to a parachute is released from an aeroplane which is moving horizontally with velocity V. If the parachute exerts a drag opposing motion which is k times the weight of the body, where k is constant, find an expression for the velocity of the body when its path is inclined at an angle ψ to the horizontal.

   If $k = 1$ show that the limiting vertical velocity is $\frac{1}{2}V$.

4. A smooth parabolic curve has equation $x^2 = 4ay$ where the y-axis is vertical and the x-axis is horizontal. Show that if a particle moves on the curve about the position of stable equilibrium the period of small oscillations is that of a simple pendulum of length $2a$.

5. A particle of mass m rests on a smooth horizontal table attached to a fixed point by a light elastic string of modulus mg and natural length a. Initially the string is just taut and the particle is projected in a direction perpendicular to the line of the string with velocity $(4ga/3)^{1/2}$. Show that the string will extend until its length is $2a$ and that the velocity is then $(ga/3)^{1/2}$.

6. A particle of mass m moves on a smooth horizontal plane under an attraction P towards a point O on the plane. Its path is a spiral whose equation in polar coordinates, with O as pole, is $r = ae^{k\theta}$. If the velocity is u when $\theta = 0$ prove that $P = \dfrac{ma^2u^2}{r^3}$.

7. A rod of length $2a$ is forced to turn in a horizontal plane about a vertical axis through one end with constant angular velocity ω. A small ring which can slide freely on the rod is released from rest at the mid-point of the rod. Show that it will leave the rod after time $\omega^{-1} \log (2 + \sqrt{3})$ and find its velocity at this instant.

8. A smooth rod is constrained to turn in a horizontal plane about a vertical axis through one end with angular velocity ω. A ring of mass m which slides on the rod is attached to the pivot by a light spring of unstretched length a and stiffness s and is released from rest with the spring unstretched. Show that if $sg = m(\omega^2 + \lambda^2)$, where λ^2 is positive, the ring will oscillate about a point of the rod. Find the distance of this point from the axis and the amplitude and period of oscillation.

9. A particle rests close to the edge of a smooth rectangular plate which begins to turn downwards about this edge with constant angular velocity ω. Show that the reaction between the particle and the plate will vanish after time t given by the equation $\cosh \omega t = 2 \cos \omega t$.

10. Two particles of equal mass m are connected by a light spring of stiffness s and unstretched length a and can slide on a smooth rod. The rod is made to turn with constant angular velocity ω in a horizontal plane about a vertical axis through one end. Initially the spring is unstretched and one of the masses is close to the axis of rotation. Show that after time $2\pi/\lambda$, where $sg = \frac{1}{2}m(\lambda^2 + \omega^2)$, the masses will each have moved a distance $\frac{1}{2}a\{\cosh (2\pi\omega/\lambda) - 1\}$.

11. Given that the greatest and least distances of the planet Mars from the sun are $8{\cdot}180 \times 10^{11}$ ft. and $6{\cdot}784 \times 10^{11}$ ft. and that its mass is negligible compared with that of the sun, find the length of the Martian year and the semi-major and minor axes of its orbit.

12. Show that if a projectile is fired with velocity V at an inclination α to the horizontal and the variation of gravity is taken into account but air resistance is neglected it will describe an ellipse whose latus rectum is $V^2 \cos^2 \alpha/g$ and whose semi-major axis is $gR^2/(2gR - V^2)$, R being the earth's radius.

CHAPTER 2

OSCILLATION OF A PARTICLE

2.1 Simple Harmonic Motion

Simple harmonic motion is characterised by an equation of motion of the form

$$\frac{d^2x}{dt^2} = -\omega^2 x + c, \tag{1}$$

$$= -\omega^2\left(x - \frac{c}{\omega^2}\right).$$

The solution of the differential equation is

$$x - \frac{c}{\omega^2} = A \cos (\omega t + \varepsilon) \tag{2}$$

where the constants A and ε are determined by the initial values of x and $\dot{x}$. Then $x = c/\omega^2$ is the *centre* of the oscillation, $2\pi/\omega$ is the *period* and A is the *amplitude* of the motion; the first two can be found from equation (1) without solving it but the amplitude is determined by the initial conditions. The *frequency* is the number of oscillations in unit time and is the inverse of the period. The quantity ε is called the *epoch* of the motion and the *phase* at any time is the time elapsed since the particle was last at the positive extremity of its path, that is $t + \varepsilon/\omega$ less a multiple of the period.

Force being proportional to acceleration, simple harmonic motion is caused by a force directed towards the centre of oscillation and proportional to the distance from the centre. Thus if a particle of weight w be attached to the end of a light spring of stiffness s which hangs freely from its other end, there is a position of equilibrium where the weight balances the extension of the spring, and if c be the extension $w = sc$. If the particle is displaced a distance x from this position the additional force in the spring is sx and we have

$$\frac{w}{g}\ddot{x} = -sx,$$

and the motion is simple harmonic with period $2\pi(w/sg)^{1/2}$.

Many kinds of oscillations of importance in science and engineering are approximately simple harmonic and may be treated as such if the amplitude is not too great.

Example 1. *A particle of mass* 1 *oz. is attached to the mid-point of an elastic string* 6 *ft. long and the ends of the string are attached to two points* 8 *ft. apart on a smooth horizontal table. The modulus of elasticity of the string is* 1 *lb.wt. The*

particle is displaced a small distance in a line perpendicular to the string and released. Show that the resulting motion is approximately simple harmonic and find its period.

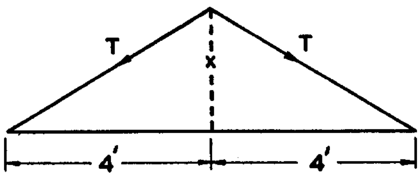

Fig. 31

When the particle is at rest the tension in each portion of the string is $1 \times \frac{1}{3}$ lb.wt. When it is displaced a small distance x (Fig. 31) the length of each part of the string is $\sqrt{(16 + x^2)}$ and the tension is

$$T = \frac{\sqrt{(16+x^2)} - 3}{3} \text{ lb.wt.}$$

The resolved part of this tension in the direction perpendicular to the string is

$$T \times \frac{x}{\sqrt{(16+x^2)}} = \frac{x}{3}\left(1 - \frac{3}{\sqrt{(16+x^2)}}\right).$$

There are two components of tension causing acceleration, therefore

$$\frac{1}{16} \times \frac{1}{g}\ddot{x} = -\frac{2x}{3}\left(1 - \frac{3}{\sqrt{(16+x^2)}}\right).$$

Since x is small we may neglect the x^2 under the square root, and we have

$$\ddot{x} = -\frac{16g}{6}x.$$

This is an equation of simple harmonic motion with period

$$2\pi\sqrt{\left(\frac{6}{16g}\right)} = 0{\cdot}68 \text{ sec.}$$

2.2 Motion of Connected Particles

When two particles are connected by a light spring (or elastic string) and move in the direction of the axis of the spring, the *difference* of their displacements is governed by the tension of the spring and, if there are no other forces affecting it, this motion is simple harmonic. The total momentum of the particles is governed by the external forces acting on the system, and if there are no such forces in the line of the displacements the total momentum is constant. Thus we have two equations from which the separate displacements of the particles may be found.

Example 2. *Two particles of masses 2 lb. and 5 lb. respectively lie on a smooth horizontal table connected by a light spring of stiffness 3 lb.wt. per foot. The 5 lb. mass is given a blow of impulse 20 pdl. sec. in the direction of the other mass. Find the displacement of each of the masses after t seconds.*

The impulse applied to the 5 lb. mass gives it a velocity v, where

$$20 = 5v,$$
$$v = 4 \text{ ft./sec.}$$

Let x be the displacement of the 5 lb. mass and y the displacement of the 2 lb. mass at time t in the direction of the blow (Fig. 32).
By the conservation of momentum

$$5\dot{x} + 2\dot{y} = 20,$$

and hence $$5x + 2y = 20t. \tag{1}$$

The thrust in the spring is $3(x-y)$ lb.wt.

and hence $\frac{5}{g}\ddot{x} = -3(x-y)$,

$$\frac{2}{g}\ddot{y} = 3(x-y).$$

Fig. 32

Multiplying the first of these equations by 2 and the second by 5 and subtracting we have

$$(\ddot{x} - \ddot{y}) = -\frac{21g}{10}(x-y),$$

that is $$\frac{d^2}{dt^2}(x-y) = -67{\cdot}2(x-y),$$
$$= -(8{\cdot}20)^2(x-y).$$

Therefore $$x - y = A\cos 8{\cdot}20t + B\sin 8{\cdot}20t.$$

When $t = 0$, $$x = y = 0 \text{ and hence } A = 0,$$
$$\dot{x} = 4,\ \dot{y} = 0 \text{ and hence } B = \frac{4}{8{\cdot}20} = 0{\cdot}49,$$

and $$x - y = 0{\cdot}49 \sin 8{\cdot}20t. \tag{2}$$

Hence, from (1) and (2) $$x = 2{\cdot}86t + 0{\cdot}14\sin 8{\cdot}20t,$$
$$y = 2{\cdot}86t - 0{\cdot}35\sin 8{\cdot}20t,$$

and the period of the harmonic part of the motion is

$$2\pi/(8{\cdot}20) = 0{\cdot}765 \text{ sec.}$$

EXERCISES 2 (*a*)

1. A light elastic string of natural length 2 ft. and modulus of elasticity 2 lb.wt. is attached to two points A and B, 3 ft. apart on a smooth horizontal table. To the middle point of the string is attached a mass of $\frac{1}{2}$ lb., which is drawn aside towards A and released from rest when 1 ft. from A. Show that the resulting motion is simple harmonic, find the period of oscillation and the greatest speed of the mass during the motion. (L.U.)

2. A light elastic string is stretched between two points A and B, distant $2a$ apart, on a smooth horizontal plane, the tension in the string being T. A particle of mass m, attached to the mid-point of the string, is displaced a distance perpendicular to AB, which is small compared with $2a$, and is then released. Show that the periodic time of oscillation of the particle is approximately $2\pi(ma/2T)^{\frac{1}{2}}$. (Q.E.)

3. Two particles A and B, each of mass m, are attached to the ends of a light stiff spiral spring AB and the system is placed on a smooth horizontal table. A blow of impulse I is applied to A in the direction AB. Prove that the greatest compression is $I(2ms)^{-1/2}$ where s is the stiffness of the spring.

Prove also, that when the spring regains its natural length for the first time it has moved forward a distance $\frac{1}{2}\pi I(2ms)^{-1/2}$.

(L.U., Pt. II)

4. Two weights, each of mass m, are connected by a light spring which exerts a force s for each unit length of extension. They are placed on a rough horizontal plane the coefficient of friction being μ. One of the weights is projected along a line directly away from the other and at the instant when the second weight begins to move the first is travelling with velocity v. Show that during the subsequent motion the tension in the spring reaches a maximum value $mg(\mu^2 + sv^2/2mg^2)^{1/2}$.

(C.U.)

5. Two particles, A and B, of equal mass m are attached to the ends of a light spring which exerts a tension of amount s per unit extension. Initially the particles are at rest on a smooth horizontal table with the spring just taut, and a constant force of magnitude sa is then applied to particle B in the direction AB.

Obtain the differential equations for the displacements, x and y, of the particles A and B respectively at time t, and show that $y + x = \frac{1}{4}a\omega^2t^2$, $y - x = \frac{1}{2}a(1 - \cos \omega t)$, where $m\omega^2 = 2s$.

(L.U., Pt. II)

6. Two particles of equal mass m and distance a apart are attached to a taut string at equal distances ka from the fixed end points. Obtain the simultaneous differential equations for small transverse displacements x, y of the particles when the tension in the string has the constant value $kman^2$.

Show that, if the particles start from rest at $t = 0$ with $x = b$, $y = 0$, then x and y can be expressed in the forms C (cos nt + cos λnt), C(cos nt − cos λnt) respectively, and evaluate the constants λ, C.

(L.U., Pt. II)

7. Two particles each of mass m lb. rest on a smooth horizontal table connected by a light spring of stiffness s lb.wt. If one of the particles is projected away from the other with velocity v ft./sec. prove that the distance each will have moved when the spring first regains its natural length is $v\pi(m/8sg)^{1/2}$.

8. A light spring has particles of masses m_1 and m_2 fixed to its ends and rests on a smooth horizontal table. The mass m_2 is placed against a fixed stop and the mass m_1 is moved towards m_2 until the spring is compressed by an amount h and then released. Show that the mass m_2 comes to rest at distances $2\pi nh\left\{\dfrac{m_1{}^2m_2}{(m_1 + m_2)^3}\right\}^{1/2}$ from the stop, where $n = 1, 2, 3, \ldots$, etc., and that the spring is then unstretched.

2.3 Damped Simple Harmonic Motion—Constant Frictional Damping

Suppose that a particle of mass m moves in a straight line $X'OX$ (Fig. 33) being attracted towards O by a force $m\omega^2 x$, where x is the distance from O, while at the same time there is a constant frictional force

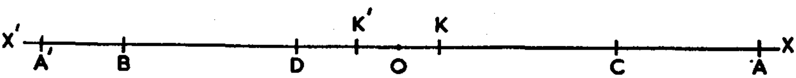

Fig. 33

μmg opposing motion. Let the particle be released from rest at a point A distant a from O, and let $A'O = OA$.

When the particle is moving from A towards A' the equation of the motion is

$$m\ddot{x} = -m\omega^2 x + \mu mg. \tag{1}$$

When the particle is moving from A' towards A the sign of the friction is changed and the equation of motion is

$$m\ddot{x} = -m\omega^2 x - \mu mg. \tag{2}$$

Equation (1) is $\qquad \ddot{x} = -\omega^2(x - k)$

where $\qquad k = \mu g/\omega^2.$

The motion is therefore simple harmonic with period $2\pi/\omega$ about the point K distant k from O, but this motion only continues for one half period. The amplitude is $a - k$ and the particle first comes to rest at a point B distant $a - k$ from K and therefore distant $2k$ from A'.

Equation (2) is $\qquad \ddot{x} = -\omega^2(x + k).$

The motion is, therefore, simple harmonic with the same period about the point K' distant k from O on the negative side and this motion continues for one half period. The amplitude is the length BK' which is $a - 3k$ and the particle comes to rest at C distant $a - 3k$ from K', that is, distant $4k$ from A.

For the next half period the oscillation is about K and the particle comes to rest at $6k$ from A'; for the next half period it is about K' and comes to rest at $8k$ from A.

Thus after n half periods the particle comes to rest at $2kn$ from A or A'. If $a - 2kn \leqslant k$ this point will be within the range KK' and then the attraction will be less than the limiting frictional resistance and the particle will remain at rest. Thus the particle comes to rest after time $n\pi/\omega$ where n is the least integer greater than or equal to $\dfrac{a - k}{2k}$, at a distance $a - 2kn$ from O on the positive side of O if n is even, on the negative side if it is odd.

2.4 Damping Proportional to Velocity

Suppose a particle of mass m is moving in a straight line $X'OX$ being attracted towards O by a force $m\omega^2 x$, where x is the distance from O, while at the same time there is a force resisting motion proportional to the velocity.

Let this resistance be $2mk\dot{x}$ with $k > O$. Then the equation of motion is

$$m\ddot{x} = -m\omega^2 x - 2mk\dot{x}.$$

The force $2mk\dot{x}$ changes sign with $\dot{x}$ and hence is always opposing motion.

We have $$\ddot{x} + 2k\dot{x} + \omega^2 x = 0.$$

This is a second order linear differential equation with constant coefficients. To find the solution we first solve the subsidiary equation

$$\alpha^2 + 2k\alpha + \omega^2 = 0.$$

We have two solutions

$$\alpha_1 = -k + (k^2 - \omega^2)^{1/2},\ \alpha_2 = -k - (k^2 - \omega^2)^{1/2},$$

and the complete solution of the differential equation is

$$x = Ae^{\alpha_1 t} + Be^{\alpha_2 t},$$

except in the case where $\alpha_1 = \alpha_2 = -k$,
when the solution is $$x = (At + B)e^{-kt}.$$

Let the initial conditions be

$$x = a \text{ and } \dot{x} = 0 \text{ when } t = 0.$$

There are three possible types of motion depending on whether k is greater, equal to or less than ω.

(i) If $k > \omega$, let $k^2 - \omega^2 = \varrho^2$.

Then
$$x = Ae^{-kt+\rho t} + Be^{-kt-\rho t}$$
$$\dot{x} = A(-k + \varrho)e^{-kt+\rho t} + B(-k - \varrho)e^{-kt-\rho t}.$$

From the initial conditions

$$A + B = a,$$
$$A(k - \varrho) + B(k + \varrho) = 0,$$

therefore

$$x = \frac{1}{2}a\left\{\left(1 + \frac{k}{\varrho}\right)e^{-kt+\rho t} + \left(1 - \frac{k}{\varrho}\right)e^{-kt-\rho t}\right\}$$

$$\dot{x} = -\frac{a(k^2 - \varrho^2)}{2\varrho}\{e^{-kt+\rho t} - e^{-kt-\rho t}\}.$$

In this motion x is essentially positive and $\dot{x}$ essentially negative and both become zero only after infinite time. The damping in this case is excessive causing the particle to return slowly to its position of equilibrium.

With other initial conditions $x = x_0$, $\dot{x} = \dot{x}_0$ when $t = 0$, the solution is easily seen to be

$$x = x_0 e^{-kt}\left\{\cosh \varrho t + \left(\frac{k}{\varrho} + \frac{\dot{x}_0}{\varrho x_0}\right)\sinh \varrho t\right\}.$$

In this case if $\dot{x}_0$ is positive there must be a turning point since $k > \varrho$ and the particle reaches the origin eventually. If $\dot{x}_0$ is negative the particle may pass through the origin before it finally returns there.

(ii) If
$$\begin{aligned} k &= \omega \\ x &= (At + B)e^{-kt} \\ \dot{x} &= (A - Bk - Akt)e^{-kt}. \end{aligned}$$

From the initial conditions $B = a$ and $A = Bk$ so that

$$\begin{aligned} x &= a(1 + kt)e^{-kt} \\ \dot{x} &= -ak^2te^{-kt}. \end{aligned}$$

Hence, as in the previous case x remains positive and the particle returns slowly to its position of equilibrium.

(iii) If $k < \omega$, let $k^2 - \omega^2 = -p^2$.

Then
$$x = Ae^{-kt+pit} + Be^{-kt-pit}.$$

The exponentials with imaginary arguments can be expressed as sines and cosines and the solution expressed in the form

$$x = e^{-kt}(C \cos pt + D \sin pt).$$

Since
$$x = a \text{ when } t = 0,\ C = a.$$

$$\dot{x} = -ke^{-kt}(C \cos pt + D \sin pt) + pe^{-kt}(-C \sin pt + D \cos pt).$$

Since
$$\begin{aligned} \dot{x} &= 0 \text{ when } t = 0 \\ -kC + pD &= 0, \\ D &= \frac{ka}{p}. \end{aligned}$$

Therefore
$$\begin{aligned} x &= \frac{a}{p}e^{-kt}(p \cos pt + k \sin pt), \\ &= ae^{-kt}(\cos pt - \tan \varepsilon \sin pt), \end{aligned}$$

where $\tan \varepsilon = -k/p$, that is

$$x = a \sec \varepsilon\, e^{-kt} \cos (pt + \varepsilon).$$

Thus the motion is periodic with period $2\pi/(\omega^2 - k^2)^{1/2}$, but the amplitude is reduced exponentially by the factor e^{-kt} so that the motion eventually dies away.

We have

$$\begin{aligned} \dot{x} &= a \sec \varepsilon\, e^{-kt}\{-k \cos (pt + \varepsilon) - p \sin (pt + \varepsilon)\} \\ &= -a \sec^2 \varepsilon\, e^{-kt} \sin pt. \end{aligned}$$

Thus $\dot{x} = 0$ for $t = \dfrac{n\pi}{p}$ and the corresponding maxima and minima values of x are

$$x = a \cos n\pi \, e^{-kn\pi/p}.$$

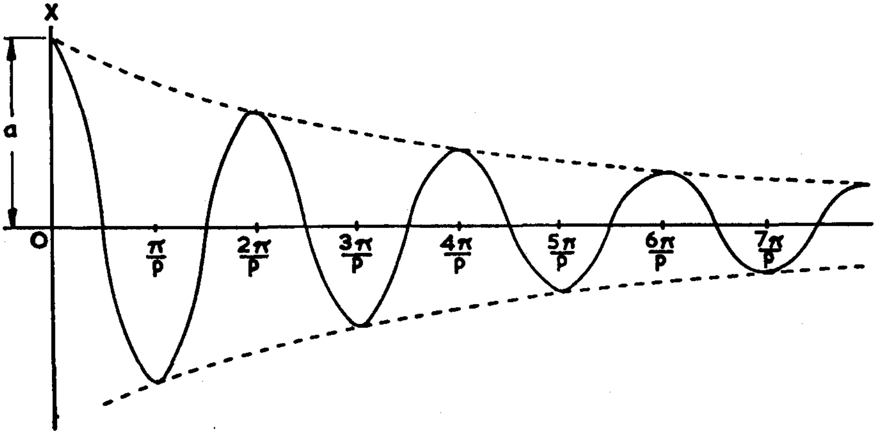

Fig. 34

Thus the graph of displacement against time (Fig. 34) is completely enclosed between the curves $x = \pm ae^{-kt}$.

The ratio of successive displacements on the positive (or negative) side is easily seen to be $e^{-2\pi k/p}$.

2.5 Forced Oscillations—without Damping

For a particle moving in a straight line attracted to a point O of the line by a force $m\omega^2 x$, where x is the distance from O, let there be a periodic disturbing force $md \cos qt$ measured in the direction of x increasing.

Then $$m\ddot{x} + m\omega^2 x = md \cos qt.$$

Such a disturbing force may be caused, for example, in the case of a particle suspended from a fixed point by a spring, by giving to the point of suspension a simple harmonic motion of period $2\pi/q$. If the displacement of the point of suspension is y and the displacement of the particle x the restoring force is proportional to $x - y$ and the equation of motion is of the form

$$m\ddot{x} + m\omega^2(x - y) = 0,$$

and if $\omega^2 y = d \cos qt$ we have an equation of the form considered.

We have $$\frac{d^2x}{dt^2} + \omega^2 x = d \cos qt.$$

The solution of an equation of this type has two parts, namely a particular integral and the complementary function.

The complementary function is found by ignoring the right-hand side of the equation and is

$$x_1 = A \cos \omega t + B \sin \omega t.$$

The particular integral is

$$x_2 = \frac{1}{D^2 + \omega^2} d \cos qt,$$

$$= \frac{1}{\omega^2 - q^2} d \cos qt.$$

The complete solution is therefore

$$x = A \cos \omega t + B \sin \omega t + \frac{d}{\omega^2 - q^2} \cos qt$$

$$= C \cos (\omega t + \varepsilon) + \frac{d}{\omega^2 - q^2} \cos qt.$$

The constants A and B, or C and ε, are determined by the initial conditions of the motion.

Thus if $x = x_0$ and $\dot{x} = \dot{x}_0$ when $t = 0$

$$A = x_0 - \frac{d}{\omega^2 - q^2},$$

$$B = \frac{\dot{x}_0}{\omega}.$$

The motion, therefore, consists of two parts, the free oscillation with the natural period of the motion and the forced oscillation with the period of the disturbing force. The free oscillation eventually disappears due to damping which is always present to some extent and we are left with the forced oscillation

$$x = \frac{d}{\omega^2 - q^2} \cos qt.$$

If $q = \omega$ we have the condition of *resonance* and the amplitude of the forced oscillation becomes indefinitely great.

It should be noted that when $q = \omega$ the differential equation of motion is

$$\ddot{x} + \omega^2 x = d \cos \omega t,$$

of which a complete solution is, as may be verified by substitution,

$$x = A \cos \omega t + B \sin \omega t + \frac{d}{2\omega} t \sin \omega t,$$

and, therefore, the amplitude of the forced oscillation builds up with the time.

2.6 Forced Oscillations with Damping Proportional to Velocity

In this case the differential equation is

$$\ddot{x} + 2k\dot{x} + \omega^2 x = d \cos qt.$$

The complementary function is

$$x_1 = e^{-kt}(A \cos pt + B \sin pt),$$

where $p^2 = \omega^2 - k^2.$

The particular integral is

$$\begin{aligned} x_2 &= \frac{d}{D^2 + 2kD + \omega^2} \cos qt \\ &= \frac{d}{\omega^2 - q^2 + 2kD} \cos qt \\ &= \frac{d(\omega^2 - q^2 - 2kD)}{(\omega^2 - q^2)^2 - 4k^2D^2} \cos qt \\ &= \frac{d(\omega^2 - q^2 - 2kD)}{(\omega^2 - q^2)^2 + 4k^2q^2} \cos qt \\ &= \frac{d}{(\omega^2 - q^2)^2 + 4k^2q^2}\{(\omega^2 - q^2) \cos qt + 2kq \sin qt\}. \end{aligned}$$

The complete solution is $x = x_1 + x_2$
and the constants A and B are determined by the initial conditions as before.

Writing $\tan \eta = \dfrac{2kq}{\omega^2 - q^2}$, the forced vibration is

$$\begin{aligned} x_2 &= d\{(\omega^2 - q^2)^2 + 4k^2q^2\}^{-1/2} \cos (qt - \eta) \\ &= \frac{d}{2kq} \sin \eta \cos (qt - \eta). \end{aligned}$$

In the case of resonance $q = \omega$ and $\eta = \dfrac{\pi}{2}$,

therefore $$x_2 = \frac{d}{2k\omega} \sin \omega t,$$

and the greatest amplitude when the free oscillation has died down is $d/(2k\omega)$. This may be large if k is small, but it does not increase indefinitely. It should be noticed that this oscillation is out of phase with the disturbing force.

EXERCISES 2 (*b*)

1. A particle of mass 2 lb. is attached to one end of a light spring whose other end is fixed to a point on a horizontal table. The stiffness of the spring is 1 lb./ft. and the coefficient of friction between the particle

and the table is 0·05. If the particle is drawn away from its equilibrium position through a distance of 18 in. and released, find the number of oscillations completed before it comes to rest and the time taken.

2. A body of mass M performs oscillations controlled by a spring of stiffness λ and subject to a frictional force of constant magnitude F, and displacement is measured from the position in which the tension in the spring is zero. If the body be released from rest with a displacement a, greater that F/λ, show that it next comes to rest with displacement $-(a - 2F/\lambda)$.

Show that whatever the initial displacement, the body comes to rest in a finite time with a displacement numerically less than F/λ.

(L.U., Pt. II)

3. A mass of m lb. hung at rest on the end of a vertical spring whose upper end is fixed produces an extension of a ft. Assuming a frictional resistance of kv/g lb.wt. when the velocity is v ft. per sec., formulate and solve the equation for the vertical oscillations of the mass.

If the amplitude of the oscillations is reduced to $1/e$ of its initial value in s seconds (e being the base of natural logarithms), find k in terms of m and s. Find also an expression for the energy in ft. lb. which must be supplied to maintain an oscillation of A ft. at the natural undamped frequency. (L.U., Pt. II)

4. The equation of motion of a particle is

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 4x = 0.$$

When $t = 0$ the distance x is 1 unit and the speed away from the origin is 2 units. Prove that the particle will be at its greatest distance from the origin after a time $\frac{1}{3}\log_e 2$, and find the greatest distance.

(L.U., Pt. I)

5. A particle of mass m moves in a straight line in a medium whose resistance is $5mnv$, where v is the velocity, and is attracted towards a fixed point O in the line by a force $4mn^2x$, where x is the distance from O. It is projected towards O with velocity u from a point at a distance a from O, at time $t = 0$. Find x in terms of t.

Discuss the motion (i) if $u < 4na$, (ii) if $u > 4na$.

6. A particle suspended by a vertical spring has its motion damped by a force proportional to the velocity so that the ratio of the amplitudes of successive oscillations in the same direction is 0·8. Show that the natural frequency of the particle is reduced by about 0·06 per cent by the damping.

7. A mass m lies at rest on a horizontal table and is attached to one end of a light spring which, when stretched, exerts a tension of amount $m\omega^2$ times its extension (where ω is constant). If the other end of the spring is now moved with uniform velocity u along the table in a direction away from the mass, and the table offers a resistance to the motion of the mass of an amount mk times its speed (where k is

constant), obtain the differential equation for the extension x of the spring after time t.

If $k = 2\omega$ show that $x = \dfrac{u}{\omega}\{2 - (2 + \omega t)e^{-\omega t}\}$. (L.U., Pt. II)

8. A 4-lb. mass hangs at rest on a spring producing in the spring an extension of 1 ft. The upper end of the spring is now made to execute a vertical simple harmonic oscillation $x = \sin 4t$, x being measured vertically downwards in feet. If the mass is subject to a frictional resistance whose magnitude in lb.wt. is one-quarter of its velocity in ft. per sec., obtain the differential equation for the motion of the mass and find the expression for its displacement at time t, when t is large. (L.U., Pt. II)

9. A mass m is supported on a horizontal platform to which it is attached by a spring of stiffness λ, and its vibration is damped by a damper which applies a force kv when the velocity of the mass relative to the platform is v. If the platform oscillates horizontally, its displacement at any time being y, while the displacement of the mass at the same time is x, show that

$$m\ddot{x} + k\dot{x} + \lambda x = k\dot{y} + \lambda y.$$

If $y = a \sin pt$ determine the amplitude of the steady oscillation of the mass and show that it attains its maximum value when

$$p^2 = \lambda^2\{(1 + 2k^2/m\lambda)^{1/2} - 1\}/k^2.$$

(L.U., Pt. I)

10. A weight is hung on a spring, the upper end of which is given a vertical simple harmonic motion of frequency equal to the natural undamped frequency of oscillation of the weight. The oscillation of the weight is restricted by means of a dashpot which provides a damping force proportional to the velocity, so that its amplitude is equal to that of the upper end of the spring.

Show that if the weight is allowed to oscillate freely under the same damping conditions the ratio of successive displacements in the same direction will be about 38 : 1. (C.U.)

2.7. Electric Circuits

In the following sections the differential equations which determine the current in an electric circuit are formed and solved. The following notation is used:

i = current in *ampères*,
q = charge on a condenser in *coulombs*,
C = capacitance of a condenser in *farads*,
L = coefficient of self-inductance in *henries*,
R = resistance in *ohms*,
e = electromotive force (e.m.f.) in *volts*,
v = voltage.

An inductance, a resistance and a condenser each causes a drop in voltage according to the following laws:

$$\text{Inductance voltage} = L\frac{di}{dt},$$

$$\text{Resistance voltage} = Ri \text{ (Ohm's law)},$$

$$\text{Condenser voltage} = \frac{q}{C}.$$

We have also that the current is the rate of change of charge, that is

$$i = \frac{dq}{dt}.$$

The differential equation of a circuit is formed by equating the voltage drop due to inductances, resistances and condensers to the voltage supplied by the e.m.f.

2.8 Circuit with Inductance and Resistance

The differential equation is

$$L\frac{di}{dt} + Ri = e.$$

(i) Let e be constant and equal to E and let the circuit be connected to the e.m.f. at time $t = 0$, so that $i = 0$ when $t = 0$ (Fig. 35).

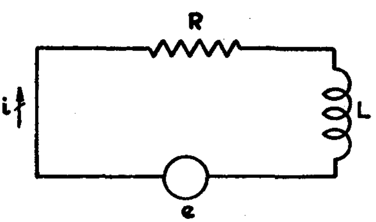

Fig. 35

The differential equation may be solved by finding a complementary function (C.F.) and a particular integral (P.I.).

$$\text{C.F.} = Ae^{-Rt/L}, \text{ where } A \text{ is a constant},$$

$$\text{P.I.} = \frac{E}{LD + R} = \frac{E}{R}.$$

Therefore $$i = Ae^{-Rt/L} + \frac{E}{R}.$$

Since $i = 0$ when $t = 0$, $A = -E/R$ and we have

$$i = \frac{E}{R}(1 - e^{-Rt/L}).$$

Thus the current increases exponentially to the value E/R.

(ii) Let the e.m.f. be alternating and equal to $E \cos \omega t$, and connected at $t = 0$, so that $i = 0$ when $t = 0$.

The differential equation is

$$L\frac{di}{dt} + Ri = E \cos \omega t.$$

The complementary function is as before and we have

$$\text{P.I.} = \frac{E}{R + LD}\cos\omega t$$
$$= \frac{E(R - LD)}{R^2 - L^2D^2}\cos\omega t$$
$$= \frac{E}{R^2 + L^2\omega^2}(R - LD)\cos\omega t$$
$$= \frac{E}{R^2 + L^2\omega^2}(R\cos\omega t + L\omega\sin\omega t)$$
$$= \frac{E}{(R^2 + L^2\omega^2)^{1/2}}\cos(\omega t - \alpha),$$

where $\tan\alpha = \dfrac{L\omega}{R}$.

Then $$i = Ae^{-Rt/L} + \frac{E}{(R^2 + L^2\omega^2)^{1/2}}\cos(\omega t - \alpha).$$

When $t = 0$, $$0 = A + \frac{E}{(R^2 + L^2\omega^2)^{1/2}}\cos\alpha,$$

and $$i = \frac{E}{(R^2 + L^2\omega^2)^{1/2}}\{\cos(\omega t - \alpha) - e^{-Rt/L}\cos\alpha\}.$$

Thus there is an alternating component of current and a transient component which quickly dies out owing to the exponential factor, and we have the steady current

$$i = \frac{E}{(R^2 + L^2\omega^2)^{1/2}}\cos(\omega t - \alpha).$$

$(R^2 + L^2\omega^2)^{1/2}$ is called the *impedance* and α is called the *phase lag.* Thus the current is as if there were a simple resistance $(R^2 + L^2\omega^2)^{1/2}$ and a phase lag.

2.9 Circuit with Resistance and Condenser

The differential equation is (Fig. 36)

$$Ri + \frac{q}{C} = e,$$

and $$i = \frac{dq}{dt},$$

so that $$R\frac{dq}{dt} + \frac{q}{C} = e.$$

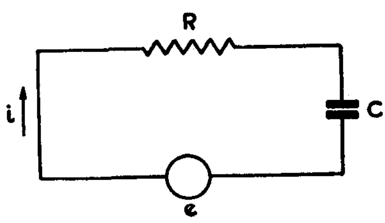

Fig. 36

(i) Let e be constant and equal to E, and let the circuit be connected at $t = 0$, so that $q = 0$ when $t = 0$.

Then
$$q = Ae^{-t/CR} + CE$$
$$= CE(1 - e^{-t/CR})$$

and
$$i = \frac{E}{R}e^{-t/CR}.$$

(ii) Let e be alternating and equal to $E \cos \omega t$, and let the circuit be connected at time $t = 0$.

We have
$$CR\frac{dq}{dt} + q = CE \cos \omega t$$
$$\text{C.F.} = Ae^{-t/CR},$$
$$\text{P.I.} = \frac{CE}{1 + CRD} \cos \omega t$$
$$= \frac{CE(1 - CRD)}{1 + C^2R^2\omega^2} \cos \omega t$$
$$= \frac{CE}{1 + C^2R^2\omega^2} (\cos \omega t + CR\omega \sin \omega t)$$
$$= \frac{CE}{(1 + C^2R^2\omega^2)^{1/2}} \sin (\omega t + \beta),$$

where
$$\tan \beta = \frac{1}{CR\omega}.$$

Since $q = 0$ when $t = 0$,
$$A = -\frac{CE \sin \beta}{(1 + C^2R^2\omega^2)^{1/2}}.$$

The transient term dies out and for the steady state we have
$$q = \frac{CE}{(1 + C^2R^2\omega^2)^{1/2}} \sin (\omega t + \beta),$$
$$i = \frac{CE\omega}{(1 + C^2R^2\omega^2)^{1/2}} \cos (\omega t + \beta).$$

Thus the impedance $= (1 + C^2R^2\omega^2)^{1/2}/C\omega$
$$= \left(R^2 + \frac{1}{C^2\omega^2}\right)^{1/2},$$

and the phase lead is β. Thus the circuit is equivalent to a simple resistance equal to the impedance with a phase lead.

Example 3. *A condenser of capacitance C is charged through a resistance R by a steady voltage v. Prove that the charge q on a plate is given by $R\dot{q} + q/C = v$. If $C = 10^{-5}$ farads, $v = 4000$ volts, $R = 5000$ ohms, calculate the current at the instant of closing the switch and after $0{\cdot}04$ seconds.*

From the previous section

$$R\frac{dq}{dt} + \frac{q}{C} = v$$

$$q = Cv(1 - e^{-t/CR})$$

$$i = \frac{v}{R}e^{-t/CR}.$$

$$1/CR = 20.$$

$$\frac{v}{R} = 0{\cdot}8 \text{ amps.}$$

$$i = 0{\cdot}8e^{-20t}.$$

The current when $t = 0$ is therefore 0·8 amps., and since $e^{-0\cdot8} = 0{\cdot}449$, the current when $t = 0{\cdot}04$ is 0·359 amps.

2.10 Circuit with Inductance, Resistance and Condenser

The differential equation is (Fig. 37)

$$L\frac{di}{dt} + Ri + \frac{q}{C} = e,$$

$$i = \frac{dq}{dt}.$$

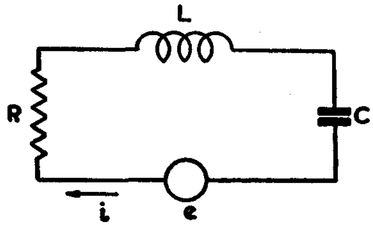

Fig. 37

The differential equation may be written as

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = e,$$

or $$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \frac{de}{dt}.$$

In this case we shall solve the differential equation for i.

(i) Let e be constant and equal to E.

We have $$CL\frac{d^2i}{dt^2} + CR\frac{di}{dt} + i = 0.$$

The auxiliary equation is

$$CLa^2 + CRa + 1 = 0$$

and $$a = \frac{-CR \pm \sqrt{(C^2R^2 - 4CL)}}{2CL}.$$

The three cases must be distinguished in which the values of a are real, coincident or complex.

If $CR^2 > 4L$, the roots a_1 and a_2 are real and distinct and negative, and we have

$$i = Ae^{a_1t} + Be^{a_2t}.$$

Thus the current diminishes exponentially to zero.

If $CR^2 = 4L$, $\qquad a_1 = a_2 = -R/2L$, and we have

$$i = (At + B)e^{-Rt/2L}.$$

In this case the damping is critical and the resistance is just large enough to prevent oscillation and the current diminishes to zero.

If $CR^2 < 4L$ the roots of the auxiliary equation are

$$-\frac{R}{2L} \pm p\sqrt{(-1)}, \text{ where } p^2 = \frac{1}{CL} - \frac{R^2}{4L^2}.$$

and we have $\qquad i = e^{-\frac{Rt}{2L}}(A \cos pt + B \sin pt).$

In this case the motion is oscillatory with an exponential damping factor so that the oscillations are eventually damped out.

(ii) Let e be alternating and equal to $E \cos \omega t$.

The differential equation for i is

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = -E\omega \sin \omega t.$$

The complementary function is as before, and this is transient, the steady current being given by the particular integral.

This is

$$i = \frac{-E\omega}{LD^2 + RD + \dfrac{1}{C}} \sin \omega t$$

$$= \frac{-E\omega}{\dfrac{1}{C} - L\omega^2 + RD} \sin \omega t$$

$$= \frac{-E\omega}{\left(\dfrac{1}{C} - L\omega^2\right)^2 + R^2\omega^2}\left(\frac{1}{C} - L\omega^2 - RD\right) \sin \omega t$$

$$= \frac{-E\omega}{\left(\dfrac{1}{C} - L\omega^2\right)^2 + R^2\omega^2}\left\{\left(\frac{1}{C} - L\omega^2\right) \sin \omega t - R\omega \cos \omega t\right\}$$

$$= \frac{E}{\left\{R^2 + \left(\dfrac{1}{C\omega} - L\omega\right)^2\right\}^{1/2}} \cos (\omega t + \beta),$$

where $$\tan \beta = \left(\frac{1}{C\omega} - L\omega\right)/R.$$

Thus the impedance is $\left\{R^2 + \left(\dfrac{1}{C\omega} - L\omega\right)^2\right\}^{1/2}$ and the phase lead is β.

The amplitude of the current in the steady state is greatest when $L\omega = 1/C\omega$ and its value is then E/R. Thus for different values of ω the amplitude is a maximum when $\omega = 1/\sqrt{(LC)}$, and this state is analogous to that of resonance in a mechanical system.

2.11 Comparison of Electrical and Mechanical Systems

The reader will have perceived the similarity between the differential equations of electrical circuits and those of mechanical systems of damped harmonic motion with a periodic disturbing force such as is given by the equation

$$m\ddot{x} + \lambda\dot{x} + \mu x = c \cos \omega t.$$

In this equation m is the mass of the particle, λ is the damping factor, μ the stiffness of the elastic restraint and c is the amplitude of the periodic disturbing force.

In the corresponding electrical circuit the coefficient of self-inductance takes the place of the mass, the resistance replaces the damping factor and the reciprocal of the capacitance replaces the stiffness. The impressed e.m.f. corresponds to the periodic disturbing force.

Many complicated problems involving electrical circuits can thus be solved by finding the solution of the corresponding mechanical problems, and vice versa.

EXERCISES 2 (*c*)

1. A condenser of capacity C discharges through a circuit of resistance R and self-inductance L. Find the condition that the discharge be just non-oscillatory. Obtain also the formulae for the time variation of charge and current in this case, when the initial voltage is E. (L.U., Pt. II)

2. A condenser of capacity C and initial charge Q_0 is discharged through a resistance R and an inductance L in series. Prove that if $R^2C < 4L$ the current at time t is $-Q_0e^{-ht}(k + h^2/k) \sin kt$, where $-h \pm ik$ are the roots of the equation $CLx^2 + CRx + 1 = 0$. (L.U., Pt. II)

3. A circuit consists of inductance L and capacity C in series. An alternating e.m.f. $E \sin nt$ is applied to the circuit commencing at time $t = 0$, the initial current and charge on the condenser being zero. Prove that the current at time t is given by $I = \dfrac{nE}{L(n^2 - \omega^2)} (\cos \omega t - \cos nt)$, where $CL\omega^2 = 1$. (L.U., Pt. II)

4. Two condensers of capacities C_1 and C_2 have one plate of each earthed and the two insulated plates joined by a wire of resistance R and self-inductance L. Initially the first condenser has charge Q and the

second is uncharged. Prove that the charge on the second condenser after time t is

$$\frac{QC_2}{C_1 + C_2}\left\{\frac{\beta e^{-\alpha t} - \alpha e^{-\beta t}}{\alpha - \beta} + 1\right\},$$

where α and β are the roots of the equation $L\lambda^2 - R\lambda - \frac{1}{C_1} - \frac{1}{C_2} = 0$.

(L.U., Pt. II)

5. An uncharged condenser of capacity C is charged by an e.m.f. of $E \sin \{t/(LC)^{1/2}\}$ through leads of self-inductance L and negligible resistance. Prove that at time t the charge on one of the plates is

$$\frac{1}{2}CE\{\sin [t/(LC)^{1/2}] - [t/(LC)^{1/2}] \cos [t/(LC)^{1/2}]\}.$$

If, in addition, there is a small resistance, in what respect is the mathematical form of the above result altered? (L.U., Pt. II)

6. An uncharged condenser of capacity C is charged by applying an e.m.f. $E \sin \pi t$ through leads of self-inductance L and small resistance R. After a time $2T$, where T is a large integer, the e.m.f. remains zero. Find the charge on the condenser at time t, where $t > 2T$.

(L.U., Pt. II)

7. An alternating e.m.f. $E \sin pt$ is applied at time $t = 0$ to a circuit. Obtain, in the usual notation, the equation

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = pE \cos pt,$$

and hence obtain an expression for the current at time t in the two cases (i) $CR^2 > 4L$, (ii) $CR^2 < 4L$. (L.U., Pt. II)

8. An e.m.f. $E \cos pt$ is switched across an electric circuit consisting of a coil of inductance L henries, resistance R ohms in series with a condenser of capacity C farads. State the differential equation for the charge q in the condenser at any time t and the form of the general solution when the resistance is just sufficient to prevent natural oscillations.

If $L = 0{\cdot}0001$, $R = 2$, find the value of C for this condition to hold and calculate the amplitude of the steady current for an imposed peak voltage of 100 at 50 cycles per second. (L.U., Pt. II)

9. A voltage $E \sin \omega t$ is applied to a circuit containing an inductance L henries, a resistance R ohms, and a capacitance C farads. Write down the differential equation for the charge q on a plate of the condenser and the current i flowing into this plate at time t. Find a formula for i in the steady state and sketch a graph of the amplitude of i for different values of ω. (L.U., Pt. II)

10. An alternating e.m.f. $E \sin \omega t$ is supplied to a circuit containing inductance L, resistance R and capacitance C. Obtain the differential equation satisfied by the current i.

Find the resistance if it is just large enough to prevent natural oscillations. For this value of R and $LC\omega^2 = 1$ prove that

$$i = E(\sin \omega t - \omega t e^{-\omega t})/2k,$$

where $k^2 = L/C$, when the current and the charge on the condenser are both zero at time $t = 0$. (L.U., Pt. II)

11. An electric circuit consists of an inductance L, resistance R and capacitance C in series. A constant e.m.f. E is applied in series with the circuit at time $t = 0$ when the current i and the potential v across the condenser are zero. Obtain the differential equation for v.

Find the values of v and i at time t, given that $5CL\omega^2 = 1$ and $5CR\omega = 2$, and prove that the greatest value of v is $E(1 + e^{-\pi/2})$. (L.U., Pt. II)

2.12 Circuit with Two Branches

When a circuit has two branches the current will not be the same in each branch but the voltage drops will be the same along each branch. The voltage drop in each branch is assigned to the appropriate inductances, resistances, etc., as before and this leads to simultaneous differential equations for the currents in the branches.

Example 4. *An e.m.f. $E \cos \omega t$ is applied through a resistance S to two branches each containing a resistance R, one a condenser of capacitance C and the other a coil of inductance L. If $L\omega^2 C = 1$ find the total current supplied in the steady state.*

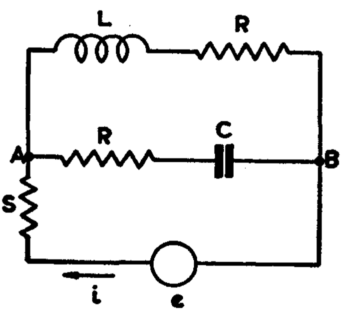

Fig. 38

Let i be the total current, i_1 and i_2 the current in the branches (Fig. 38). The voltage drop between A and B is the same for each branch and we have

$$V_A - V_B = L\frac{di_2}{dt} + Ri_2 = Ri_1 + \frac{q_1}{C},$$

where $\dfrac{dq_1}{dt} = i_1$.

We have therefore

$$Si + Ri_1 + \frac{q_1}{C} = E \cos \omega t,$$

$$Si + L\frac{di_2}{dt} + Ri_2 = E \cos \omega t.$$

Differentiating the first equation and replacing i_1 by $i - i_2$ we have

$$(S + R)\frac{di}{dt} - R\frac{di_2}{dt} + \frac{i - i_2}{C} = -E\omega \sin \omega t.$$

Therefore $$\left\{(S + R)D + \frac{1}{C}\right\}i - \left(RD + \frac{1}{C}\right)i_2 = -E\omega \sin \omega t$$

$$Si + (LD + R)i_2 = E \cos \omega t.$$

That is

$$(LD + R)\left\{(S + R)D + \frac{1}{C}\right\}i - (LD + R)\left(RD + \frac{1}{C}\right)i_2$$
$$= -E\omega(L\omega \cos \omega t + R \sin \omega t),$$

$$\left(RD + \frac{1}{C}\right)Si + \left(RD + \frac{1}{C}\right)(LD + R)i_2 = E\left(- R\omega \sin \omega t + \frac{1}{C} \cos \omega t\right).$$

Adding, we have

$$\{L(S + R)D^2 + (R^2 + 2SR + L/C)D + (R + S)/C\}i = -2ER\omega \sin \omega t.$$

Hence, for the steady state we have

$$i = \frac{-2ER\omega}{-L(S + R)\omega^2 + (R^2 + 2SR + L/C)D + (R + S)/C} \sin \omega t$$
$$= \frac{-2ER\omega}{(R^2 + 2SR + L^2\omega^2)D} \sin \omega t$$
$$= \frac{2R}{R^2 + 2SR + L^2\omega^2} E \cos \omega t.$$

2.13 Coupled Circuits

Two circuits may be coupled so that they have a mutual inductance. If the mutual inductance is M and the currents in the two circuits are i_1 and i_2 the voltage drop in the circuits will be $M\frac{di_1}{dt}$ and $M\frac{di_2}{dt}$ respectively.

The differential equation for each circuit thus involves the current in the other and we have simultaneous differential equations.

Example 5. *Two coupled circuits have resistances R_1, R_2 self-inductance L_1, L_2, e.m.f. e_1, e_2 and mutual inductance M. If $L_1 = L_2 = 0{\cdot}1$ henry, $R_1 = R_2 = 4$ ohms, $M = 0{\cdot}06$ henry, $e_1 = 8$ volts, $e_2 = 0$, find the currents at time t, given that they are initially zero.*

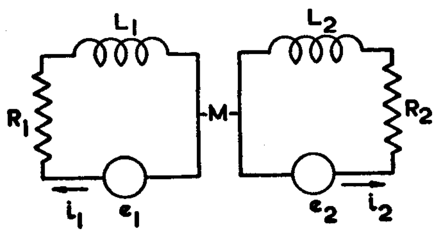

Fig. 39

If i_1 and i_2 are the currents in the circuits (Fig. 39), their differential equations are

$$L_1\frac{di_1}{dt} + R_1i_1 + M\frac{di_2}{dt} = e_1,$$

$$L_2\frac{di_2}{dt} + R_2i_2 + M\frac{di_1}{dt} = e_2.$$

That is
$$(L_1D + R_1)i_1 + MDi_2 = e_1,$$
$$MDi_1 + (L_2D + R_2)i_2 = e_2.$$
Hence
$$\{(L_1D + R_1)(L_2D + R_2) - M^2D^2\}i_1 = (L_2D + R_2)e_1 - MDe_2,$$
$$\{(L_1D + R_1)(L_2D + R_2) - M^2D^2\}i_2 = (L_1D + R_1)e_2 - MDe_1.$$
The operator on i_1 and i_2
$$\begin{aligned} &= (L_1L_2 - M^2)D^2 + (L_1R_2 + R_1L_2)D + R_1R_2 \\ &= 10^{-4} \times 64D^2 + 0{\cdot}8D + 16 \\ &= 10^{-4} \times 64(D^2 + 125D + 2500) \\ &= 10^{-4} \times 64(D + 25)(D + 100). \end{aligned}$$
The complementary function is therefore
$$\text{C.F.} = Ae^{-25t} + Be^{-100t}.$$
We have
$$10^{-4} \times 64(D + 25)(D + 100)i_1 = 32$$
$$10^{-4} \times 64(D + 25)(D + 100)i_2 = 0.$$
The particular integral for i_1 is 2 and we have
$$i_1 = Ae^{-25t} + Be^{-100t} + 2,$$
and since $i_1 = 0$ when $t = 0$, $A + B + 2 = 0$.

We have also
$$(L_1D + R_1)i_1 + MDi_2 = e_1,$$
that is
$$(D + 40)i_1 + 0{\cdot}6Di_2 = 80.$$
$$\begin{aligned} 0{\cdot}6Di_2 &= 80 + \{(25A - 40A)e^{-25t} + (100B - 40B)e^{-100t} - 80\} \\ &= -15Ae^{-25t} + 60Be^{-100t} \\ Di_2 &= -25Ae^{-25t} + 100Be^{-100t} \\ i_2 &= Ae^{-25t} - Be^{-100t} + \text{constant}. \end{aligned}$$
From the differential equation for i_2 the constant must be zero, and since $i_2 = 0$ when $t = 0$, $A - B = 0$. Hence, $A = B = -1$, and we have
$$i_1 = 2 - e^{-25t} - e^{-100t},$$
$$i_2 = -e^{-25t} + e^{-100t}.$$

EXERCISES 2 (*d*)

1. An electric circuit consists of two branches in parallel containing respectively a resistance R and an inductance L, and a resistance R and a capacitance C. Construct the differential equations for the currents in the branches due to an applied e.m.f. and solve them for a steady e.m.f. E suddenly applied at $t = 0$. Show that if $CR^2 = L$ the circuit behaves, as far as total current is concerned, as a pure resistance of magnitude R. (L.U., Pt. II)

2. An e.m.f. $E \sin \omega t$ is applied through a resistance R to two branches each containing a resistance R, one a condenser of capacitance C and the other a coil of inductance L. Determine the peak value and the phase lag or lead of the total current supplied. Show that if $LC\omega^2 = 1$, the circuit behaves as a pure resistance of amount $(3R^2 + L^2\omega^2)/2R$. (L.U., Pt. II)

3. Two points A, B are joined by a wire of resistance R without self-induction; B is joined to a third point C by two wires each of resistance R, of which one is without self-induction and the other has a co-efficient of self-induction L. If the ends A, C are kept at a potential

difference $E \cos \omega t$, and if there are no mutual inductances, prove that the current in AB is

$$\frac{E}{R}\left\{\frac{4R^2 + L^2\omega^2}{9R^2 + 4L^2\omega^2}\right\}^{1/2} \cos(\omega t - \alpha) \text{ where } \tan \alpha = \frac{RL\omega}{6R^2 + 2L^2\omega^2}.$$

Find also the difference of potential at B and C. (L.U., Pt. II)

4. An alternating e.m.f. of amplitude E and frequency $\omega/2\pi$ is supplied to a coil of inductance L and resistance R. Write down the differential equation for the current in the coil and solve it, indicating the transient term.

   If the coil is shunted by a condenser of capacitance C and resistance S show that the circuit can be replaced (as far as permanent current is concerned) by a non-inductive resistance provided that $CR^2 - L = \omega^2 CL(CS^2 - L)$. (L.U., Pt. II)

5. Show that a combination of an inductance L and a resistance R in parallel with a resistance R and a capacitance C is equivalent to a simple resistance R for all applied e.m.f.'s if $R^2C = L$.

6. The primary circuit of a transformer consists of resistance R_1, inductance L_1 and a constant applied voltage E. The secondary circuit contains resistance R_2 and inductance L_2 only. Show that the current in the primary circuit is

$$Ae^{\alpha_1 t} + Be^{\alpha_2 t} + E/R_1$$

   where A and B are constants and α_1 and α_2 are the roots of the equation $(L_1L_2 - M^2)x^2 + (L_1R_2 + L_2R_1)x + R_1R_2 = 0$, M being the mutual inductance of the circuits.

SERVOMECHANISMS

A servomechanism is generally understood to be an automatic control system involving some amplification of power, and, therefore, incorporating an external source of power, whose output is made to follow predetermined behaviour. Thus a searchlight made to follow an aircraft located by radar and an automatic profiling machine tool reproducing a contour are examples of servomechanisms.

2.14 Second Order Servomechanisms

The second order differential equation

$$\frac{d^2\theta_0}{dt^2} + 2\zeta\omega_n\frac{d\theta_0}{dt} = \omega_n^2(\theta_i - \theta_0), \tag{1}$$

that is $$(D^2 + 2\zeta\omega_n D + \omega_n^2)\theta_0 = \omega_n^2\theta_i, \tag{2}$$

where ζ and ω_n are constants, describes a servomechanism in which θ_0 is the output corresponding to a given input θ_i. The mechanism is designed to ensure that the output θ_0 remains equal to θ_i, and if $\theta_0 \neq \theta_i$, the error $\theta_i - \theta_0$ is used to generate a force proportional to the error which will eventually reduce it to zero. Such a mechanism is said to be

error actuated. When θ_i is known as a function of the time the differential equation may be solved to find the manner and speed with which θ_0 approaches θ_i.

In (1) ω_n is called the *undamped natural angular frequency* of the mechanism and ζ is called the *damping ratio.*

For example, suppose it is desired to control the alignment of two rotating shafts so that the angular displacement θ_0 of one of them is equal to the angular displacement θ_i of the other. The input and output shafts may be mechanically coupled to a differential gear (Fig. 40) so

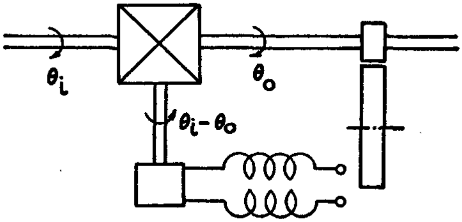

Fig. 40

that the angular displacement of a third shaft coupled to the gear is $\theta_i - \theta_0$. This third shaft may be used to move the control of a potentiometer which will supply a voltage proportional to $\theta_i - \theta_0$ to an electric motor coupled to the output shaft. The torque applied to the motor is approximately proportional to the voltage, and hence if J be the moment of inertia of the armature, f the coefficient of viscous friction we have

$$J\ddot{\theta}_0 + f\dot{\theta}_0 = k(\theta_i - \theta_0),$$

where k is a constant. Thus we have an equation connecting θ_0 and θ_i of the form (1) with

$$\omega_n = \sqrt{(k/J)},$$
$$\zeta = f/\sqrt{(4Jk)}.$$

It is also possible in a system of this sort to arrange for the torque to be partly proportional to the derivative of the error.

Thus if the torque is $(k + lD)(\theta_i - \theta_0)$ we have

$$(JD^2 + fD)\theta_0 = (k + lD)(\theta_i - \theta_0),$$
$$\{JD^2 + (f + l)D + k\}\theta_0 = (k + lD)\theta_i.$$

2.15 Solution of the Differential Equation

To find the complementary function in the solution of the differential equation § 2.14 (2) we consider the auxiliary equation
$\alpha^2 + 2\zeta\omega_n\alpha + \omega_n^2 = 0$ whose roots are

$$\alpha = -\zeta\omega_n \pm \omega_n\sqrt{(\zeta^2 - 1)}.$$

Then the complementary function is

if $\zeta > 1$, $\theta_0 = e^{-\zeta\omega_n t}(A \cosh \lambda t + B \sinh \lambda t)$, $\lambda = \omega_n \sqrt{(\zeta^2 - 1)}$,

if $\zeta = 1$, $\theta_0 = e^{-\zeta\omega_n t}(A + Bt)$,

if $\zeta < 1$, $\theta_0 = e^{-\zeta\omega_n t}(A \cos \mu t + B \sin \mu t)$, $\mu = \omega_n (1 - \zeta^2)$.

The constants A and B are determined by the initial values of θ_0 and $\dot{\theta}_0$. It is easily seen that for $\zeta > 0$ this part of the solution diminishes to zero as t increases. This is called the *transient* part of the solution and does not affect the ultimate value of θ_0. A system in which the complementary function is transient is said to be *stable*.

The particular integral may be written as

$$\theta_0 = f(D)\theta_i,$$

where
$$f(D) = \frac{\omega_n^2}{D^2 + 2\zeta\omega_n D + \omega_n^2}.$$

The operator $f(D)$ is called the *transfer function* of the system and the solution is written in symbolic form as

$$\frac{\theta_0}{\theta_i} = f(D).$$

The advantage of this notation is that if there is more than one mechanism in series in a system the transfer functions of the mechanisms can be combined. Thus if a second mechanism with transfer function $f_1(D)$ is joined to the mechanism whose transfer function is $f(D)$ the relation between input and final output is

$$\frac{\theta_0}{\theta_i} = f_1(D) . f(D),$$

that is, the transfer function $f_1(D) . f(D)$ applied to the input θ_i gives the output θ_0.

As an example, suppose that the input shaft in the system considered in § 2.14 is turning with constant angular velocity Ω, so that $\theta_i = \Omega t$. Then the particular integral is

$$\theta_0 = \frac{\omega_n^2}{D^2 + 2\zeta\omega_n D + \omega_n^2}\Omega t,$$
$$= \left(1 - \frac{2\zeta}{\omega_n}D + \ldots\right)\Omega t,$$
$$= \Omega\left(t - \frac{2\zeta}{\omega_n}\right).$$

Thus when the transient effect has died away we have

$$\theta_0 = \Omega\left(t - \frac{2\zeta}{\omega_n}\right) = \Omega\left(t - \frac{f}{k}\right),$$
$$\dot{\theta}_0 = \Omega.$$

The lag in the position of the output shaft is proportional to the angular velocity of the system and is called the *velocity lag*. It is desirable that this velocity lag should be as small as possible, but at the same time the rate at which the transient solution dies away is governed by $e^{-\zeta\omega_n t} = e^{-1/2ft/J}$, so that f must not be unduly small.

2.16 Higher Order Servomechanisms

The mechanism described in § 2.14 leads to a second order differential equation on the assumption that the torque is proportional to the voltage. In fact, the torque is proportional to the current i and if L be the self-inductance of the coils, R the resistance and $K(\theta_i - \theta_0)$ the voltage, we have

$$L\frac{di}{dt} + Ri = K(\theta_i - \theta_0),$$

that is

$$(TD + 1)i = (K/R)(\theta_i - \theta_0), \tag{1}$$

where $T = L/R$.

We have also

$$J\ddot{\theta}_0 + f\dot{\theta}_0 = Ci, \tag{2}$$

where C is a constant, and hence, eliminating i between (1) and (2) we have

$$\{TJD^3 + (J + fT)D^2 + fD + G\}\theta_0 = G\theta_i,$$

where $G = CK/R$. The transfer function is thus

$$f(D) = G\{TJD^3 + (J + fT)D^2 + fD + G\}^{-1}.$$

It is essential that this transfer function should represent a stable system with the complementary function transient. The complementary function will be

$$Ae^{\alpha t} + Be^{\beta t} + Ce^{\gamma t},$$

where α, β and γ are the roots of the auxiliary equation

$$TJa^3 + (J + fT)a^2 + fa + G = 0.$$

It is therefore essential that if a root of this equation is real it should be negative, and if it is complex of the form $p + iq$ its real part p should be negative. The condition that a cubic equation $x^3 + ax^2 + bx + c = 0$, where a, b, c are positive and real should have roots whose real parts are negative is $ab - c > 0$.

Hence the above system is stable if

$$f(J + fT) > TJG.$$

For higher order transfer functions there are criteria for stability based on the theory of the roots of algebraic equations.

2.17 Harmonic Response of a System

The *response* of a stable system to an input θ_i is the output θ_0. The nature of the response may be studied by considering the output

corresponding to the input $\theta_i = a \sin \omega t$. The eventual output due to θ_i, when the transient part of the solution has disappeared, is called the *harmonic response* of the system.

In the case of the second order servomechanism the harmonic response is

$$\begin{aligned}\theta_0 &= \frac{\omega_n^2}{D^2 + 2\zeta\omega_n D + \omega_n^2} a \sin \omega t,\\ &= \frac{a\omega_n^2}{\omega_n^2 - \omega^2 + 2\zeta\omega_n D} a \sin \omega t,\\ &= \frac{a\omega_n^2}{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2}\{(\omega_n^2 - \omega^2)\sin \omega t - 2\zeta\omega\omega_n \cos \omega t\},\\ &= am \sin(\omega t - \phi),\end{aligned}$$

where

$$\phi = \tan^{-1} \frac{2\zeta\omega_n\omega}{\omega_n^2 - \omega^2},$$

$$m = \omega_n^2/\{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2\}^{1/2}.$$

Thus the harmonic response θ_0 is an oscillation with the same frequency as the input but lagging behind it by the phase angle ϕ. The amplitude of the response is m times that of the input, and m is called the *amplitude magnification factor* of the system.

The amplitude magnification factor and the phase lag may be shown diagrammatically for different values of the ratio ω/ω_n.

2.18 Unit Response

Another standard input which is used to study the transient output of a system is the unit step function $h(t)$ shown in Fig. 41. The function

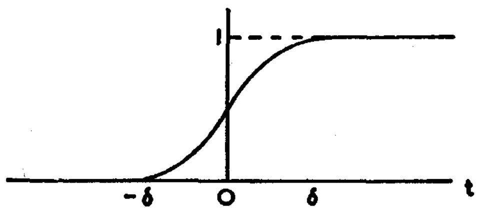

Fig. 41

$h(t)$ is a function of the time t; it is zero for $t < -\delta$ and is unity for $t > \delta$, where δ is small, changing continuously from 0 to 1 in the interval $-\delta$ to δ. By finding the value of θ_0 and its derivatives for $t = \delta$ and letting δ tend to zero we find the initial values of θ_0 and its derivatives for $t = 0$ and hence the constants for the transient part of the output are found.

Consider the system given by the transfer function

$$\frac{\theta_0}{\theta_i} = \frac{5D + 26}{D^3 + 6D^2 + 21D + 26}.$$

If $\theta_i = h(t) = 1$, the particular integral in the solution of the differential equation is $\theta_0 = 1$ and the complementary function is

$$Ae^{-2t} + e^{-2t}(B \cos 3t + C \sin 3t).$$

We have $\quad (D^3 + 6D^2 + 21D + 26)\theta_0 = (5D + 26)h(t).$

Integrating this equation from $-\delta$ to t, where $-\delta \leqslant t \leqslant \delta$, we have

$$(D^2 + 6D + 21)\theta_0 + 26\int_{-\delta}^{t} \theta_0 dt = 5h(t) + 26\int_{-\delta}^{t} h(t)dt.$$

The values of θ_0 and $h(t)$ being bounded in the interval the integrals of these quantities tend to zero as δ tends to zero, and hence in the limit we have for $t = 0$

$$(D^2 + 6D + 21)\theta_0 = 5.$$

Integrating again twice and using the same limiting process we have

$$\begin{aligned}(D + 6)\theta_0 &= 0,\\ \theta_0 &= 0.\end{aligned}$$

Thus we find that initially on application of the input

$$\theta_0 = 0,\ \dot{\theta}_0 = 0,\ \ddot{\theta}_0 = 5.$$

The complete solution is

$$\theta_0 = Ae^{-2t} + e^{-2t}(B \cos 3t + C \sin 3t) + 1,$$

and

$$\dot{\theta}_0 = -2Ae^{-2t} + e^{-2t}\{(3C - 2B) \cos 3t - (3B + 2C) \sin 3t\},$$

$$\ddot{\theta}_0 = 4Ae^{-2t} + e^{-2t}\{-(12C + 5B) \cos 3t + (12B - 5C) \sin 3t\}.$$

Inserting the initial values of θ_0, $\dot{\theta}_0$, $\ddot{\theta}_0$ we have

$$\begin{aligned}A + B + 1 &= 0,\\ -2A - 2B + 3C &= 0,\\ 4A - 12C - 5B &= 5.\end{aligned}$$

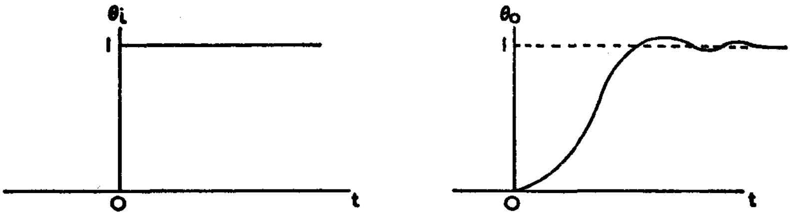

Fig. 42

Hence $A = -\frac{8}{9}, B = -\frac{1}{9}, C = -\frac{2}{3},$

and $$\theta_0 = -\frac{8}{9}e^{-2t} - e^{-2t}\left\{\frac{1}{9}\cos 3t + \frac{2}{3}\sin 3t\right\} + 1$$

is the unit response of the system.

The response is shown graphically in Fig. 42.

EXERCISES 2 (*e*)

1. Show that the response of the second order servomechanism whose transfer function is $\omega_n^2(\omega_n^2 + 2\zeta\omega_n D + D^2)^{-1}$ to the input Ωt^2 is $\Omega\{(t - 2\zeta/\omega_n)^2 + (4\zeta^2 - 2)/\omega_n^2\}$.
2. Show that the harmonic response of the system whose transfer function is $G\{TJD^3 + (J + fT)D^2 + fD + G\}^{-1}$ has phase lag $\tan^{-1}\omega(f - TJ\omega^2)/(G - J\omega^2 - fT\omega^2)$ and amplitude magnification factor $G\{(G - J\omega^2 - fT\omega^2)^2 + \omega^2(f - TJ\omega^2)^2\}^{-1/2}$.
3. Show that the harmonic response of the system whose transfer function is $\omega_n^2(1 + CD)/(\omega_n^2 + 2\zeta\omega_n D + D^2)$ has phase lag $\tan^{-1} 2\zeta\omega\omega_n/(\omega_n^2 - \omega^2) + \tan^{-1} C\omega$ and amplitude magnification factor $\omega_n^2(1 + C^2\omega^2)^{1/2}\{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega_n^2\omega^2\}^{-1/2}$.
4. Show that the unit response of the system whose transfer function is $\omega_n^2(1 + CD)/(\omega_n^2 + 2\zeta\omega_n D + D^2)$ is
$$1 + \frac{1}{\mu}e^{-\zeta\omega_n t}\{\omega_n(C\omega_n - \zeta)\sin \mu t - \mu\cos \mu t\},$$
where $\mu^2 = \omega_n^2(1 - \zeta^2)$.
5. Show that the unit response of the system whose transfer function is $(3D + 2)/(D^2 + 3D + 2)$ is $1 + e^{-t} - 2e^{-2t}$ and sketch its graph.

CHAPTER 3

MOMENTS OF INERTIA

3.1 Second Moments and Moments of Inertia

Let δA be an element of an area at a distance x from an axis in its plane. Then $\Sigma x\delta A$, where the sum is taken for all elements of the area, is called the *first moment of area* about the axis. If A be the total area and $\bar{x}$ the distance of the centroid from the axis we have

$$\Sigma x\delta A = A\bar{x}.$$

The sum $\Sigma x^2\delta A$, in which each element of an area is multiplied by the square of its distance from an axis, is called the *second moment of area* about the axis. If the surface density of the area be ϱ, so that the mass of an element δA is $\varrho\delta A$, then $\Sigma\varrho x^2\delta A$ is called the *moment of inertia* of the area about the given axis.

The units of a moment of inertia are mass $\times$ (distance)^2, and hence writing

$$Mk^2 = \Sigma\varrho x^2\delta A,$$

where M is the total mass, k is a length and is called the *radius of gyration* of the area about the axis. If the surface density is constant so that $M = A\varrho$, the second moment of area is

$$\Sigma x^2\delta A = Ak^2.$$

Similarly, if δV be an element of volume of a solid distant x from a given axis, $\Sigma x^2\delta V$ is called the *second moment of volume* about the axis, and if ϱ be the volume density so that the mass is

$$M = \Sigma\varrho\delta V,$$

then

$$\Sigma\varrho x^2\delta V = Mk^2$$

is the moment of inertia and k is the radius of gyration of the solid about the axis. If the density is constant so that $M = \varrho V$, where V is the volume of the solid, we have

$$\Sigma x^2\delta V = Vk^2.$$

The problem of finding a second moment of area or volume of a uniform area or volume about an axis is the same as that of finding the moment of inertia and the radius of gyration found from the second moment is the same as that found from the moment of inertia. It is common practice to refer to the second moment of area of a lamina as its moment of inertia taking its area as its equivalent mass.

3.2 Parallel Axes Theorem

If the moment of inertia of a body of mass M about an axis through its centre of mass is Mk^2, the moment of inertia about a parallel axis distant a from this axis is $M(k^2 + a^2)$.

Let any section of the body perpendicular to the given axis meet the axis through the centre of mass in G and the other axis in O, so that $OG = a$ (Fig. 43).

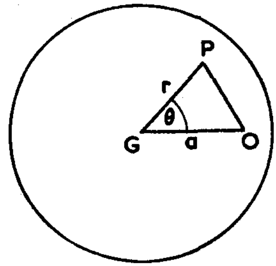

Fig. 43

Let P be any particle of mass m in the section distant r from G and let the angle OGP be θ.

Then $OP^2 = r^2 + a^2 - 2ra \cos\theta$.

The moment of inertia about the axis through G is given by $Mk^2 = \Sigma mr^2$, where the summation is taken over the whole body.

The moment of inertia about the axis through O is

$$\begin{aligned}\Sigma m(OP)^2 &= \Sigma m(r^2 + a^2 - 2ra\cos\theta),\\ &= \Sigma mr^2 + a^2\Sigma m - 2a\Sigma mr\cos\theta,\\ &= Mk^2 + Ma^2 - 2a\Sigma mr\cos\theta.\end{aligned}$$

The distance of the centre of mass of the body from a plane through G perpendicular to OG is $\Sigma mr\cos\theta/M$, and since the centre of mass G is in this plane we have $\Sigma mr\cos\theta = 0$.

Therefore $\Sigma m\,(OP)^2 = M(k^2 + a^2)$, and the theorem is proved.

3.3 Moments of Inertia of a Lamina

If the moments of inertia of a lamina about two perpendicular axes in its plane which meet in a point O are A and B respectively, its moment of inertia about an axis through O perpendicular to its plane is $A + B$.

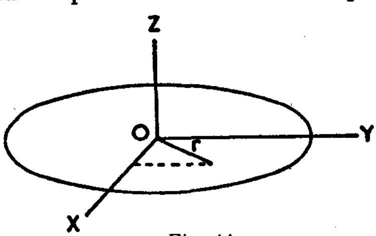

Fig. 44

Let OX and OY be the axes in the plane of the lamina and OZ the perpendicular axis (Fig. 44).

Let an element of mass m have coordinates (x, y) with respect to OX, OY.

Then we have $\quad A = \Sigma my^2, \quad B = \Sigma mz^2,$

for the moments of inertia about OX and OY respectively.
If the distance of the element from OZ be r we have $r^2 = x^2 + y^2$, and the moment of inertia about OZ is

$$\begin{aligned}&= \Sigma mr^2,\\ &= \Sigma m(x^2 + y^2),\\ &= A + B.\end{aligned}$$

3.4 Standard Forms

Moments of inertia of areas and volumes are found by the methods of the calculus. The following moments of inertia are given for reference.

Uniform thin rod, length $2a$, about axis through its centre perpendicular to rod	$\frac{1}{3}Ma^2$
Uniform rectangular lamina, sides $2a$ and $2b$ about axis through its centre parallel to the side of length $2b$	$\frac{1}{3}Ma^2$
Uniform rectangular parallelepiped of edges $2a$, $2b$ and $2c$ about axis through its centre parallel to the edge of length $2a$	$\frac{1}{3}M(b^2 + c^2)$
Uniform triangular lamina of base a and height h about an axis parallel to the base through the centroid	$\frac{1}{18}Mh^2$
Uniform circular ring of radius a about a diameter	$\frac{1}{2}Ma^2$
Uniform circular disc of radius a about an axis through its centre perpendicular to the disc and about any diameter	$\frac{1}{2}Ma^2$, $\frac{1}{4}Ma^2$
Uniform elliptic lamina of semi-axes a and b, about the axes	$\frac{1}{4}Mb^2$, $\frac{1}{4}Ma^2$
Uniform solid sphere of radius a about a diameter	$\frac{2}{5}Ma^2$
Uniform solid hemisphere of radius a about a diameter of its plane face	$\frac{2}{5}Ma^2$
Uniform thin spherical shell of radius a about a diameter	$\frac{2}{3}Ma^2$
Uniform solid ellipsoid of semi-axes $2a$, $2b$ and $2c$ about the axes	$\frac{1}{5}M(b^2 + c^2)$, $\frac{1}{5}M(c^2 + a^2)$, $\frac{1}{5}M(a^2 + b^2)$
Uniform solid circular cylinder of radius a and length h about its axis and about a perpendicular to its axis through its centre	$\frac{1}{2}Ma^2$, $\frac{1}{4}M(a^2 + \frac{1}{3}h^2)$
Uniform thin cylindrical shell of radius a and length h about its axis and about a perpendicular through its centre	Ma^2, $M(\frac{1}{2}a^2 + \frac{1}{12}h^2)$
Uniform solid cone of base radius a and height h about its axis and about a perpendicular through its centre of gravity	$\frac{3}{10}Ma^2$, $\frac{3}{20}M(a^2 + \frac{1}{4}h^2)$

Uniform thin conical shell of base radius a and height h about its axis and about a perpendicular through its centre of gravity $\frac{1}{2}Ma^2$, $M(\frac{1}{4}a^2 + \frac{1}{18}h^2)$

3.5 Moments of Inertia of Composite Bodies

It is sometimes required to find the moment of inertia of a composite body about an axis through its centre of gravity when the moments of inertia of the component bodies are known. The parallel axes theorem is used to find the moments of inertia of the component bodies about the axis through the centre of gravity of the whole before the moments are added.

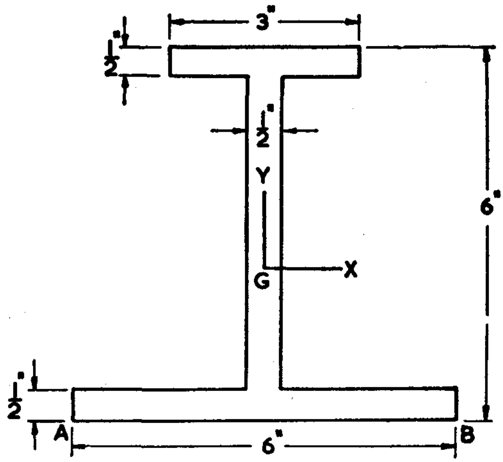

Fig. 45

Example 1. *Find the moments of inertia of the I-section shown in Fig. 45 about axes through its centroid parallel and perpendicular to the flanges.*

The area is 7 in.² and the distance of the centroid G from AB is given by

$$7\bar{y} = 3 \times 0{\cdot}25 + 2{\cdot}5 \times 3 + 1{\cdot}5 \times 5{\cdot}75,$$
$$\bar{y} = 2{\cdot}41 \text{ in.}$$

About GX the moments of the upper flange, the web and the lower flange are respectively

$$1{\cdot}5\left\{\frac{1}{48} + (5{\cdot}75 - 2{\cdot}41)^2\right\} = 16{\cdot}77,$$

$$2{\cdot}5\left\{\frac{25}{12} + (3 - 2{\cdot}41)^2\right\} = 6{\cdot}08,$$

$$3\left\{\frac{1}{48} + (0{\cdot}25 - 2{\cdot}41)^2\right\} = 14{\cdot}06.$$

Hence, the total moment of inertia about $GX = 36{\cdot}91$ in.^{4}, and the radius of gyration is $2{\cdot}30$ in.

The axis GY passes through the centroids of the three parts and hence the moment about GY is

$$1{\cdot}5 \times \frac{9}{12} + 2{\cdot}5 \times \frac{1}{48} + 3 \times 3 = 10{\cdot}18 \text{ in.}^4$$

and the radius of gyration $= 1{\cdot}21$ in.

EXERCISES 3 (*a*)

1. A hole of 1 ft. diameter is punched in a uniform circular plate of 5 ft. diameter, the centre of the hole being 18 in. from the centre of the plate. Find the moment of inertia of the plate, (i) about the diameter through the centre of the hole, (ii) about the diameter which is perpendicular to this. (L.U.)

2. AB and BC are two uniform rods of the same material of lengths 6 ft. and 4 ft. respectively, joined at B so that the angle ABC is a right-angle. Find the radii of gyration of the rods about the axes through the centre of gravity of the rods parallel to AB and BC respectively.

3. Three rectangular areas, 2 ft. by 2 in., 3 ft. by 2 in., and 1 ft. by $1\frac{1}{2}$ in., are fitted together to form an I-figure, the longest and shortest areas forming the cross-pieces. Find the moment of inertia of the area about the outer edge of the shortest area. (L.U.)

4. $ABCD$ is a uniform square lamina of side $2a$ and E is the mid-point of BC. The triangular portion ABE is removed; find the second moment of the remaining area about AD.

5. The radii of the ends of a frustum of a uniform cone are a and $2a$, its mass is M and its height h. Find its moment of inertia (i) about its axis, (ii) about the diameter of its larger face.

6. A uniform body of mass M consists of a hemisphere of radius a and a right circular cylinder of radius a and height a with one of its circular faces attached to the plane face of the hemisphere. Find the moments of inertia of the body about its axis of symmetry and about a diameter of the free circular face of the cylinder.

7. A wind-measuring instrument consists of two hemispherical thin shells, each of radius a and mass m, attached to the ends of a uniform thin rod of mass m and length $4a$, the rims of the hemispheres being in the same plane and their centres $6a$ apart. Find the moment of inertia of the body about an axis through its centre of gravity perpendicular to the rod and in the same plane as the rims.

8. A uniform circular disc of radius $2a$ has a concentric hole of radius a. The mass of the annulus is M. Find the moment of inertia of the annulus about an axis through a point on the circumference of the circle of radius a (i) if the axis is perpendicular to the plane of the annulus, (ii) if the axis lies in the plane of the annulus and touches the circle. (L.U.)

9. A uniform hemisphere of radius a has its plane face joined to the base of a uniform circular cylinder of radius a and height $2a$, ϱ being the density of each body. Find the moments of inertia of the composite body about its axis of symmetry and a perpendicular axis through its centre of gravity.

10. The figure (Fig. 46) represents the section of a girder of the given dimensions. Find its radius of gyration about AB, and about a parallel axis through the centroid of the section. (L.U., Pt. I)

11. A box girder is made of plates 5 in. wide and $\frac{1}{2}$ in. thick. The plates enclose a rectangle 5 in. by 3 in., the flange plates overlapping $\frac{1}{2}$ in. on each side. Determine the radius of gyration of a section about an axis through the centroid parallel to the flanges. (L.U., Pt. I)

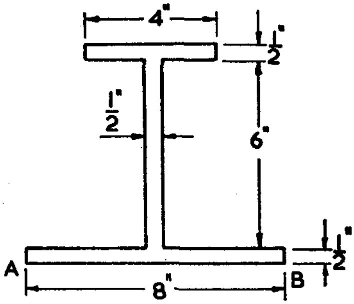

Fig. 46

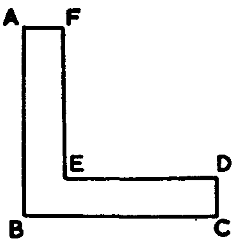

Fig. 47

12. AB is a uniform bar, 26 in. long, pivoted at its ends to parallel arms, AC and BD perpendicular to an axis CD, and of lengths 8 in. and 18 in. respectively. Find the radius of gyration of the bar AB about the axis CD. (L.U., Pt. I)

13. In the uniform thin metal plate shown in the diagram (Fig. 47) $AB = BC = 10$ in., $AF = CD = 2$ in.
Find in inches, (i) the distance of the centre of mass from AB, (ii) the radius of gyration about AB, (iii) the radius of gyration about an axis through the centre of mass perpendicular to the plane of the plate. (L.U., Pt. I)

14. A uniform T-shaped area consists of two rectangles $ABCD$, $PQRS$, where AB, BC, PQ, QR are 24, 8, 12, 4 in. respectively, PS lies in DC and DP is 10 in. Calculate the distance from AB of the centroid, G, of the area and also the second moment of the area about an axis through G parallel to AB.

If this area is revolved through 360° about AB show that the radius of gyration of the resulting solid is approximately 11 in. about the axis of revolution. (L.U., Pt. I)

15. Prove that the moment of inertia of a uniform triangular plate of mass M and height h about the base is $\frac{1}{6}Mh^2$.

A uniform lamina is in the form of an isosceles triangle with base angles θ, equal sides a and area density s. Find the moments of inertia

about (i) the axis of symmetry and (ii) a line through the vertex parallel to the base.

Find also, their greatest values if θ is the only variable.

(L.U., Pt. I)

16. An anchor ring of mass M is formed by rotating a disc of radius a about an axis in its plane distant h ($> a$) from its centre. Show that its moment of inertia about the axis of rotation is $M\left(h^2 + \frac{3}{4}a^2\right)$, and about a perpendicular axis through the centre $M\left(\frac{1}{2}h^2 + \frac{5}{8}a^2\right)$.

3.6 Product of Inertia for a Lamina

Let the coordinates of an element of mass m of a lamina with respect to axes OX and OY be (x, y) (Fig. 48).

The moments of inertia of the lamina about OX and OY are respectively

$$A = \Sigma my^2,$$
$$B = \Sigma mx^2.$$

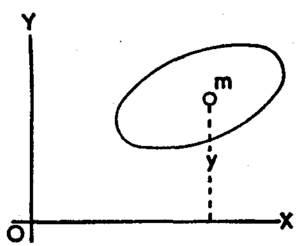

Fig. 48

The product of inertia about OX and OY is defined as

$$H = \Sigma mxy.$$

Products of inertia are required in expressions for the energy and momentum of a rigid body. They are also required for the calculation of moments of inertia about other axes of the body.

It is evident that if the lamina is symmetrical about either of the axes OX, OY, the product of inertia about the pair of axes is zero.

The product of inertia may be calculated by integration using a double integral. Thus

$$H = \iint mxydxdy,$$

where m is the density, the integration being over the area of the lamina. It is rarely necessary to perform this integration since the product of inertia may be obtained by a parallel axes theorem similar to that for moments of inertia.

3.7 Parallel Axes Theorem for Products

Let H be the product of inertia of a lamina of mass M about axes OX, OY and let H_1 be the product of inertia about parallel axes GX', GY' through the centre of gravity, the coordinates of G being $(\bar{x}, \bar{y})$ (Fig. 49).

Let (x', y') be the coordinates of an element m with respect to GX', GY'; with respect to OX, OY the coordinates will be $(x' + \bar{x}, y' + \bar{y})$.

Then
$$H_1 = \Sigma mx'y'.$$
$$H = \Sigma m(x' + \bar{x})(y' + \bar{y})$$
$$= \Sigma mx'y' + \bar{y}\Sigma mx' + \bar{x}\Sigma my' + \bar{x}\bar{y}\Sigma m.$$

Now since G is the centre of gravity $\Sigma mx' = \Sigma my' = 0$ and we have

$$H = H_1 + M\bar{x}\bar{y}.$$

In particular, if OX or OY is an axis of symmetry,

$$H_1 = -M\bar{x}\bar{y}.$$

If GX' or GY' is an axis of symmetry,

$$H = M\bar{x}\bar{y}.$$

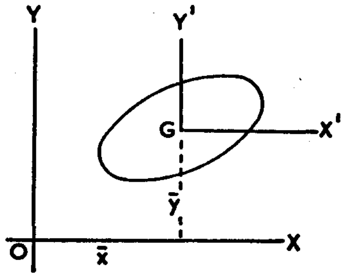

Fig. 49.

Example 2. *Find the product of inertia for the given L-section (Fig. 50), about axes parallel to its sides through its centre of gravity.*

Let axes OX, OY, be drawn parallel to the sides through the centres of gravity of the rectangles. The coordinates of the centre of gravity of the section with respect to these axes are

$$\bar{x} = \frac{\frac{7}{4} \times 2}{3 + \frac{7}{4}} = \frac{14}{19},$$

$$\bar{y} = \frac{3 \times 2\frac{3}{4}}{3 + \frac{7}{4}} = \frac{33}{19}.$$

Now each of the rectangles is symmetrical about one of the axes OX, OY, therefore the product of inertia of the whole section about OX, OY is zero. Therefore, about parallel axes GX', GY', through the centre of gravity the product of inertia is, taking the area of the section 4·75 in.^{2} as its mass,

$$H_1 = -4{\cdot}75 \times \frac{14}{19} \times \frac{33}{19}$$
$$= -6{\cdot}08 \text{ in.}^4.$$

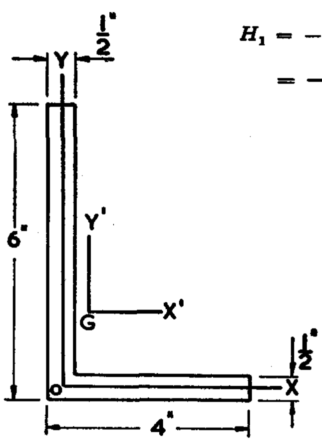

Fig. 50

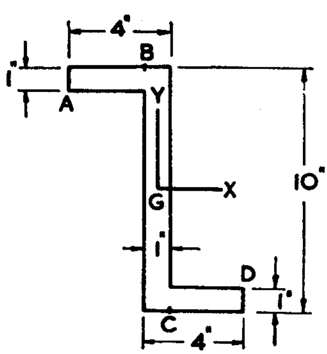

Fig. 51

Example 3. *Find the moments and products of inertia of the given section (Fig. 51) about axes parallel to its sides through its centre of gravity.*

The section may be divided into three rectangles AB, BC, CD, the centre of gravity G being at the centre of BC.
The moment of inertia about GX of

$$BC = \frac{1 \times 10^3}{12} = 83{\cdot}33$$

$$AB = \frac{3 \times 1^3}{12} + 3(4{\cdot}5)^2 = 61{\cdot}00$$

$$CD = \frac{3 \times 1^3}{12} + 3(4{\cdot}5)^2 = 61{\cdot}00$$

$$\text{Total} = 205{\cdot}33 \text{ in.}^4$$

The moment of inertia about GY of

$$BC = \frac{10 \times 1^3}{12} = 0{\cdot}83$$

$$AB = \frac{1 \times 3^3}{12} + 3(2)^2 = 14{\cdot}25$$

$$CD = \frac{1 \times 3^3}{12} + 3(2)^2 = 14{\cdot}25$$

$$\text{Total} = 29{\cdot}33 \text{ in.}^4$$

The product of inertia about GX, GY of $BC = 0$.
The product of inertia of AB about parallel axes through its centre of gravity is zero and the coordinates of its centre of gravity with respect to GX, GY are $(-2, 4{\cdot}5)$. Therefore its product of inertia about GX, GY

$$= 3 \times (-2) \times (4{\cdot}5)$$
$$= -27 \text{ in.}^4.$$

CD has the same product of inertia and hence the total product about GX, $GY = -54$ in.^{4}.
Hence the moments and product of inertia about GX, GY are

$$A = 205{\cdot}33 \text{ in.}^4,$$
$$B = 29{\cdot}33 \text{ in.}^4,$$
$$H = -54{\cdot}00 \text{ in.}^4.$$

EXERCISES 3 (*b*)

1. Find by integration the product of inertia of a quadrant of a circle of radius a and mass M about its bounding radii and deduce the product about parallel axes through the centroid.

2. A lamina is in the form of a rectangle of sides $2a$ and $2b$ extended by a semicircle whose bounding diameter is a side of length $2b$ of the rectangle. Find the product of inertia about a side of length $2a$ and a side of length $2b$ remote from the semicircle.

3. A circular disc of radius 4 in. has a concentric hole of radius 1 in. Find the product of inertia about two tangents to the disc which are at right-angles.

4. Show that the product of inertia of a lamina in the form of a right-angled triangle about sides of length a and b which are at right-angles is $a^2b^2/24$.

5. Find the moments and product of inertia of the L-section shown in Fig. 52 about axes parallel to the edges through the centroid.

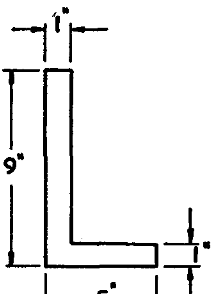

Fig. 52

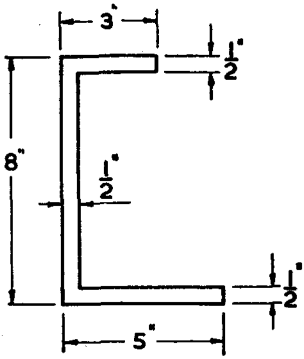

Fig. 53

6. Find the moments and product of inertia of the section shown in Fig. 53 about axes parallel to the edges through the centroid.

3.8 Change in Direction of Axes

Given that the moments and product of inertia of a lamina with reference to axes OX, OY are A, B and H respectively we can find the moments and product A', B' and H' with respect to axes OX', OY' inclined at θ to the original axes (Fig. 54).

Let the coordinates of an element be (x, y) with respect to OX, OY, and (x', y') with respect to OX', OY'.

Then

$$x' = x \cos\theta + y \sin\theta,$$
$$y' = -x \sin\theta + y \cos\theta.$$

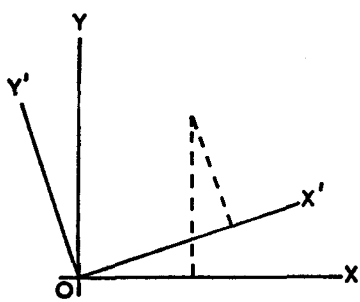

Fig. 54

Therefore

$$\begin{aligned} A' &= \Sigma m(y')^2 \\ &= \Sigma m(-x \sin\theta + y \cos\theta)^2 \\ &= \cos^2\theta \Sigma my^2 + \sin^2\theta \Sigma mx^2 - 2 \sin\theta \cos\theta \Sigma mxy \\ &= A \cos^2\theta + B \sin^2\theta - 2H \sin\theta \cos\theta. \\ B' &= \Sigma m(x \cos\theta + y \sin\theta)^2 \\ &= A \sin^2\theta + B \cos^2\theta + 2H \sin\theta \cos\theta. \end{aligned}$$

$$\begin{aligned}H' &= \Sigma m(x \cos\theta + y \sin\theta)(-x \sin\theta + y \cos\theta)\\ &= \sin\theta\cos\theta\Sigma my^2 - \sin\theta\cos\theta\Sigma mx^2 + (\cos^2\theta - \sin^2\theta)\Sigma mxy\\ &= \frac{1}{2}(A - B)\sin 2\theta + H\cos 2\theta.\end{aligned}$$

It is easily seen that

$$A' + B' = A + B,$$
$$A'B' - (H')^2 = AB - H^2,$$

and therefore that the values of $A + B$ and $AB - H^2$ are the same for all pairs of axes through O.

3.9 Principal Axes of a Lamina

We have from the previous section,

$$\begin{aligned}A' &= A\cos^2\theta + B\sin^2\theta - 2H\sin\theta\cos\theta\\ &= \frac{1}{2}(A + B) - \frac{1}{2}(B - A)\cos 2\theta - H\sin 2\theta\\ &= \frac{1}{2}(A + B) - \frac{1}{2}\{(B - A)^2 + 4H^2\}^{1/2}\cos(\alpha - 2\theta),\end{aligned}$$

where $\tan\alpha = \dfrac{2H}{B - A}$.

Similarly,

$$B' = \frac{1}{2}(A + B) + \frac{1}{2}\{(B - A)^2 + 4H^2\}^{1/2}\cos(\alpha - 2\theta),$$

$$H' = \frac{1}{2}\{(B - A)^2 + 4H^2\}^{1/2}\sin(\alpha - 2\theta).$$

When $\theta = \frac{1}{2}\alpha$, $\cos(\alpha - 2\theta) = 1$ and A' is a minimum, B' is a maximum and $H' = 0$.

Then
$$A' = \frac{1}{2}(A + B) - \frac{1}{2}\{(B - A)^2 + 4H^2\}^{1/2},$$

$$B' = \frac{1}{2}(A + B) + \frac{1}{2}\{(B - A)^2 + 4H^2\}^{1/2},$$

and the axis OX' is inclined at an angle $\dfrac{1}{2}\tan^{-1}\dfrac{2H}{B - A}$ to OX.

The axes about which the moments of inertia have their maximum and minimum values and the product of inertia is zero are called the principal axes of the lamina at the point.

When principal axes OX', OY' have been found for any point and the moments of inertia A' and B' calculated, the moment of inertia about any axis OP inclined at an angle ϕ to OX' is, since $H' = 0$,

$$A'' = A'\cos^2\phi + B'\sin^2\phi.$$

Example 4. *Find the directions of the principal axes at the centroid of the given section (Fig. 55), and the moments of inertia about these axes.*

Let axes OX, OY be drawn parallel to the sides through the centroids of the rectangles $DEFH$ and $ABCH$ respectively. The coordinates of the centroid G of the section with respect to these axes are

$$\bar{x} = \frac{14}{19}, \quad \bar{y} = \frac{33}{19}.$$

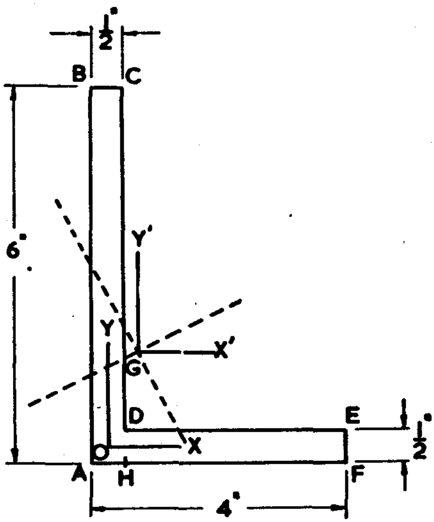

Fig. 55

Let parallel axes GX', GY' be drawn through G. The moment of inertia about GX' of

$$ABCH = 3\left\{\frac{6^2}{12} + \left(\frac{77}{76}\right)^2\right\} = 12{\cdot}08$$

$$DEFH = \frac{7}{4}\left\{\frac{(1/2)^2}{12} + \left(\frac{33}{19}\right)^2\right\} = 5{\cdot}28$$

$$\text{Total} = 17{\cdot}36 \text{ in.}^4$$

The moment of inertia about GY' of

$$ABCH = 3\left\{\frac{(1/2)^2}{12} + \left(\frac{14}{19}\right)^2\right\} = 1{\cdot}68$$

$$DEFH = \frac{7}{4}\left\{\frac{(7/2)^2}{12} + \left(\frac{24}{19}\right)^2\right\} = 4{\cdot}58$$

$$\text{Total} = 6{\cdot}26 \text{ in.}^4$$

The product of inertia is zero about OX, OY, hence, about GX', GY' it is

$$-4{\cdot}75 \times \frac{14}{19} \times \frac{33}{19} = -6{\cdot}08 \text{ in.}^4.$$

Hence, for the axes GX', GY' we have

$$A = 17{\cdot}36, \quad B = 6{\cdot}26, \quad H = -6{\cdot}08.$$

The direction of the principal axes is given by

$$\frac{1}{2}\tan^{-1}\frac{2H}{B-A} = \frac{1}{2}\tan^{-1}\frac{12\cdot16}{11\cdot10} = 23^\circ\,48',$$

that is, the principal axes are inclined at angles $23^\circ\,48'$ and $113^\circ\,48'$ to GX'.

Also
$$\frac{1}{2}(A+B) = 11\cdot81,$$
$$\frac{1}{2}\{(B-A)^2 + 4H^2\}^{1/2} = 8\cdot23.$$

Hence the moments of inertia about the principal axes are

$$A' = 20\cdot04 \text{ in.}^4,$$
$$B' = 3\cdot58 \text{ in.}^4.$$

EXERCISES 3 (*c*)

1. Find the moment of inertia of a rectangular lamina of sides $2a$ and $2b$, (i) about an axis through its centre inclined at 30° to the side of length $2b$, (ii) about a diagonal.
2. Find the moments of inertia of a quadrant of a circle about a radius through the centroid and about a perpendicular axis through the centre.
3. Find the inclination to the sides of a rectangle of lengths $2a$ and a of the principal axes through a corner of the rectangle, and the moments of inertia about them.
4. Find the moment of inertia of the I-section shown in Fig. 56 about an axis through its centroid inclined at 25° to the flanges.
5. Given that the moments and product of inertia of an L-section (Exercises 3 (*b*), 5) about axes through its centroid parallel to its edges are $105\cdot4$, $23\cdot4$ and $-27\cdot7$ in.^{4}, find the inclination to these axes of the principal axes at the centroid and the moments of inertia about them.
6. Find the moment of inertia of the L-section shown in Fig. 57 about an axis AB through the point A inclined at 30° to the edge.

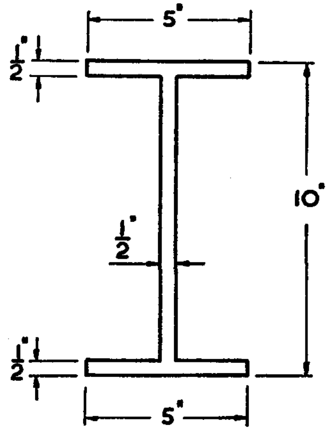

Fig. 56

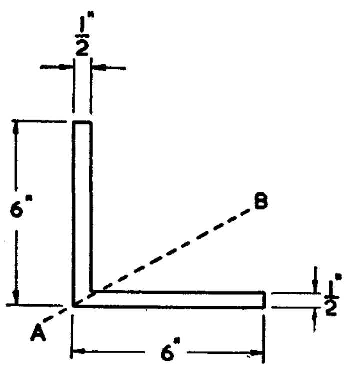

Fig. 57

3.10 Momental Ellipse

The moments and products of inertia of a lamina being A, B, H respectively about axes OX, OY, the equation

$$Ax^2 + By^2 - 2Hxy = \text{constant},$$

represents an ellipse, since $AB - H^2 > 0$, which is called the momental ellipse. The equation is made more precise by defining the momental ellipse by the equation

$$Ax^2 + By^2 - 2Hxy = \frac{AB - H^2}{M},$$

M being the mass of the lamina.

If the equation is referred to other axes OX', OY' inclined at θ to OX, OY (Fig. 58) and $(x'y')$ be the coordinates of a point with respect to these axes, we have

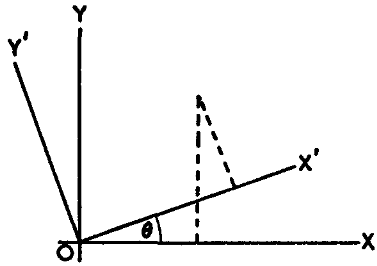

Fig. 58

$$x = x' \cos\theta - y' \sin\theta,$$
$$y = x' \sin\theta + y' \cos\theta,$$

and the equation referred to the new axes becomes

$$A(x' \cos\theta - y' \sin\theta)^2 + B(x' \sin\theta + y' \cos\theta)^2 - 2H(x' \cos\theta - y' \sin\theta)(x' \sin\theta + y' \cos\theta) = \frac{AB - H^2}{M},$$

that is

$$(x')^2(A\cos^2\theta + B\sin^2\theta - 2H\sin\theta\cos\theta) + (y')^2(A\sin^2\theta + B\cos^2\theta + 2H\sin\theta\cos\theta) - 2x'y'\{(A - B)\sin\theta\cos\theta + H\cos 2\theta\} = \frac{AB - H^2}{M}.$$

If A', B', H' be the moments and products of inertia with respect to the axes OX', OY', this becomes

$$A'x'^2 + B'y'^2 - 2H'x'y' = \frac{AB - H^2}{M} = \frac{A'B' - H'^2}{M}.$$

If the axes OX', OY' are principal axes so that $H' = 0$ we have

$$A'x'^2 + B'y'^2 = \frac{A'B'}{M}.$$

Writing $A' = Mk_x^2$ and $B' = Mk_y^2$ we see that the equation of the ellipse referred to principal axes is

$$\frac{x^2}{k_y^2} + \frac{y^2}{k_x^2} = 1.$$

Thus the major and minor semi-axes of the ellipse are principal axes

of the lamina at the point and each is equal to the radius of gyration about the other.

The radius of this ellipse which is inclined at an angle ϕ to the x-axis is found by substituting $x = r\cos\phi$, $y = r\sin\phi$ giving

$$r^2(k_x^2\cos^2\phi + k_y^2\sin^2\phi) = k_x^2 k_y^2.$$

But $k_x^2\cos^2\phi + k_y^2\sin^2\phi$ is the square of the radius of gyration about the radius, therefore each radius is inversely proportional to the radius of gyration about it.

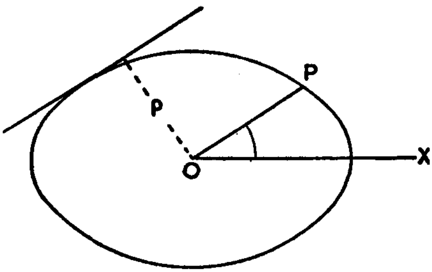

Fig. 59

If k_ϕ be the radius of gyration about the radius we have

$$k_\phi = \frac{k_x k_y}{r}.$$

A well-known property of an ellipse is that if OP be any radius and p a perpendicular drawn from O to a tangent parallel to OP, then $p \times OP = ab$, where a and b are the semi-axes of the ellipse (Fig. 59).

In the case of the momental ellipse this gives

$$pr = k_x k_y$$

$$p^2 = \frac{k_x^2 k_y^2}{r^2}$$

$$= k_x^2\cos^2\phi + k_y^2\sin^2\phi,$$

therefore p is the radius of gyration about OP.

If the momental ellipse be drawn for the centre of gravity of a lamina the principal axes are clearly shown and the radius of gyration about an axis through the centre of gravity in any direction easily found.

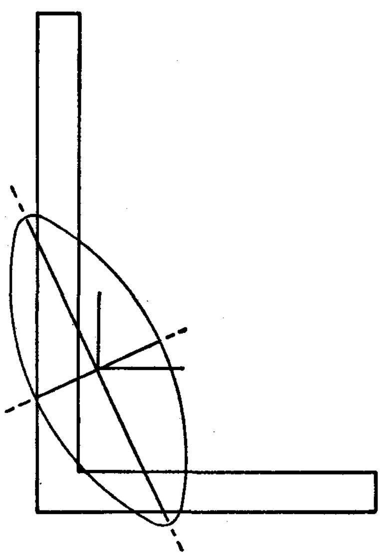

Fig. 60

Example 5. *Construct the momental ellipse for the L-section considered in Example* 4.

The inclination of the principal axes to the lower edge are 23° 48′ and 113° 48′ and the moments of inertia about them are 20·04 and 3·58 in.^{4} respectively. The corresponding radii of gyration are

$$\left(\frac{20\cdot04}{4\cdot75}\right)^{1/2} = 2\cdot05 \text{ in.}, \qquad \left(\frac{3\cdot58}{4\cdot75}\right)^{1/2} = 0\cdot87 \text{ in.}$$

The ellipse is as shown in Fig. 60.

3.11 Moments and Products for a Triangle

Let PQR be a triangular lamina of mass M, PO the perpendicular from P on QR, and let $OP = h$, $OQ = b$, $OR = c$ (Fig. 61). We shall take OR and OP as the axes of x and y respectively and find the moments and product of inertia A, B and H, about these axes.

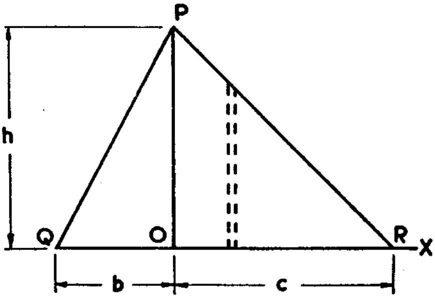

Fig. 61

For the moment about OX we consider a strip of length $(h - y)(b + c)/h$ distant y from OX and we have

$$A = \int_0^h y^2\left\{\frac{(h-y)(b+c)}{h}\right\}dy = \frac{(b+c)h^3}{12} = M\frac{h^2}{6}.$$

Similarly the moments of the triangles OPR and OPQ about OY are respectively $hc^3/12$ and $hb^3/12$, therefore

$$B = \frac{h}{12}(b^3 + c^3) = M\frac{b^2 - bc + c^2}{6}.$$

To find the product of inertia for the triangle OPR we consider a strip of width δx distant x from OY, whose length is $h(c - x)/c$. The product of inertia for this strip is, x being constant,

$$\int_0^{h(c-x)/c} xydxdy$$

$$= xdx\left[\frac{1}{2}y^2\right]_0^{h(c-x)/c}$$

$$= \frac{1}{2}\frac{h^2}{c^2}x(c - x)^2dx.$$

For the whole triangle OPR the product is

$$\int_0^c \frac{1}{2}\frac{h^2}{c^2}x(c - x)^2dx$$

$$= \frac{1}{2}\frac{h^2}{c^2}\left[\frac{c^2x^2}{2} - \frac{2cx^3}{3} + \frac{x^4}{4}\right]_0^c$$

$$= \frac{h^2c^2}{24}.$$

For each element of the triangle OPQ, x is negative and the product of inertia is $-\frac{h^2b^2}{24}$.

Therefore
$$H = \frac{1}{24}h^2(c^2 - b^2)$$
$$= M\frac{h(c-b)}{12}.$$

Now consider the moments and products of inertia about the same axes of masses $\frac{1}{3}M$ placed at the mid-point of each side.

We have

$$A = \frac{1}{3}M\left(\frac{h}{2}\right)^2 + \frac{1}{3}M\left(\frac{h}{2}\right)^2 = M\frac{h^2}{6}.$$

$$B = \frac{1}{3}M\left(\frac{b}{2}\right)^2 + \frac{1}{3}M\left(\frac{c}{2}\right)^2 + \frac{1}{3}M\left(\frac{c-b}{2}\right)^2 = M\frac{b^2 - bc + c^2}{6}.$$

$$H = \frac{1}{3}M\left(\frac{c}{2}\cdot\frac{h}{2}\right) + \frac{1}{3}M\left(-\frac{b}{2}\cdot\frac{h}{2}\right) = M\frac{h(c-b)}{12}.$$

Therefore, these three masses have the same moments and product of inertia as the lamina about OX, OY.

Further, since the centre of gravity of the masses is the same as that of the lamina, the moments and products of inertia of the lamina and of the masses about any other axes in the plane are obtained in the same way and are identical.

This is expressed by saying that the lamina is *equimomental* with masses each equal to one-third of its mass placed at the mid-points of its sides. It is thus a simple matter to calculate the moment of inertia or product of inertia of the lamina about any axes in its plane.

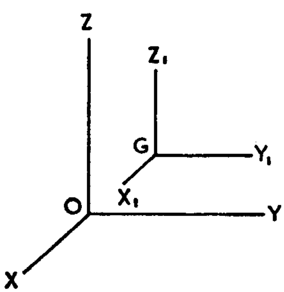

Fig. 62

3.12 Moments and Products in Three Dimensions

If m be the mass of an element of a body and (x, y, z) its coordinates with respect to rectangular axes OX, OY, OZ (Fig. 62) the moments of inertia of the body about OX, OY, OZ respectively are

$$A = \Sigma m(y^2 + z^2).$$
$$B = \Sigma m(z^2 + x^2).$$
$$C = \Sigma m(x^2 + y^2).$$

The products of inertia about the pairs of axes OY, OZ; OZ, OX; OX, OY are respectively,

$$F = \Sigma myz.$$
$$G = \Sigma mzx.$$
$$H = \Sigma mxy.$$

The *polar moment of inertia* about O is defined as

$$\begin{aligned} I &= \Sigma m(x^2 + y^2 + z^2) \\ &= \frac{1}{2}(A + B + C). \end{aligned}$$

The parallel axis theorems hold for both moments and products of inertia. Let $G(\bar{x}, \bar{y}, \bar{z})$ be the centre of gravity and (x_1, y_1, z_1) the coordinates of the element with respect to parallel axes through G. If A_1, B_1, C_1, F_1, G_1 H_1 be the moments and products about these axes we have

$$\begin{aligned} A &= \Sigma m\{(\bar{y} + y_1)^2 + (\bar{z} + z_1)^2\} \\ &= \Sigma m(y_1{}^2 + z_1{}^2) + 2\bar{y}\Sigma my_1 + 2\bar{z}\Sigma mz_1 + (\bar{y}^2 + \bar{z}^2)\Sigma m \\ &= A_1 + M(\bar{y}^2 + \bar{z}^2). \\ B &= B_1 + M(\bar{z}^2 + \bar{x}^2). \\ C &= C_1 + M(\bar{x}^2 + \bar{y}^2). \\ F &= \Sigma m(\bar{y} + y_1)(\bar{z} + z_1) \\ &= \Sigma my_1z_1 + \bar{y}\Sigma mz_1 + \bar{z}\Sigma my_1 + \bar{y}\bar{z}\Sigma m, \\ &= F_1 + M\bar{y}\bar{z}. \\ G &= G_1 + M\bar{z}\bar{x}. \\ H &= H_1 + M\bar{x}\bar{y}. \end{aligned}$$

If the products F, G, H are all zero the axes are called principal axes.

3.13 Momental Ellipsoid

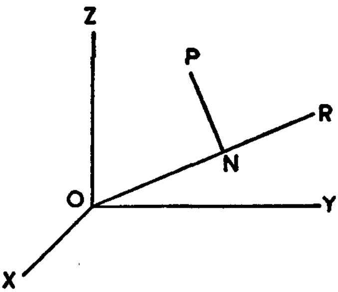

Fig. 63

Let l, m, n be the direction cosines of any line through O and m' the mass of an element of the body at $P(x, y, z)$. Then if PN be the perpendicular from P to OR (Fig. 63), N is the point $(l\varrho$, $m\varrho$, $n\varrho)$ where the directions $x - l\varrho$: $y - m\varrho$: $z - n\varrho$ and $l : m : n$ are perpendicular, that is $\varrho = lx + my + nz$.

Hence $ON = lx + my + nz$.

The moment of inertia of the body about OR

$$\begin{aligned} &= \Sigma m'(PN)^2 \\ &= \Sigma m'\{x^2 + y^2 + z^2 - (lx + my + nz)^2\} \\ &= l^2\Sigma m'(y^2 + z^2) + m^2\Sigma m'(z^2 + x^2) + n^2\Sigma m'(x^2 + y^2) \\ &\qquad - 2mn\Sigma m'yz - 2nl\Sigma m'zx - 2lm\Sigma m'xy, \\ &= Al^2 + Bm^2 + Cn^2 - 2Fmn - 2Gnl - 2Hlm. \end{aligned}$$

Consider the quadric surface whose equation is

$$Ax^2 + By^2 + Cz^2 - 2Fyz - 2Gzx - 2Hxy = K,$$

where K is any constant.

The length r of a radius vector of the quadric whose direction cosines are l, m, n, is found by substituting $x = lr$, $y = mr$, $z = nr$, in the equation of the quadric, giving

$$r^2\{Al^2 + Bm^2 + Cn^2 - 2Fmn - 2Gnl - 2Hlm\} = K.$$

Hence the moment of inertia about the radius is $\dfrac{K}{r^2}$.

It follows that the quadric has a real finite radius vector in any direction and hence it is an ellipsoid, and it is called the momental ellipsoid of the body at the point.

In books dealing with the coordinate geometry of ellipsoids it is shown that it is always possible to choose a set of coordinate axes with origin at O such that when the ellipsoid is referred to these axes the product terms disappear and the equation is of the form

$$A'x^2 + B'y^2 + C'z^2 = K.$$

These axes are the principal axes of the body at the point O and A', B', C' are called the principal moments of inertia at the point.

In particular, if $A' = B' = C'$ the momental ellipsoid is a sphere for which all axes are principal axes and the moment of inertia is the same about any axis. For example, a cube of side a has moment of inertia $Ma^2/6$ about axes through its centre perpendicular to a face. By symmetry these are principal axes and hence the moment of inertia about any axis through the centre is $Ma^2/6$.

EXERCISES 3 (*d*)

1. Show that for the I-section considered in Exercises 3 (*c*), 4, the principal radii of gyration at the centroid are 3·88 in. and 1·05 in. Draw the momental ellipse at the centroid and read off the values of the radii of gyration about axes inclined at 30° and 60° to the principal axes.

2. Show that for the L-section of Exercise 3 (*c*), 6, the principal radii of gyration at the centroid are 2·35 and 1·18 in. Draw the momental ellipse and measure the radius of gyration about an axis inclined at 30° to the axis of symmetry.

3. Prove that the moment of inertia of a regular hexagon of side a about any axis through its centre is $5Ma^2/24$.

4. ABC is a triangular lamina and $AB = 4$ in., $BC = 5$ in., $CA = 3$ in. Find the inclination to AB of the principal axes at the centroid and the moments of inertia about them.

5. A right-angled isosceles triangular lamina ABC, whose sides AB, BC are each of length 9 ft., is immersed in a liquid with the corners A, B, C, at depths 10 ft., 10 ft., and 4 ft., respectively. Find the moment of inertia of the lamina about the line in which its plane cuts the surface of the liquid.

6. Show that the moment of inertia of a rectangular prism of sides $2a$, $2b$, $2c$ and mass M about a diagonal is

$$\frac{2}{3}M(b^2c^2 + c^2a^2 + a^2b^2)/(a^2 + b^2 + c^2).$$

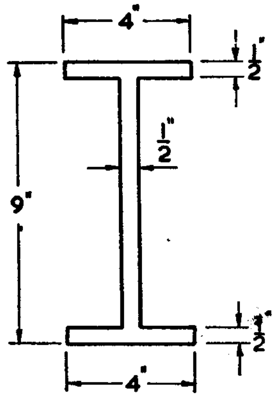

Fig. 64

7. Fig. 64 shows the cross-section of a rolled-steel joist 10 ft. long. Taking the weight of the steel as 480 lb./ft.^{3}, find in lb. ft.^{2} the moments of inertia about the three axes of symmetry.

8. A beam 8 ft. long has a cross-section which is a right-angled isosceles triangle, the equal sides being 1 ft. The material weighs 144 lb./ft.^{3}. Find the moments of inertia about the three principal axes through the centre of gravity.

CHAPTER 4

MOTION OF A RIGID BODY ABOUT A FIXED AXIS

4.1. Angular Velocity

In discussing the motion of a rigid body we assume that such a body is completely inelastic and is made up of an indefinitely large number of particles whose relative positions are unaltered. There may be action between the particles of the body so that two particles exert on each other forces which are equal and opposite. These are the *internal* forces of the body, but since they occur in pairs and the relative positions of the particles are unchanged the work done by any such pair will be zero. It follows that the total work done by the internal forces will be zero and the motion is determined by the *external* forces acting on the body.

The position of a rigid body turning about an axis is known if the position of a section of the body by a plane perpendicular to the axis through the centre of gravity is known. The position of this plane section may be fixed by one coordinate, usually the angle which the perpendicular from the centre of gravity to the axis makes with a fixed direction in its plane. If this angle be θ and any other line in the section makes an angle ϕ with the fixed direction then $\phi - \theta$ is constant and therefore $\dot{\phi} = \dot{\theta}$. Thus we have an angular velocity $\dot{\theta}$ common to all particles of the body.

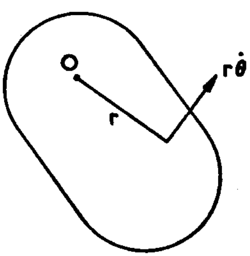

Fig. 65

4.2. Kinetic Energy of Rotation

Let a body of mass M be rotating with angular velocity $\dot{\theta}$ about a fixed axis; let M be the mass of the body and $I = Mk^2$ its moment of inertia about the axis (Fig. 65).

Let a particle of mass m be distant r from the axis. Since it is moving in a circle of radius r its velocity is $r\dot{\theta}$ and its kinetic energy is $\frac{1}{2}mr^2\dot{\theta}^2$ in absolute units. Hence we have the total kinetic energy of the body,

$$T = \Sigma\frac{1}{2}mr^2\dot{\theta}^2,$$

the summation being over all the particles of the body.

Therefore

$$T = \frac{1}{2}\dot{\theta}^2\Sigma mr^2$$
$$= \frac{1}{2}I\dot{\theta}^2$$
$$= \frac{1}{2}Mk^2\dot{\theta}^2.$$

This is the kinetic energy in absolute units; in gravitational units it is $\frac{1}{2}Mk^2\dot{\theta}^2/g$.

It follows that the kinetic energy of the body is the same as that of a particle of mass M placed at a distance k from the axis and turning about it with angular velocity $\dot{\theta}$.

4.3. Energy Equation

The increase of kinetic energy of any particle of the body as the body rotates is equal to the work done by the forces both internal and external which act on it. Hence, since the total work done by the internal forces is zero, the increase of kinetic energy of the whole body is equal to the work done by the external forces acting on it.

If these forces are conservative the work done by the external forces is equal to the loss of potential energy, and we have that the sum of the kinetic and potential energy is constant.

Let a body of mass M have radius of gyration k about a horizontal axis and let its centre of gravity G be distant h from the axis of rotation at O (Fig. 66). Let the line OG make an angle θ with the downward vertical at O. Then the angular velocity is $\dot{\theta}$ and the kinetic energy $\frac{1}{2}Mk^2\dot{\theta}^2$.

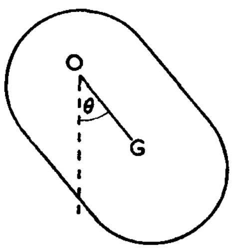

Fig. 66

If the body can turn freely about the axis, the forces acting on it at O do no work as it rotates, and the potential energy of the body due to its weight is $Mgh(1 - \cos\theta)$, since $h - h\cos\theta$ is the height of the centre of gravity above its lowest position.

We have therefore

$$\frac{1}{2}Mk^2\dot{\theta}^2 + Mgh(1 - \cos\theta) = \text{constant}.$$

If the body is released from rest in a position in which $\theta = \alpha$, $\dot{\theta}$ is initially zero and we have

$$\frac{1}{2}Mk^2\dot{\theta}^2 + Mgh(1 - \cos\theta) = Mgh(1 - \cos\alpha)$$

$$\dot{\theta}^2 = \frac{2gh}{k^2}(\cos\theta - \cos\alpha).$$

The angular acceleration of the body is found by differentiating this equation with respect to θ, and since $\frac{d}{d\theta}\dot{\theta}^2 = 2\ddot{\theta}$, we have

$$\ddot{\theta} = -\frac{gh}{k^2}\sin\theta.$$

If there is a frictional couple of magnitude G at the axis of rotation the work done by this couple against the motion in a displacement θ_1 to θ_2 is

$$\int_{\theta_1}^{\theta_2} G d\theta.$$

If G is constant, this is $G(\theta_2 - \theta_1)$ and if $\dot{\theta}_1$ and $\dot{\theta}_2$ be the angular velocities in the two positions we have for the change of kinetic energy

$$\frac{1}{2}Mk^2\dot{\theta}_2^2 - \frac{1}{2}Mk^2\dot{\theta}_1^2 = Mgh(\cos\theta_2 - \cos\theta_1) - G(\theta_2 - \theta_1).$$

Example 1. *A uniform rod of length 4 ft. and mass 2 lb. is free to turn about a horizontal axis through the rod at a point 1 ft. from one end. The rod is held in a horizontal position and released; what should be its angular velocity when it becomes vertical?*

If there is a constant frictional couple at the axis of rotation, and the rod after passing through the vertical position rises only until it is inclined at 60° to the vertical, find the frictional couple and the angular velocity when it first reached the vertical position.

The moment of inertia of the rod about an axis through its centre is $2 \times 2^2/3$, and about the axis of rotation it is

$$2\left(\frac{2^2}{3} + 1\right) = \frac{14}{3} \text{ lb.ft.}^2.$$

The kinetic energy when the angular velocity is $\dot{\theta}$ is

$$\frac{7}{3}\dot{\theta}^2 \text{ ft.pdl.}$$

The work done by gravity as the rod falls to the vertical position is $2g \times 1$ ft.pdl. and we have in this position

$$\frac{7}{3}\dot{\theta}^2 = 2g,$$

$$\dot{\theta} = 5{\cdot}24 \text{ rad./sec.}$$

If the rod comes to rest inclined at 60° to the vertical the loss of potential energy in this position is $2g \cos 60° = 32$ ft.pdl.

This is, therefore, the work done by the frictional couple in the rotation through 150°. Hence, if G be the couple

$$G \times \frac{5}{6}\pi = 32,$$
$$G = 12{\cdot}22 \text{ ft.pdl.}$$

With this couple the velocity in the vertical position is given by

$$\frac{7}{3}\dot{\theta}^2 = 2g - \frac{1}{2}\pi \times 12{\cdot}22$$
$$\dot{\theta}^2 = 19{\cdot}2$$
$$\dot{\theta} = 4{\cdot}38 \text{ rad./sec.}$$

Example 2. *A uniform circular cylinder of mass M and radius a can turn freely about its axis which is horizontal. A chain of length $2\pi a$ and mass m per unit length is fixed to the cylinder at the end of a horizontal diameter and encircles the cylinder. A weight of mass M_1 is attached to the free end of the chain and released. Find an expression for the velocity of the weight when the cylinder has turned through an angle θ.*

The change in potential energy of the chain is that a length θ of the circumference is freed of chain and a length $a\theta$ of the chain hangs vertically (Fig. 67).

The centre of gravity of the circular arc is on a central radius at a distance $(2a \sin \frac{1}{2}\theta)/\theta$ from the centre, and therefore at a depth $(2a \sin^2 \frac{1}{2}\theta)/\theta$ below the centre. Therefore the loss of potential energy by the chain is

$$mga\theta\left(\frac{a\theta}{2} - \frac{2a \sin^2 \frac{1}{2}\theta}{\theta}\right).$$

The energy equation is

$$\frac{1}{4}Ma^2\dot{\theta}^2 + \frac{1}{2}(2\pi am + M_1)a^2\dot{\theta}^2$$
$$= mga\left(\frac{1}{2}a\theta^2 - 2a \sin^2 \frac{1}{2}\theta\right) + M_1ga\theta,$$

and the velocity of the weight is $a\dot{\theta}$.

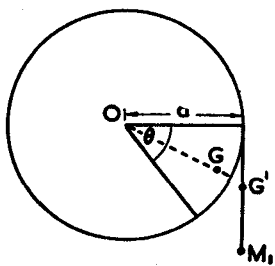

Fig. 67

4.4 Moment of Inertia of a Flywheel

The moment of inertia of a flywheel is found experimentally by mounting it in a horizontal bearing and attaching to it a weight by a light string wrapped round the wheel. From the time taken by the weight to fall a measured distance the moment of inertia can be calculated. If the bearing is not smooth the frictional torque can be eliminated by finding the time taken by two different weights.

Let I be the moment of inertia of the flywheel, r its radius, and w the attached weight. When the wheel has turned through an angle θ the angular velocity of the wheel is $\dot{\theta}$ and the velocity of the weight is $r\dot{\theta}$ (Fig. 68).

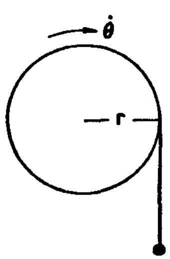

Fig. 68

The kinetic energy of the system in gravitational units is

$$\frac{I}{2g}\dot{\theta}^2 + \frac{w}{2g}r^2\dot{\theta}^2.$$

If F be the frictional torque the work done in turning through an angle θ is $wr\theta - F\theta$ and we have

$$\frac{(I + wr^2)}{2g}\dot{\theta}^2 = wr\theta - F\theta.$$

Differentiating this equation with respect to θ we have

$$(I + wr^2)\ddot{\theta} = (wr - F)g.$$

Thus the acceleration $r\ddot{\theta}$ with which the weight descends is seen to be constant. This acceleration is found by timing the fall of the weight and an equation involving I and F is obtained.

Example 3. *A flywheel on a horizontal axis* 2 *in. in diameter is set in motion by weights attached to a cord round the axle and allowed to fall vertically. The time to fall* 5 *ft. from rest for weights of* 10 *lb. and* 20 *lb. are* 9·5 *and* 5·5 *sec. respectively. Show that the moment of inertia of the flywheel is approximately* 1440 *lb.in.*^{2} *and determine the frictional torque in the bearings.* (L.U., Pt.I)

For the weight of 10 lb. falling 5 ft. in 9·5 sec., the acceleration is $\frac{1}{12}\ddot{\theta}_1$ and

$$5 = \frac{1}{2} \times \frac{1}{12}\ddot{\theta}_1 \times (9{\cdot}5)^2.$$

For the weight of 20 lb., falling 5 ft. in 5·5 sec., the acceleration is $\frac{1}{12}\ddot{\theta}_2$ and

$$5 = \frac{1}{2} \times \frac{1}{12}\ddot{\theta}_2 \times (5{\cdot}5)^2.$$

Therefore $\ddot{\theta}_1 = 1{\cdot}33, \quad \ddot{\theta}_2 = 3{\cdot}97$ rad./sec.^{2}

Substituting in the equation

$$(I + wr^2)\ddot{\theta} = (wr - F)g,$$

$$\left(I + \frac{10}{144}\right)1{\cdot}33 = \left(\frac{10}{12} - F\right)g,$$

$$\left(I + \frac{20}{144}\right)3{\cdot}97 = \left(\frac{20}{12} - F\right)g.$$

Subtracting we have

$$2{\cdot}64I + \frac{66{\cdot}1}{144} = \frac{10}{12}g.$$

$$\begin{aligned} I &= 9{\cdot}93 \text{ lb.ft.}^2 \\ &= 10 \text{ lb.ft.}^2 \text{ approximately.} \end{aligned}$$

$$\begin{aligned} F &= \frac{10}{12} - \left(9{\cdot}93 + \frac{10}{144}\right) \times \frac{1{\cdot}33}{32} \\ &= 0{\cdot}42 \text{ ft.lb.} \end{aligned}$$

4.5 Flywheel of a Reciprocating Engine

When the crank of a reciprocating engine is turning with a certain angular velocity the torque exerted on it by the connecting rod will vary, being zero when the crank and the connecting rod are in line. Also the piston and connecting rod have a certain kinetic energy which varies with their position and the loss or gain of energy of these moving parts is reflected in an increase or decrease of the torque on the crank.

Thus the torque will vary about its mean value during a revolution in some way. The variation may be plotted from a knowledge of the energy of the moving parts and the piston thrust and a diagram of the variation such as is shown in Fig. 69 obtained.

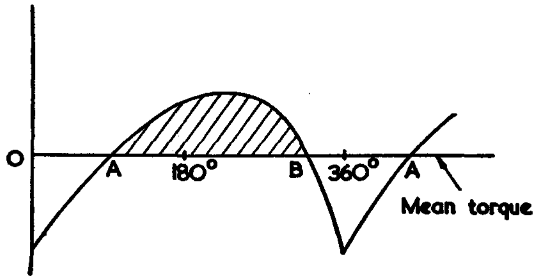

Fig. 69

From A to B there is an excess of torque giving a maximum speed of rotation at B, similarly, there is a minimum speed at A.

The excess work done by the torque at B is energy which has to be absorbed by the flywheel without excessive variation in speed.

Thus if Mk^2 be the moment of inertia of the flywheel, ω the mean angular velocity, ω_B the angular velocity at B and W_B the excess work at B we have

$$\frac{1}{2g}Mk^2(\omega_B{}^2 - \omega^2) = W_B.$$

The larger the flywheel the smaller will be the variation of angular velocity. Thus if a variation of not more than one per cent from an angular velocity ω is desired, we have $\omega^2\{(1{\cdot}01)^2 - 1\} = 0{\cdot}02\omega^2$, approximately, so that for $\dfrac{1}{2g}Mk^2 \times 0{\cdot}02\omega^2 \geqslant W_B$ we must have $Mk^2 \geqslant \dfrac{100gW_B}{\omega^2}$.

EXERCISES 4 (*a*)

1. A wheel has a cord of length 10 ft. coiled round its axle; the cord is pulled with a constant force of 25 lb.wt., and when the cord leaves the axle, the wheel is rotating five times a second. Calculate the moment of inertia of the wheel and axle. (L.U.)

2. A uniform straight rod AB of length $2a$ is smoothly jointed at A to a horizontal table and is allowed to fall, in a vertical plane through A from rest when AB makes an angle α with the vertical. Prove that when the angle which AB makes with the vertical is θ, then $2a\dot{\theta}^2 = 3g\ (\cos\alpha - \cos\theta)$. Find also the angular acceleration of the rod just before it becomes horizontal.

3. A uniform circular disc of mass 120 lb. and radius 10 in. is free to rotate about a horizontal axis through its centre perpendicular to its plane. A particle of mass 30 lb. is attached to the highest point of the rim of the disc and the equilibrium of the system slightly disturbed. Find the angular velocity of the particle, in revolutions per minute, when the particle is passing through its lowest point. (L.U.)

4. A uniform thin rod of length $6a$ and mass m is suspended so as to rotate freely about a fixed axis perpendicular to the rod and through a point of trisection. It is allowed to rotate freely from the horizontal position. What should be its greatest angular velocity? If there is a constant frictional couple, so that the rod turns through an angle of 60° beyond the vertical before coming to instantaneous rest, show that the greatest angular velocity is $\sqrt{(7g/20a)}$.

5. A torpedo is driven by expending the energy stored in a flywheel, initially rotating at 10,000 r.p.m. If the mass of the flywheel is 200 lb. and it is regarded as a uniform circular disc of diameter 2 ft., show that it will be rotating at half the initial rate after about 685 yd. run at 30 m.p.h. assuming that the average power necessary for this speed is 50 h.p. (L.U.)

6. A uniform solid wheel of mass 20 lb. and radius 9 in., is free to turn about a horizontal axis through its centre. A light inextensible string is wrapped three times around the wheel and carries a mass of 5 lb. hanging vertically from the other end. If the system is released from rest, find the angular velocity of the wheel when the string has left it, assuming that the string does not slip on the wheel.

7. A flywheel is carried on a shaft of radius 1·5 in., mounted horizontally in frictionless bearings. A thin cord has one end attached to the shaft and is then wound round it, a weight of 40 lb. being attached to the other end, which leaves the shaft vertically.

On releasing the weight it is found that it takes 2 sec. to fall 8 ft. Find the moment of inertia of the flywheel and shaft about their axis and the tension in the cord during the motion. (Q.E.)

8. A solid flywheel 10 in. in diameter is mounted on a horizontal axle 2 in. in diameter running in frictionless bearings. The mass of the wheel is 44 lb. and of the axle 8·8 lb. One end of a string is fixed to the axle and twenty turns of the string are wound round it and a mass of 30 lb. is hung from the free end. If the system is released from rest, find the angular velocity when the string is just unwound.

9. A flywheel and shaft at a particular instant are rotating at 200 r.p.m. If the moment of inertia is 5400 lb.ft.^{2}, what is the kinetic energy of the wheel and shaft?

If during the next ten revolutions the kinetic energy decreases by 10 per cent, what is the average retarding torque for this amount of rotation?

If at the end of the ten revolutions the torque remains constant at the previous average value, how long will it take for the flywheel and shaft to come to rest? (Q.E.)

10. A pulley whose radius is r and moment of inertia about its axis I is mounted in a smooth horizontal bearing. A chain of weight W and length $3\pi r$ has one end fixed to the pulley at an extremity of a horizontal diameter and then passes completely round the pulley, its other end hanging freely. If the system is allowed to move from rest, show that the angular velocity of the pulley after turning through half a revolution will be

$$\{Wrg(3\pi^2 - 4)/3\pi(I + Wr^2)\}^{\frac{1}{2}}. \qquad \text{(L.U., Pt. I)}$$

11. A flywheel on a horizontal axle 3 in. in diameter has weights attached to a light string wound round the axle. When the weight is 20 lb. the time it takes to fall 6 ft. is 10 sec., when the weight is 30 lb. the time to fall the same distance is 7·5 sec. Find how long a weight of 40 lb. will take to fall the same distance.

12. An engine, working against a steady resistance, develops 200 h.p. at 400 r.p.m., and the greatest fluctuation of energy is 40 per cent of the energy developed in a revolution. If the radius of gyration of the flywheel is 3 ft., calculate its mass if the revolutions of the engine are to be kept within the limits 399 and 401 r.p.m. (C.U.)

13. A wheel of moment of inertia I is fixed on a shaft of small radius r and round the shaft is wound a length l of thin chain of mass m per unit length, one end being fixed to the shaft, the other carrying a mass equal to that of the whole chain. Initially the shaft is at rest with the attached mass level with the axis. If the mass is released, show that the length of chain unwound after t seconds is $2l \sinh^2 \lambda t$, where

$$\lambda = \tfrac{1}{2}\{mgr^2/(I + 2mlr^2)\}^{\frac{1}{2}}. \qquad \text{(L.U., Pt. II)}$$

4.6 Angular Momentum

Let a body of mass M rotating about a fixed axis with angular velocity $\dot{\theta}$ have moment of inertia $I = Mk^2$ about the axis (Fig. 70).

A particle of mass m at a point P distant r from the axis at O, is moving in a circle of radius r about O and has velocity $r\dot{\theta}$ perpendicular to OP. Its momentum is $mr\dot{\theta}$ and the moment of this momentum about O is $mr^2\dot{\theta}$. This moment of momentum is called the *angular momentum* of the particle about O.

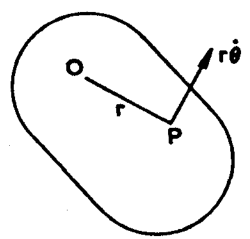

Fig. 70

The angular momentum of the body about the axis is the sum of the angular momenta of all the particles of the body, that is

$$\begin{aligned}&\Sigma mr^2\dot{\theta}\\&= \dot{\theta}\Sigma mr^2\\&= I\dot{\theta}.\end{aligned}$$

Let the internal and external forces which act on the particle at P have component X perpendicular to OP. Then since the particle is moving in a circle we have

$$X = mr\ddot{\theta}.$$

Therefore $$Xr = mr^2\ddot{\theta}.$$

Summing for all the particles of the body we have

$$\begin{aligned}\Sigma Xr &= \Sigma mr^2\ddot{\theta}\\&= I\ddot{\theta}.\end{aligned}$$

Now ΣXr is the sum of the moments of all the forces acting on the particles of the body about the axis. In this sum the moment of the internal forces of the body, which occur in pairs, vanishes. Hence, if G be the moment of the external forces about the axis we have

$$\begin{aligned}G &= I\ddot{\theta}\\&= \frac{d}{dt}I\dot{\theta}.\end{aligned}$$

That is, the sum of the moments of the external forces acting on the body about the axis of rotation is equal to the rate of change of angular momentum about the axis. It is clear that in this equation the forces and their moment are expressed in absolute units.

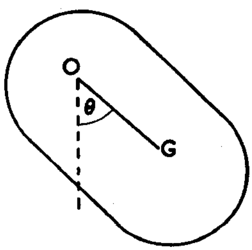

Fig. 71

If the external forces have no moment about the axis, $G = 0$ and $I\dot{\theta}$ is constant, that is, the angular momentum remains constant.

Let the centre of gravity of the body G be distant h from the axis of rotation and OG be inclined at θ to the vertical (Fig. 71) and let the body be free to turn under its own weight about the axis. The forces at O have no moment about the axis and the external forces have therefore a moment—$Mgh \sin\theta$ about the axis.

We have, therefore,

$$Mk^2\ddot{\theta} = -Mgh\sin\theta.$$

If there is in addition a frictional couple F opposing motion, we have

$$Mk^2\ddot{\theta} = -Mgh\sin\theta - F.$$

These equations are, of course, the derivatives of the energy equations obtained in § 4.3.

Example 4. *A uniform solid wheel of diameter 2 ft. and mass 60 lb. is turning in frictionless bearings at 600 r.p.m. If the wheel is to be stopped in 40 seconds by a brake pressing on the rim, calculate the pressure required assuming that the coefficient of friction is 0·1 between the brake block and the rim.*

Let X lb.wt. be the pressure required. The frictional force is $0{\cdot}1Xg$ pdl. and its moment about the axis of rotation $1 \times 0{\cdot}1Xg$ ft.pdl.

Therefore
$$60 \times \frac{1}{2} \times \ddot{\theta} = -1 \times 0{\cdot}1Xg,$$
$$\ddot{\theta} = -0{\cdot}1067X \text{ rad./sec.}^2.$$

The wheel is stopped from a speed of 20π rad./sec. in 40 sec. and the deceleration is therefore $\frac{1}{2}\pi$ rad./sec.^{2}

Therefore
$$0{\cdot}1067X = \frac{1}{2}\pi = 1{\cdot}5708$$
$$X = 14{\cdot}7 \text{ lb.wt.}$$

4.7 Pulley Systems

When a light string passes over a pulley which is in motion the tension in the string is not in general the same on either side and it is the difference in the tensions on either side of the pulley which provides the torque which changes the angular momentum of the pulley.

Thus if T_1 and T_2 be the tensions either side of the pulley (Fig. 72), r its radius and I its moment of inertia, its angular acceleration is given by the equation

$$(T_2 - T_1)r = I\ddot{\theta}.$$

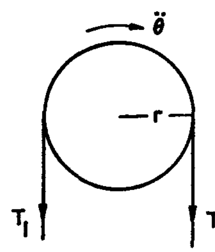

Fig. 72

In problems involving complicated systems of weights and pulleys the acceleration of the parts of the system can usually be found by means of the energy equation, the tensions in the parts of the string being internal forces of the system. To find these tensions, however, equations of motion for the parts of the system must be considered.

Example 5. *Two bodies of equal mass are attached to the ends of an inextensible string which passes over a pulley which can be regarded as a uniform circular disc, whose plane is perpendicular to the edge of a rough horizontal table, coefficient of friction 1/2. One body rests on the table and the other hangs vertically, and the mass of the pulley is equal to mass of each of the bodies. Assuming that the string does not slip on the pulley and that the part of the string over the table is parallel to the table, determine the acceleration of the system and prove that the ratio of the tensions in the string on the two sides of the pulley is 7 : 8.* (L.U., Pt. I)

Let T_1 and T_2 be tensions in the horizontal and vertical portions of the string respectively (Fig. 73). If m be the mass of each body the frictional force on the table is $\frac{1}{2}mg$.

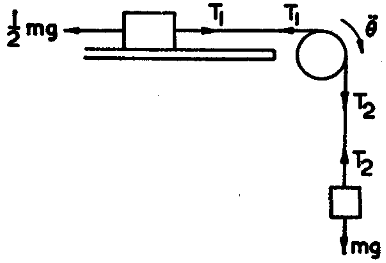

Fig. 73

Let $\ddot{\theta}$ be the angular acceleration of the pulley and r its radius. Then if the string does not slip on the pulley the linear acceleration of each of the weights is $r\ddot{\theta}$.

The equations of motion for the three masses are

$$T_1 - \frac{1}{2}mg = mr\ddot{\theta},$$

$$(T_2 - T_1)r = \frac{1}{2}mr^2\ddot{\theta},$$

$$mg - T_2 = mr\ddot{\theta}.$$

Dividing the second equation by r and adding it to the first and third equations we have

$$\frac{1}{2}mg = \frac{5}{2}mr\ddot{\theta}.$$

Hence the linear acceleration $r\ddot{\theta} = \frac{1}{5}g.$

We have $$T_1 = \frac{1}{2}mg + mr\ddot{\theta} = \frac{7}{10}mg,$$

$$T_2 = mg - mr\ddot{\theta} = \frac{4}{5}mg,$$

and hence $$T_1 : T_2 = 7 : 8.$$

Example 6. *Two masses 2m and m are attached to the ends of a light inelastic string passing over a pulley of mass m with its axis fixed horizontally. From m is suspended another mass m by means of an elastic string of unstretched length a and modulus mg. If the system is released from rest with the elastic string unstretched and the pulley is considered as a solid circular disc, prove that each mass will move with simple harmonic motion, and find the period and the distance over which the mass 2m oscillates.* (L.U., Pt. II)

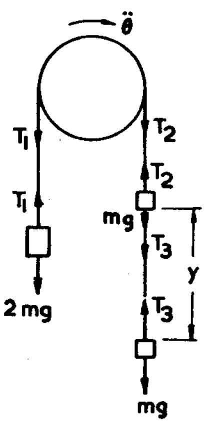

Fig. 74

Let T_1, T_2, T_3 be the tensions in the three portions of string (Fig. 74), θ the angle through which the wheel has turned and y the length of the elastic string at time t.

The accelerations of the $2m$ mass and of the upper m mass are each $r\ddot{\theta}$, r being the radius of the pulley, and the acceleration of the other mass is $r\ddot{\theta} + \ddot{y}$.

The equations of motion are

$$T_1 - 2mg = 2mr\ddot{\theta}, \tag{1}$$

$$r(T_2 - T_1) = \frac{1}{2}mr^2\ddot{\theta}, \tag{2}$$

$$mg + T_3 - T_2 = mr\ddot{\theta}, \tag{3}$$

$$mg - T_3 = m(r\ddot{\theta} + \ddot{y}), \tag{4}$$

and $$T_3 = mg\left(\frac{y - a}{a}\right). \tag{5}$$

Adding equations (1) to (4), having divided (2) by r, we have

$$0 = m\left(\frac{9}{2}r\ddot{\theta} + \ddot{y}\right).$$

Hence

$$\begin{aligned} 2\ddot{y} + 9r\ddot{\theta} &= 0, \\ 2\dot{y} + 9r\dot{\theta} &= 0, \\ 2y + 9r\theta &= 2a. \end{aligned}$$

Adding equations (4) and (5) and substituting for $r\ddot{\theta}$ we have

$$mg = m\left(\ddot{y} - \frac{2}{9}\ddot{y}\right) + mg\left(\frac{y-a}{a}\right),$$

that is

$$\ddot{y} + \frac{9g}{7a}(y - 2a) = 0.$$

This is an equation of simple harmonic motion and initially $y = a$, $\dot{y} = 0$, therefore

$$y = 2a - a\cos(9g/7a)^{1/2}t.$$

Hence

$$r\theta = -\frac{2}{9}a\{1 - \cos(9g/7a)^{1/2}t\}.$$

The period is therefore $2\pi(7a/9g)^{1/2}$ and the $2m$ mass oscillates over a distance $4a/9$.

EXERCISES 4 (*b*)

1. A uniform cylindrical wheel of mass 100 lb. and radius 2 ft. is fixed on a light horizontal axle of radius 6 in., which turns in frictionless bearings. A fine string wrapped round the axle carries a weight of 10 lb. at its free end and a similar string wrapped round the wheel in the opposite sense carries a mass of 12 lb. at its free end. If the system is released from rest, find the velocity of the 12 lb.wt. when it has fallen through 6 ft., and the tension in the string from which it hangs.

2. A flywheel whose moment of inertia is 5400 lb. ft.^{2} is rotating at 100 r.p.m. Find its kinetic energy. During ten revolutions the kinetic energy decreases by 10 per cent, find the average retarding torque. Find how long it would take this torque to bring the flywheel to rest from 200 r.p.m., assuming it to be constant.

3. Two particles each of mass m are joined by a light inextensible string of length πa which passes over a uniform rough circular cylinder of radius a and mass $2m$. The cylinder is free to rotate about its axis which is horizontal. Initially the particles are in equilibrium at opposite ends of a horizontal diameter of the cylinder. One particle then receives a small displacement vertically downwards. When the second particle reaches the top of the cylinder, the first particle strikes the floor and remains there. Find the velocity of the first particle on striking the floor, and prove that the second particle leaves the cylinder when it has travelled a further distance.

$$a\cos^{-1}\left\{\frac{1}{6}(\pi + 1)\right\}.$$

4. A uniform circular disc of radius a and mass M with a rough edge is mounted on a horizontal axis. A string is passed over the disc and carries masses m_1 and m_2 ($m_1 > m_2$) at its ends. The system starts from rest, and the string does not slip on the disc. Show that the velocity v of each weight is expressible in terms of the distance x each has moved by the equation

$$(M + 2m_1 + 2m_2)v^2 = 4(m_1 - m_2)gx.$$

Find the acceleration if $m_1 = 4$ lb., $m_2 = 1$ lb. and $M = 2$ lb. (L.U.)

5. A wheel and axle having a moment of inertia of 4 lb.ft.^{2} is free to rotate about a horizontal axis. A mass of 3 lb. is fixed to the wheel at a distance of 1 ft. from its axis. A weight of 18 lb. is suspended from a light string wound round the axle which is 2 in. in diameter. The 3-lb. mass is held level with the axis of the axle and released, thus causing the 18-lb. mass to be wound up. What is the greatest angular velocity of the wheel and axle in the subsequent motion? (Q.E.)

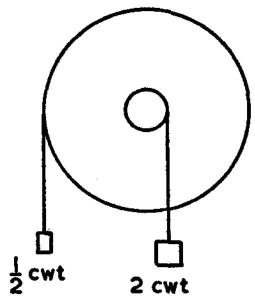

Fig. 75

6. The diagram (Fig. 75) represents a wheel and axle, of radii 2 ft. and 3 in. respectively, total weight 3 cwt., and radius of gyration 18 in., free to turn about a horizontal axis. Neglecting friction, find the angular acceleration. The observed angular acceleration was 1·74 radian/sec.^{2}. Calculate the frictional torque. (L.U., Pt. I)

7. An engine working at constant power P ft. lb. per sec., drives a fan whose moment of inertia about its axis is I lb.ft.^{2}, and which is subject to a resistive torque Kn ft. lb., where n is the angular velocity in radians per second. Find the maximum speed at which the fan can be driven, and show that the time taken to reach a speed n from rest is

$$\frac{I}{2Kg} \cdot \log \frac{P}{P - Kn^2}. \qquad \text{(L.U., Pt. I)}$$

8. Two gear-wheels A and B, of radii a and b and moments of inertia I_1 and I_2 respectively, are mounted on parallel axes and run permanently in mesh. When in motion, there are constant frictional torques P and Q on A and B respectively. A constant torque G is applied to A. Find
   (i) the tangential force between the wheels,
   (ii) the angular acceleration of A,
   (iii) the number of revolutions made by B in acquiring from rest a speed of N revolutions per unit time.

   Prove also that, for motion to be possible, G must be greater than $P + Qa/b$. (L.U., Pt. I)

9. A fly wheel has a moment of inertia of 3600 lb.ft.^{2} and its speed varies with time as shown in the table. Show that the retarding torque on

the flywheel is approximately constant and find its value in lb.ft. What is the mean horse-power absorbed by this torque during the period covered by the table?

Time, secs.	0	10	20	30	40	50	60
Speed, r.p.m.	240	224	206	189	173	155	140

(Q.E.)

10. A friction band is passed round a pulley of radius r keyed to the shaft of a dynamo armature. One end of the band is connected to a helical spring whose other end is fixed, while the other end of the band carries a weight W, the extension of the spring when the system is in equilibrium being a.

Neglecting friction in the armature bearings and assuming that the band does not slip, obtain the equations of motion of the weight W and the armature when the system is oscillating freely, and hence find the moment of inertia of the armature in terms of the time of oscillation. (L.U., Pt. II)

11. A thin uniform rod of mass m and length $2a$ is free to rotate in the vertical plane about a horizontal axis through one end. Each element of the rod is subject to a resistance per unit length of amount mk/a times its velocity. Obtain the differential equation of motion if the rod is released from rest in a position making a small angle with the downward vertical.

State the condition for the motion to be oscillatory and solve the differential equation when this condition is satisfied.

If a torque of magnitude L and frequency the same as the natural undamped frequency of the rod is applied, prove that the resulting final oscillation (assumed small) is of amplitude $(L/4mka)(3/ga)^{1/2}$. (L.U., Pt. II)

4.8 Impulsive Torques

If a force P acts on a body which is free to rotate about a fixed axis and pP be its moment about the axis, we have Mk^2 being the moment of inertia and $\ddot{\theta}$ the angular acceleration,

$$pP = Mk^2\ddot{\theta}.$$

Therefore
$$\int_{t_1}^{t_2} pPdt = \left[Mk^2\dot{\theta}\right]_{t_1}^{t_2}.$$

Hence, the time integral of the torque over a given period is the change of angular momentum. If p is constant

$$\int_{t_1}^{t_2} pPdt = p\int_{t_1}^{t_2} Pdt,$$

and the impulsive torque is the moment of the impulse about the axis of rotation.

When we are dealing with large and possibly varying forces acting for a short period we can measure the impulsive torque by the change

of angular momentum without attempting to evaluate directly the time integral of the force causing the torque.

When dealing with impulses of short duration we may suppose that the effect of the ordinary forces which act on the body is negligible while the impulse lasts, that is, we treat the impulse as if it were due to an infinitely large force acting for an infinitesimal period of time.

Example 7. *Two cog-wheels A and B are mounted on parallel shafts and turn in opposite directions at 120 and 240 r.p.m. respectively. The wheels may be taken as uniform circular discs of radii r and 2r and masses m and 8m respectively. If the wheels engage find the new rate of revolution of each wheel and the proportion of the original kinetic energy that is lost.*

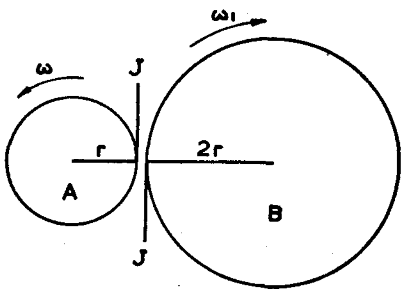

Fig. 76

Let the angular speeds of the wheels after engagement be ω and ω_1 r.p.m. respectively in opposite directions (Fig. 76).

Since the rims have the same speed $\omega_1 = \frac{1}{2}\omega$.

Let J be the impulse on the teeth as they engage, the impulsive torque on A is Jr and on B it is $2Jr$.

The impulsive torque on A will have the effect of increasing its angular speed from 120 to ω r.p.m., the moment of inertia is $\frac{1}{2}mr^2$ and we have

$$Jr = \frac{1}{2}mr^2(\omega - 120).$$

The impulsive torque on B will reduce its angular speed from 240 to $\frac{1}{2}\omega$ r.p.m., the moment of inertia is $4mr^2$ and we have

$$2Jr = 4mr^2\left(240 - \frac{1}{2}\omega\right).$$

Eliminating J we find $\omega = 360$, $\frac{1}{2}\omega = 180$ r.p.m.

The original kinetic energy was

$$\frac{1}{2}\left\{\frac{1}{2}mr^2 \times 120^2 + 4mr^2 \times 240^2\right\} = 118{,}800mr^2.$$

The new kinetic energy is

$$\frac{1}{2}\left\{\frac{1}{2}mr^2 \times 360^2 + 4mr^2 \times 180^2\right\} = 97{,}200mr^2.$$

The proportion lost $= \dfrac{21{,}600}{118{,}800} = \dfrac{2}{11}$.

EXERCISES 4 (*c*)

1. A uniform rod of mass m and length l is free to turn about a horizontal axis through one end of the rod. When the rod is at rest in its position of stable equilibrium it receives a blow J perpendicular to the rod and to the axis of rotation at the lower end of the rod. If the rod subsequently turns through an angle of 60° before coming to rest, show that $J = m\sqrt{(gl/6)}$.

2. A uniform solid flywheel of mass 25 lb. and diameter 1 ft. is set in motion by a pull on a rope wrapped round the wheel. If the impulsive tension in the rope is 50 lb. sec., find the kinetic energy imparted to the flywheel.

3. A uniform solid flywheel of mass 20 lb. and diameter 1 ft. is rotating at 1200 r.p.m. A particle of mass 2 lb. which is at rest becomes attached to a point on the rim of the wheel. Find the loss of speed of the wheel.

4. A uniform rod of length $2a$ and mass m can turn freely about a horizontal axis through one end. The rod is released from a horizontal position, and when it is vertical it strikes a particle of mass m which becomes attached to the extremity of the rod. Find the angle through which the rod turns from the vertical before coming to instantaneous rest.

5. A uniform rod of length 2 ft. and mass 2 lb. is free to oscillate about a horizontal axis through one end. When the rod is hanging vertically it receives a blow in a direction perpendicular to the rod and to the axis of rotation at the mid-point of the rod. Subsequently the rod turns through a right-angle before coming to rest. Find the magnitude of the blow.

6. A flywheel whose moment of inertia is 2400 lb.ft.^{2} is rotating at 1000 r.p.m. when it is suddenly put into gear with a stationary wheel of equal radius whose moment of inertia is 750 lb.ft.^{2} Find the rate at which the two wheels rotate and the loss of kinetic energy.

7. A uniform steel plate of side 2 ft. and mass 80 lb. can turn freely about one side which is horizontal. When the plate is hanging vertically a bullet of mass 2 oz. moving horizontally strikes the plate at a distance of 15 in. from the axis of rotation and the plate turns through an angle of 30°. If the momentum of the bullet is destroyed by the impact, find the velocity with which it strikes the plate.

8. In an impact shear test a heavy pendulum carrying a hammer at its lower end is released from rest at an inclination of 60° to the

vertical. At the bottom of its swing the hammer meets the test-piece at a point 3 ft. below the pivot and, after shearing through it, rises to an inclination of 30° to the vertical. If the mass of the pendulum and hammer is 50 lb., the distance of its centre of gravity from the pivot 2·25 ft., and its moment of inertia about the pivot 400 lb.ft.^{2}, find (*a*) the energy dissipated during the impulse and (*b*) the total impulse of test-piece on hammer. (L.U., Pt. II)

9. Two coaxial flywheels, A and B, of moment of inertia 32 lb.ft.^{2} and 96 lb.ft.^{2} are rotating in opposite directions at 1000 r.p.m. and 500 r.p.m. respectively, when they are suddenly connected by a friction clutch which slips when the couple applied to it exceeds 50 lb.ft. Calculate the speed and direction of motion of the wheels when slipping ceases and find the time for which slipping persists. (Q.E.)

10. Two cog-wheels A and B are mounted on parallel shafts and are rotating in opposite directions at the rate of one and two revolutions per second respectively. The distance between the shafts is diminished so that the wheels engage. The wheels may be treated as uniform circular discs of the same material and thickness, the radius of B being three times the radius of A. Find the new rate of revolution of each wheel, and prove that 9/130 of the energy is lost. (L.U., Pt. I)

11. The moments of inertia of three gear-wheels A, B and C are in the ratios 4 : 2 : 1, and the radii of their pitch circles in the ratios 4 : 3 : 2. Initially B and C are in mesh and at rest, while A is rotating at n r.p.m. If A is suddenly brought into mesh with B show that its speed is reduced to $9n/26$, and find the ratio of the impulse between A and B to that between B and C. (The shafts on which the wheels are mounted remain fixed.) (L.U., Pt. I)

12. Two wheels A and B are mounted on parallel axes in a fixed frame. Their rims press against one another and the friction at the point of contact is such that when B is held fixed a couple L must be applied to A to cause slipping. The wheels are rotating without slip, A having an angular velocity ω_1, when a third wheel mounted on the same axle as B and at rest is suddenly connected to B. Find the angular velocity of A after slip has ceased, and show that this slip continues for a time

$$II_1\omega_1 r_1^2/\{L(I_1 r_2^2 + I_2 r_1^2 + Ir_1^2)\},$$

where I_1, I_2, r_1, r_2 are the moments of inertia and radii of A and B and I is the moment of inertia of the third wheel. (C.U.)

13. Two wheels A and B, rotating in fixed parallel bearings are of radii R_1 and R_2 respectively and their moments of inertia about their axes of rotation are I_1 and I_2. Initially the angular velocity of A is ω_1 and B is at rest. The rims of the wheels are then suddenly pressed together, the total reaction between them being P. Show that after the rims have acquired the same speed, the angular velocity of A has been reduced to

$$I_1R_2^2\omega_1/(I_1R_2^2 + I_2R_1^2).$$

If the coefficient of friction between the rims is μ, obtain an expression for the length of time during which slip occurs. (C.U.)

14. A beam AB of uniform cross-section is hinged at the end A to a fixed point and supported horizontally at the other by a vertical helical spring. The mass of the beam is M and its length l.

When the beam is at rest, a mass m is suddenly applied to it at a section distant a from A, the downward vertical velocity of m at the instant of application being u. If, in the subsequent motion, m moves with the beam and the beam is assumed to be rigid, show that its angular velocity just after the blow is $\dfrac{mua}{ma^2 + \dfrac{Ml^2}{3}}$.

If the spring is such that μ is the force required to give it unit elongation, show that d the maximum vertical displacement of B is the positive root of the equation

$$d^2 - \frac{2mag}{\mu l}d = \frac{1}{\mu} \times \frac{m^2u^2a^2}{ma^2 + \dfrac{Ml^2}{3}}. \qquad \text{(C.U.)}$$

4.9 Compound Pendulum

A rigid body oscillating freely under gravity about an axis is called a compound pendulum.

Let M be the mass of the body, Mk^2 its moment of inertia about the axis of rotation and h the distance of its centre of gravity G from the axis at O (Fig. 77).

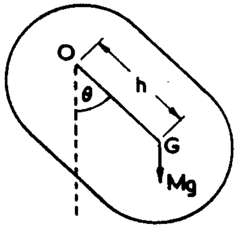

Fig. 77

When OG is inclined at an angle θ to the downward vertical, the moment of the weight of the body about the axis is $Mgh \sin \theta$ in the sense tending to diminish θ.

Therefore
$$Mk^2\ddot{\theta} = -Mgh \sin \theta,$$
$$\frac{k^2\ddot{\theta}}{h} = -g \sin \theta.$$

For a simple pendulum of length l we have the equation

$$l\ddot{\theta} = -g \sin \theta,$$

and for small oscillations about the equilibrium position, the motion is simple harmonic with period $2\pi \sqrt{(l/g)}$.

The equation of motion of the compound pendulum is, therefore, exactly the same as that of a simple pendulum of length k^2/h, and the quantity k^2/h is known as the length of the *equivalent simple pendulum*.

The period for small oscillations about the equilibrium position is $2\pi \sqrt{(k^2/gh)}$.

If the axis of rotation were inclined at an angle α to the horizontal, the moment of the weight about the axis would be $Mg \cos \alpha\, h \sin \theta$ and the length of the equivalent simple pendulum would be $k^2/(h \cos \alpha)$.

The length of the equivalent simple pendulum is the second moment of the body about the axis divided by the first moment about the axis and problems involving the compound pendulum are largely problems of finding second moments or moments of inertia.

Example 8. *A compound pendulum consists of a uniform rod of length* 6 *in. attached to the rim of a uniform disc of diameter* 3 *in., the rod lying in the plane of the disc and the centre of the disc in the line of the rod produced. If the mass of the disc is four times that of the rod and the pendulum oscillates in the plane of the disc about an axis through the other end of the rod, find the length of the equivalent simple pendulum.*

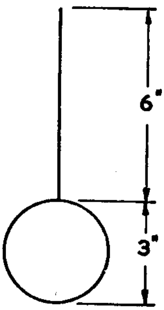

Fig. 78

If the mass of the rod be m lb., its moment of inertia about the axis of rotation (Fig. 78) at O

$$= \frac{4}{3}m \times 3^2 = 12m \text{ lb.in.}^2.$$

The moment of inertia of the disc about a parallel axis through its centre is $4m \times \frac{3^2}{8}$, and about the axis of rotation it is

$$4m\left(\frac{9}{8} + \left(\frac{15}{2}\right)^2\right) = \frac{459}{2}m \text{ lb.in.}^2.$$

The total moment of inertia $= \frac{483}{2}m$ lb.in.^{2}.

The first moment about the axis is

$$m \times 3 + 4m \times \frac{15}{2} = 33m \text{ lb.in.}$$

The length of the equivalent simple pendulum

$$= \frac{483m}{2 \times 33m},$$

$$= 7{\cdot}32 \text{ in.}$$

Example 9. *A uniform rod of mass* 3*m and length* 2 *ft. can turn freely in a vertical plane about a horizontal axis through one end. Find at what distance from the axis of rotation a particle of mass* 4*m must be attached to the rod so that the period of small oscillations of the combined mass shall be a minimum, and find this period.*

Let the particle be x ft. from the axis of rotation.
The total moment of inertia about the axis

$$= 3m \times \frac{4}{3} + 4mx^2$$

$$= 4m(x^2 + 1).$$

The first moment about the axis

$$= 3m \times 1 + 4mx$$
$$= m(4x + 3).$$

The length l of the equivalent simple pendulum is

$$l = \frac{4(x^2 + 1)}{4x + 3}$$

$$= x - \frac{3}{4} + \frac{25}{4(4x + 3)}.$$

This length has to be a minimum, and we have

$$\frac{dl}{dx} = 1 - \frac{25}{(4x + 3)^2},$$

$$\frac{d^2l}{dx^2} = \frac{200}{(4x + 3)^3}.$$

For a turning value $4x + 3 = \pm 5$, and $x = \frac{1}{2}$ gives l its minimum value of 1 ft.

The minimum period is

$$2\pi\sqrt{(1/g)} = 1{\cdot}11 \text{ sec.}$$

4.10 Centre of Oscillation

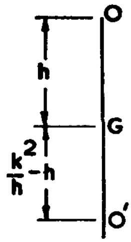

Fig. 79

Let GO (Fig. 79) be the perpendicular from the centre of gravity to the axis of rotation of a compound pendulum, so that $OG = h$, and the radius of gyration of the body about the axis through O is k.

Let O' be a point on OG produced, such that $OO' = k^2/h$, the length of the equivalent simple pendulum. Then the radius of gyration squared about an axis parallel to the axis of rotation through G is $k^2 - h^2$.

About a parallel axis through O' it is

$$\begin{aligned} &k^2 - h^2 + [(k^2/h) - h]^2, \\ &= [(k^2/h) - h][h + (k^2/h) - h], \\ &= (k^2/h)(k^2/h - h). \end{aligned}$$

But the length $O'G$ is $k^2/h - h$, therefore if the body should oscillate about the axis through O', the length of the equivalent simple pendulum is again k^2/h. The point O' is called the centre of oscillation of the pendulum.

This property of an alternative point of suspension with the same period is used in *Kater's pendulum* to determine the value of g. This pendulum has two knife-edges, whose distance apart is accurately known, and a movable mass is adjusted until the time of oscillation about the two knife-edges is the same. Then if l be their distance apart, the period is $2\pi(l/g)^{1/2}$, and by measuring the period g may be found. Thus the quantity to be measured in determining g is a time which may be found accurately from a number of oscillations.

4.11 Experimental Method of Finding Moments of Inertia

If a body can be made to oscillate about a knife-edge, its period as a compound pendulum can be determined with some accuracy and hence

the ratio k^2/h found. If h, the distance of the centre of gravity from the knife-edge, is known the value of k^2 is found and the square of the radius of gyration about a parallel axis through the centre of gravity is $k^2 - h^2$.

If the position of the centre of gravity is not accurately known, the moment of inertia may be determined by finding the periods of oscillation about two knife-edges such that the centre of gravity lies on the line connecting them.

Let k_1 be the radius of gyration about the centre of gravity and let the knife-edges be distant a and $l - a$ from the centre of gravity, l being known.

The periods of oscillation are

$$t_1 = 2\pi\sqrt{\left(\frac{k_1^2 + a^2}{ag}\right)}, \quad t_2 = 2\pi\sqrt{\left(\frac{k_1^2 + (l-a)^2}{(l-a)g}\right)}.$$

Hence the periods t_1 and t_2 having been found, k_1 and a can be determined.

4.12 Thrust on the Axis of Rotation

Since a particle of mass m, distant r from the axis of rotation, is moving in a circle of radius r with angular velocity $\dot{\theta}$, it has an acceleration $r\dot{\theta}^2$ towards the centre of the circle and an acceleration $r\ddot{\theta}$ in a perpendicular direction.

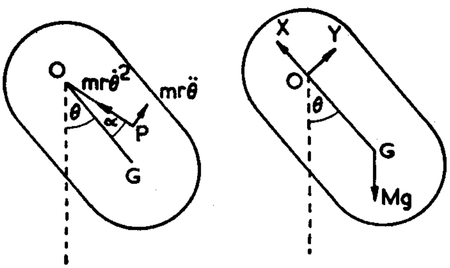

Fig. 80

Let the radius OP (Fig. 80) make an angle a with the line OG joining the centre of gravity to the axis of rotation. The effective force acting on the particle has components $mr\dot{\theta}^2$ along PO and $mr\ddot{\theta}$ perpendicular to OP.

Resolving these components along and perpendicular to GO we have

$$mr\dot{\theta}^2 \cos a + mr\ddot{\theta} \sin a, \text{ parallel to } GO,$$
$$mr\ddot{\theta} \cos a - mr\dot{\theta}^2 \sin a, \text{ perpendicular to } GO.$$

If the summation is taken over all the particles of the body, since these sums determine the position of the centre of gravity with respect to O,

$$\Sigma mr \cos a = Mh,$$
$$\Sigma mr \sin a = 0,$$

M being the total mass of the body and h the length OG.

Hence the total effective force acting on the body has components

$$Mh\dot{\theta}^2, \text{ along } GO,$$
$$Mh\ddot{\theta}, \text{ perpendicular to } GO.$$

Let X and Y be the components of the thrust of the axis of rotation on the body along and perpendicular to GO. If the only other force acting on the body is the weight acting at G we have

$$X - Mg\cos\theta = Mh\dot{\theta}^2$$
$$Y - Mg\sin\theta = Mh\ddot{\theta}.$$

The values of $\ddot{\theta}$ and $\dot{\theta}^2$ for any value of θ are given by the equations

$$\frac{k^2}{h}\ddot{\theta} = -g\sin\theta,$$

$$\frac{k^2}{h}\dot{\theta}^2 = 2g(\cos\theta - \cos\beta),$$

where β is the amplitude of swing.

Hence $$X = Mg\cos\theta + 2Mg\frac{h^2}{k^2}(\cos\theta - \cos\beta),$$

$$Y = Mg\left(1 - \frac{h^2}{k^2}\right)\sin\theta.$$

Example 10. *A rod of length 2a and mass m has a particle of mass m attached to it at one end and oscillates about a horizontal axis through the other end. If the rod oscillates through an angle of* 60° *each side of the vertical find the thrust on the axis (i) when the rod is vertical, (ii) when it is instantaneously at rest.*

The length of the equivalent simple pendulum (Fig. 81)

$$= \left(\frac{4}{3}ma^2 + 4ma^2\right) \div (ma + 2ma)$$

$$= \frac{16}{9}a.$$

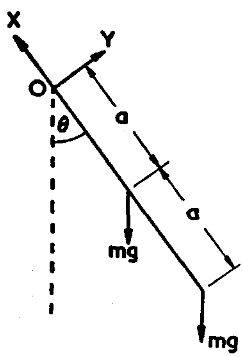

Fig. 81

When the rod is vertical $\theta = 0$, $\ddot{\theta} = 0$, and

$$\frac{16a}{9}\dot{\theta}^2 = 2g\left(1 - \frac{1}{2}\right)$$

$$\dot{\theta}^2 = \frac{9g}{16a}.$$

Hence $$Y = 0$$

$$X = 2mg + 2m \times \frac{3}{2}a \times \frac{9g}{16a}$$

$$= \frac{59mg}{16}.$$

When the rod is instantaneously at rest $\theta = 60°$, $\dot{\theta} = 0$, and

$$\frac{16a}{9}\ddot{\theta} = -g\frac{\sqrt{3}}{2}.$$

Hence $$X = mg,$$

$$Y = 2mg\frac{\sqrt{3}}{2} - 2m \times \frac{3}{2}a \times \left(\frac{9\sqrt{3}}{32a}\right)g$$

$$= \frac{5\sqrt{3}}{32}mg,$$

$$\sqrt{(X^2 + Y^2)} = \sqrt{(1099)}mg/32.$$

EXERCISES 4 (*d*)

1. A uniform rod of length 3 ft. and mass m is rigidly attached to a thin uniform disc of diameter 2 ft. and mass $4m$. The rod is in the plane of the disc and lies along a radius produced. The rod and the disc are suspended freely from the outer end of the rod. Determine the periodic time of small oscillations in the plane of the disc. (Q.E.)

2. A pendulum consists of a thin uniform rod 5 ft. long pivoted 1 ft. below its upper end. If it is released from a position 2° from vertical, calculate its inclination 20 sec. later, assuming successive swings all have the same amplitude. (Q.E.)

3. A symmetrical bicycle wheel has a moment of inertia of 7 lb.ft.^{2} about its axis. It is supported in frictionless horizontal bearings, and a concentrated mass of 2 lb. is rigidly attached to its rim, 15 in. from its axis.

   (*a*) If it is released from the position in which the mass is as high as possible and the equilibrium is just disturbed, find the greatest angular velocity the wheel will have during its subsequent motion.

   (*b*) What is the period of small oscillations of the wheel and weight about their position of stable equilibrium? (Q.E.)

4. A thin uniform rod of length $4a$ and mass m has a uniform disc of mass m and radius a attached to one end, the centre being in line with the rod. Find the radius of gyration about an axis perpendicular to the plane of the disc through the mass centre.

   If the body oscillates as a pendulum about an axis through one end of the rod perpendicular to the plane of the disc, find the period.

   Show that there is another parallel axis through a point of the rod about which the body would swing with the same period. (L.U., Pt. I)

5. A cylinder of radius r and height $4r$ is joined to another of radius $2r$ and height r end to end and with the axes in line. The body oscillates as a compound pendulum about a horizontal axis through a diameter of the outer end of the long cylinder. Find the period of oscillation. Prove that there is a parallel axis about which the body would oscillate with the same period and determine its position. (L.U., Pt. I)

6. A straight uniform thin rod of mass $3m$ and length $2a$ has a particle of mass m attached to one end. It swings under gravity in a vertical plane about a horizontal axis through a point on it. Find all the possible positions of the axis for which the length of the equivalent simple pendulum is $5a/3$. (L.U., Pt. I)

7. A compound pendulum is made by suspending from one end a straight uniform rod of length l and mass m. At a distance x from the point of suspension a particle of mass nm is fixed to the rod. As x varies from 0 to l, show that there is a value of x which renders the time of oscillation a minimum. Show that for a minimum time of oscillation $x = l/3$ nearly, when n is small and $l/\sqrt{3n}$ nearly when n is large. By ascriblng suitable values to n, determine the time of oscillation of a

uniform rod oscillating about one end and the time of oscillation of a simple pendulum of length x. (L.U., Pt. I)

8. A straight thin uniform rod of mass 20 lb. is pivoted at one end and swings in a vertical plane. It makes 30 complete oscillations per minute. A small mass of 5 lb. is then attached to the rod at a point 1 ft. from the point of suspension. Determine the number of oscillations made by the new arrangement in one minute. (L.U., Pt. I)

9. A uniform piece of wire of length $18a$ and mass m is bent to form a right-angle ABC in which $AB = 12a$, $BC = 6a$. It is smoothly hinged at A and oscillates in its own plane, which is vertical.

   Prove (i) that the moment of inertia about the axis of rotation is $84ma^2$, and (ii) that the length of the equivalent simple pendulum is $84a/\sqrt{65}$. (L.U., Pt. I)

10. A circular washer of uniform material has internal radius a and external radius b. Find the period of small oscillations, in the plane of the washer, about a knife-edge in contact with the inner surface. (L.U., Pt. I)

11. A uniform heavy rectangular plate 30 cm. by 40 cm., oscillates as a compound pendulum about a pivot through one corner and perpendicular to its plane. Find the period of the small oscillations.

    Find the locus of points in the lamina at which it may be pivoted in order that the period of its small oscillations in its own plane may be a minimum, and find this minimum period. (L.U., Pt. I)

12. Four uniform thin bars, each of mass m per unit length, are joined rigidly together to form a rectangle of sides $2a$, $2b$. This body performs under gravity, small oscillations in its plane about a horizontal axis through the mid-point of one of the bars of length $2a$. Find the period. (L.U., Pt. I)

13. A uniform disc of mass M, radius a and centre C oscillates under gravity about a fixed horizontal tangent through the point O on the circumference. If the angular velocity of the disc in its lowest position is ω, find the total pressure on the axis for a deflection θ from this position.

    Find the value of ω if, in a position of instantaneous rest, the resultant pressure is perpendicular to the line OC. (L.U., Pt. I)

14. A thin uniform rod of mass m and length $2a$ is pivoted freely about a horizontal axis through one end. It is slightly displaced from the position of unstable equilibrium and falls under gravity. Show that there is an oblique position for which the reaction at the pivot is entirely vertical and find the reaction in this position. (L.U., Pt. I)

4.13 Torsional Vibrations

When a body is suspended from a fixed point by a wire and is turned through an angle θ about a vertical axis through the point of suspension the wire exerts a restoring couple tending to bring the body back to its

original position. It is found that this restoring couple is proportional to θ and we may take it as

$$\frac{\lambda}{l}\theta,$$

where l is the length of the wire and the constant λ is independent of θ and, within certain limits, of the weight of the body.

The quantity λ is called the torsional rigidity or stiffness of the wire.

If Mk^2 be the moment of inertia of the body about the axis of rotation we have in gravitational units

$$\frac{Mk^2}{g}\ddot{\theta} = -\frac{\lambda}{l}\theta.$$

This is an equation of simple harmonic motion with period $2\pi(Mk^2l/\lambda g)^{1/2}$. Hence, if the body is turned through some angle α and then released it will oscillate about its equilibrium position with this period, and at time t, $\theta = \alpha \cos (\lambda g/Mk^2l)^{1/2}t$. The isochronism, that is, the equal period of swing, of a torsion pendulum may be used to find the moment of inertia of a body by measuring its period when suspended by a wire whose torsional rigidity is known.

4.14 Torsional Rigidity of Shafts

If a length of circular shaft is subjected to two equal and opposite torques at its ends one end will rotate slightly with respect to the other and the angle of twist is proportional to the torque. Thus for an angle of twist θ the torque is $\frac{\lambda\theta}{l}$. λ is called the torsional rigidity of the shaft.

If the shaft has diameter d, the second moment of area about the axis of the shaft is $\pi d^4/32$, and, denoting this quantity by J, the torsional rigidity is written as

$$\lambda = GJ,$$

and G is called the modulus of rigidity of the material.

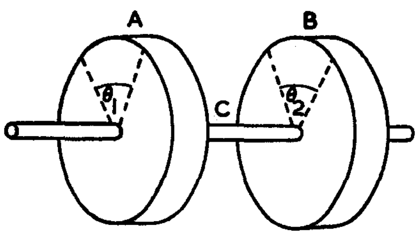

Fig. 82

4.15 Oscillation of a Shaft with Two Flywheels

Suppose that a light shaft turns without friction and carries two flywheels A and B of moments of inertia I_1 and I_2, the length of shaft between the wheels being l.

When the shaft is rotating let θ_1 and θ_2 be the angular displacements of A and B respectively from their mean position (Fig. 82). The twist of the shaft is $\theta_1 - \theta_2$ and if GJ be its torsional rigidity we have a torque $GJ(\theta_1 - \theta_2)/l$ in the shaft.

Therefore
$$I_1\ddot{\theta}_1 = -\frac{GJ}{l}(\theta_1 - \theta_2), \tag{1}$$
$$I_2\ddot{\theta}_2 = \frac{GJ}{l}(\theta_1 - \theta_2). \tag{2}$$

Hence
$$I_1\ddot{\theta}_1 + I_2\ddot{\theta}_2 = 0,$$
$$I_1\theta_1 + I_2\theta_2 = \text{constant}.$$

Now the twist in the shaft varies uniformly from θ_1 at A to θ_2 at B. Therefore at a point C distant x from A the twist is

$$\phi = \theta_1 - \frac{x}{l}(\theta_1 - \theta_2).$$

If
$$x = \frac{I_2 l}{I_1 + I_2},$$
$$\phi = \theta_1 - \frac{I_2}{I_1 + I_2}(\theta_1 - \theta_2)$$
$$= \frac{I_1\theta_1 + I_2\theta_2}{I_1 + I_2} = \text{constant}.$$

Therefore at the point where

$$x = \frac{I_2 l}{I_1 + I_2},\ l - x = \frac{I_1 l}{I_1 + I_2},$$

that is at a point which divides the shaft inversely in the ratio of the moments of inertia there is a constant angular displacement from the mean.

We have also from (1) and (2)

$$I_1 I_2(\ddot{\theta}_1 - \ddot{\theta}_2) = -\frac{GJ}{l}(I_1 + I_2)(\theta_1 - \theta_2).$$

Therefore the twist $\theta_1 - \theta_2$ varies harmonically with period

$$2\pi\sqrt{\left(\frac{I_1 I_2 l}{GJ(I_1 + I_2)}\right)}.$$

4.16 Suspension by Several Strings

If a body is suspended symmetrically by several strings and made to oscillate about a vertical axis the period of oscillation is also approximately constant.

Suppose that a uniform circular disc of radius a and mass m is suspended in a horizontal position by three vertical strings of length l, the strings being attached to points on the rim of the disc.

When the disc turns through an angle θ about a vertical axis through its centre, the lower end of each string is displaced horizontally through a distance $AB = 2a \sin \frac{1}{2}\theta$ (Fig. 83).

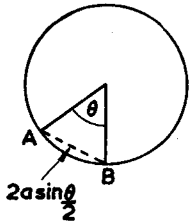

Fig. 83

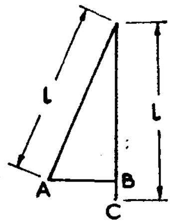

Fig. 84

Now consider the vertical plane through one of the strings in its equilibrium position and in its displaced position (Fig. 84). If T be the tension in the string in its displaced position the horizontal component of this force is $T\dfrac{2a \sin \frac{1}{2}\theta}{l}$.

The perpendicular distance from the centre of the disc on to the line AB is $a \cos \dfrac{1}{2}\theta$ and hence the turning moment of the tension is

$$T \times \frac{2a \sin \frac{1}{2}\theta}{l} \times a \cos \frac{1}{2}\theta = T\frac{a^2}{l} \sin \theta.$$

If we assume that when θ is small T has the same value as in its equilibrium position we have $T = \dfrac{1}{3}mg$, and the restoring couple due to one string is $\dfrac{1}{3}mg\dfrac{a^2}{l} \sin \theta = \dfrac{1}{3}mg\dfrac{a^2}{l}\theta$, approximately.

Thus for the three strings we have

$$m\frac{a^2}{2}\ddot{\theta} = -mg\frac{a^2}{l}\theta,$$

$$\frac{1}{2}l\ddot{\theta} = -g\theta,$$

and the period is $2\pi\sqrt{(l/2g)}$. It is evident that the numbers of strings by which the disc is suspended is immaterial.

The assumption that the tension in the strings is unchanged in the displaced position is equivalent to neglecting the rise and fall of the disc during its rotation.

In fact, if x be the vertical displacement of the disc due to a rotation θ

$$x = l - \left\{ l^2 - 4a^2 \sin^2 \frac{1}{2}\theta \right\}^{1/2}$$

$$= \frac{a^2}{2l}\theta^2, \text{ approximately.}$$

Thus if θ is small, the vertical displacement of the disc is of the second

order of small quantities and its vertical velocity and acceleration are of the same order and may be neglected.

EXERCISES 4 (*e*)

1. A uniform circular disc of radius 6 in. and mass 1 lb. is suspended in a horizontal position by a long vertical wire attached to its centre and the period of its torsional oscillations is 2 sec. If the disc is replaced by a body of mass 4 lb. with an axis of symmetry which is in the line of the wire and the period of oscillation is now 3 sec. find the radius of gyration of the body about its axis of symmetry.

2. The balance wheel of a chronometer has a moment of inertia I about its axis. What must be the torsional stiffness of the controlling spiral spring if the wheel is to perform one complete oscillation per second?

   If the wheel oscillates with a maximum angular deflection a degrees from its equilibrium position, what is the total energy of the motion? (L.U., Pt. I)

3. Two pulleys, of moments of inertia I_1 and I_2 are mounted at the end of a shaft of length a and torsional rigidity GJ which runs in tee-bearings with negligible friction. Show that the torsional oscillations have a node at a point in the shaft which divides its length inversely in the ratio of the moments of inertia, and find the frequency of the oscillations. (L.U., Pt. I)

4. A light shaft whose torsion modulus is k ft.lb. per radian is mounted in smooth bearings and carries a flywheel of moment of inertia I_1 lb.ft.^{2} at one end and one of moment of inertia I_2 lb.ft.^{2} at the other. An impulsive torque Q ft.lb.sec. is applied to the first wheel. Find the time-period T, of the oscillations in the subsequent motion, and show that the amplitude of the oscillations of the second wheel is
$$\frac{T}{2\pi}\cdot\frac{Qg}{I_1+I_2}. \qquad \text{(L.U., Pt. II)}$$

5. Two flywheels of moments of inertia I_1 and I_2 are mounted at the ends of a light shaft of length l and torsional rigidity GJ which runs in bearings with negligible friction. A variable torque of magnitude $M \sin \omega t$ is applied to the wheel whose moment is I_1. Show that in the state of steady motion the twist of the shaft is
$$\frac{M \sin \omega t}{I_1(\omega_1^2-\omega^2)}, \text{ where } \omega_1^2 = \frac{GJ}{l}\left(\frac{1}{I_1}+\frac{1}{I_2}\right).$$

6. A uniform circular disc of mass M and radius r hangs with its plane horizontal supported by three vertical strings, each of length l, attached to points on its rim. Show that its small oscillations about its axis will be approximately simple harmonic of time-period $2\pi(l/2g)^{1/2}$. (L.U., Pt. I)

7. A uniform rod of length $2a$ is suspended in a horizontal position by two vertical strings of length l attached to the ends of the rod. Find the period of the small oscillations of the rod about a vertical axis through its centre.

8. A uniform circular disc of radius a is suspended with its plane horizontal by a number of vertical strings, each of length l, attached to points which are the vertices of a regular polygon concentric with the circle. Show that the period of a small oscillation about a vertical axis is

$$2\pi(a^2l/2R^2g)^{1/2},$$

where R is the radius of the circle circumscribing the polygon.

9. A circular platform of mass 2 lb. and radius 1 ft. is suspended in a horizontal position by four vertical strings each of length 6 ft. attached to points on its rim. A shell of mass 20 lb. is placed on the platform with its axis of symmetry vertical and central and the period of the small oscillations of the platform and shell about the axis of symmetry is 2·01 sec. Find the radius of gyration of the shell about its axis.

10. A flywheel is suspended with its axis vertical by long ropes attached to its rim. It is found that a torque of 45 ft.lb. is required to hold the wheel when it has been turned about its axis through an angle of 5 degrees. When it is released it makes a complete oscillation in 4 seconds. Find its moment of inertia.

CHAPTER 5

PLANE MOTION OF A RIGID BODY

5.1 Coordinates and Degrees of Freedom

When a rigid body is moving in a plane, that is in such a way that the motion of any particle of the body is confined to a plane, the position of any particle of the body is known at a given time if the position of a plane section through the centre of gravity is known. Thus the motion of the body may be described as that of a lamina moving in its own plane.

The position of every point of the lamina is known if the position of one point and the direction of one line in the lamina are known. Thus the position of the body is determined by three coordinates, usually the x and y coordinates of the centre of gravity with reference to fixed rectangular axes in the plane of its motion and the angle θ which a line through the centre of gravity in the plane of motion makes with the x-axis.

These three coordinates need not be independent. For example, if a disc of radius r is rolling in a straight line on a horizontal plane (Fig. 85) its centre is always vertically above the point of contact. When the disc has rotated through an angle θ a length of arc $r\theta$ has been in contact with the ground and so the point of contact has moved a distance $r\theta$ along the plane. Hence the coordinates of the centre of the disc are

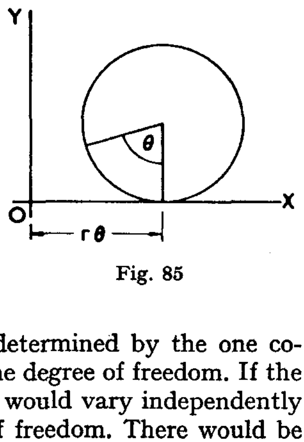

Fig. 85

$$x = r\theta, \quad y = r.$$

In this case the position of the disc is determined by the one coordinate θ and the motion is said to have one degree of freedom. If the disc were to slide instead of rolling x and θ would vary independently and the motion would have two degrees of freedom. There would be three degrees of freedom if x, y and θ should all vary independently.

If x, y and θ are known for a set of values of the time t, their derivatives $\dot{x}$, $\dot{y}$ and $\dot{\theta}$ may be found. Then $\dot{x}$ and $\dot{y}$ are the components of the velocity of the centre of gravity and $\dot{\theta}$ is the angular velocity of the lamina. The angular velocity $\dot{\theta}$ is the rate at which *any* line in the lamina is changing its direction, for if one line makes an angle θ with the x-axis and another line makes an angle $\theta + \alpha$ with the same axis, α is the angle between the lines and is constant.

Therefore the derivative of $\theta + \alpha$ is the same as the derivative of θ and is $\dot{\theta}$. It follows that if any two points of the lamina are at a distance l apart the relative velocity of either with respect to the other is $l\dot{\theta}$ in a direction perpendicular to the line joining the points.

Similarly, $\ddot{\theta}$ is the angular acceleration of the lamina as a whole and the relative acceleration of any two points l apart has components $l\dot{\theta}^2$ and $l\ddot{\theta}$ along and perpendicular to the line joining the points.

5.2 Motion of the Centre of Gravity

Let a particle of mass m of the lamina have coordinates (x_1, y_1) with respect to fixed axes OX, OY, and let the forces acting on this particle, which may be internal or external forces of the body, have components X_1 and Y_1 parallel to OX and OY respectively (Fig. 86).

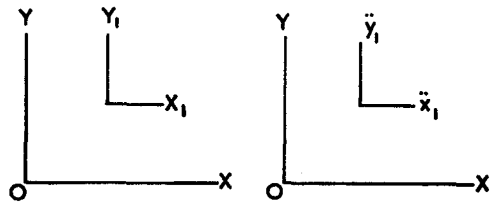

Fig. 86

Then
$$m\ddot{x}_1 = X_1,$$
$$m\ddot{y}_1 = Y_1.$$

Summing these equations of motion over all the particles of the body, we have
$$\Sigma m\ddot{x}_1 = \Sigma X_1 = X \text{ (say)},$$
$$\Sigma m\ddot{y}_1 = \Sigma Y_1 = Y \text{ (say)}.$$

Now if M be the total mass and (x, y) the coordinates of the centre of gravity
$$Mx = \Sigma m x_1,$$
$$My = \Sigma m y_1.$$

These equations hold for all values of t and therefore may be differentiated with respect to t, giving
$$M\ddot{x} = \Sigma m\ddot{x}_1,$$
$$M\ddot{y} = \Sigma m\ddot{y}_1.$$

Also, in the summation of the components of force acting on the particles the internal forces, which occur in pairs, vanish and X and Y are the sums of the components of the *external* forces acting on the body.

We have therefore
$$M\ddot{x} = X,$$
$$M\ddot{y} = Y.$$

These equations, which are fundamental in rigid dynamics, show that the centre of gravity moves as if the whole mass of the body were concentrated there and all the external forces acted upon it there in directions parallel to their actual directions. By virtue of this principle we have been able to discuss the motion of large bodies as if the bodies were particles when we were concerned only with their motion of translation.

5.3 Motion about the Centre of Gravity

If $G(x, y)$ be the centre of gravity and $P(x_1, y_1)$ be a particle of the lamina, let $x_1 - x = r\cos a$, $y_1 - y = r\sin a$. Then r and a are polar coordinates of P relative to the centre of gravity and r is constant. Also, $\dot{a}$ is the rate at which the line PG is changing direction and, therefore $\dot{a} = \dot{\theta}$, where $\dot{\theta}$ is the angular velocity of the lamina.

The relative velocity of P with respect to G is, therefore, $r\dot{\theta}$ in a direction perpendicular to PG.

The relative acceleration of P with respect to G has components $r\ddot{\theta}$ perpendicular to PG and $r\dot{\theta}^2$ along PG.

The components of the acceleration of the centre of gravity are $\ddot{x}$ and $\ddot{y}$ parallel to the axes of coordinates and hence the acceleration of P is the vector sum of the components $\ddot{x}$, $\ddot{y}$, $r\ddot{\theta}$ and $r\dot{\theta}^2$ in the appropriate directions (Fig. 87).

Let the force acting on the particle of mass m at P have components R along GP and S perpendicular to GP (Fig. 88).

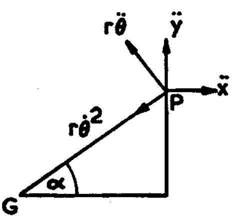

Fig. 87

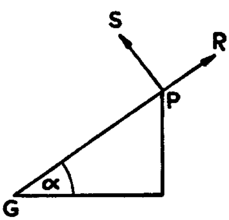

Fig. 88

Then
$$S = mr\ddot{\theta} + m\ddot{y}\cos a - m\ddot{x}\sin a,$$
$$Sr = mr^2\ddot{\theta} + \ddot{y}mr\cos a - \ddot{x}mr\sin a.$$

Adding for all the particles of the lamina we have
$$\Sigma Sr = \ddot{\theta}\Sigma mr^2 + \ddot{y}\Sigma mr\cos a - \ddot{x}\Sigma mr\sin a.$$

Now ΣSr is the sum of the moments of all the forces acting on the particles of the lamina about the centre of gravity, and in this sum the moments of the internal forces cancel. Therefore $G = \Sigma Sr$ is the sum of the moments of the external forces about the centre of gravity. Also $\Sigma mr^2 = Mk^2$ is the moment of inertia of the lamina about the centre of

gravity; $\Sigma mr \cos a = \Sigma mr \sin a = 0$, since these are the sums of moments of mass about the centre of gravity.

Thus we have the third equation of motion

$$G = Mk^2\ddot{\theta}. \tag{1}$$

The velocity of the particle at P is the vector sum of the components of velocity $\dot{x}$ and $\dot{y}$ of the centre of gravity and the relative velocity $r\dot{\theta}$ (Fig. 89). The component of velocity perpendicular to GP is $r\dot{\theta} + \dot{y}\cos a - \dot{x}\sin a$ and the corresponding momentum is $m(r\dot{\theta} + \dot{y}\cos a - \dot{x}\sin a)$. The moment of this momentum about the centre of gravity is $mr^2\dot{\theta} + \dot{y}mr\cos a - \dot{x}mr\sin a$, and the total moment of momentum of all the particles of the lamina is

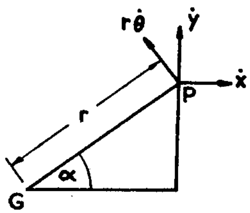

Fig. 89

$$\dot{\theta}\Sigma mr^2 + \dot{y}\Sigma mr\cos a - \dot{x}\Sigma mr\sin a.$$

Since $\Sigma mr^2 = Mk^2$ and $\Sigma mr\cos a = \Sigma mr\sin a = 0$ we see that the total moment of momentum about the centre of gravity is $Mk^2\dot{\theta}$. This moment of momentum is called the angular momentum of the body and equation (1) shows that, Mk^2 being constant, the rate of change of angular momentum about the centre of gravity is equal to the moment of the external forces about the centre of gravity.

If $G = 0$ it follows that $Mk^2\dot{\theta}$ is constant and we have *conservation of angular momentum.*

The equation (1) is similar to that obtained for the rotation of a body about a fixed axis and shows that the rotation about the centre of gravity is the same as it would be if the centre of gravity were a fixed point. It also shows the independence of the motions of translation and rotation.

5.4 Equations of Motion

We thus have three equations of motion for the plane motion of a rigid body:

$$X = M\ddot{x} \tag{1}$$
$$Y = M\ddot{y} \tag{2}$$
$$G = Mk^2\ddot{\theta} \tag{3}$$

Here, X and Y are the sums of the components of the external forces parallel to the axes of reference and G is the moment of the external forces about the centre of gravity.

M is the mass of the body, k the radius of gyration about a perpendicular to the plane of motion through the centre of gravity.

$\ddot{x}$ and $\ddot{y}$ are the components of the acceleration of the centre of gravity parallel to the axes of reference, and $\ddot{\theta}$ is the angular acceleration of the body.

5.5 Kinetic Energy

The coordinates of a typical particle of mass m with respect to the centre of gravity being $(r\cos\alpha, r\sin\alpha)$, the velocity of the particle is the vector sum of the velocity $(\dot{x}, \dot{y})$ of the centre of gravity and the relative velocity $r\dot{\theta}$ with respect to the centre of gravity (Fig. 89).

The velocity of the particle has therefore components

$$\dot{x} - r\dot{\theta}\sin\alpha, \quad \dot{y} + r\dot{\theta}\cos\alpha,$$

parallel to the axes of reference.

The kinetic energy of the lamina is

$$\begin{aligned} T &= \tfrac{1}{2}\Sigma m\{(\dot{x} - r\dot{\theta}\sin\alpha)^2 + (\dot{y} + r\dot{\theta}\cos\alpha)^2\} \\ &= \tfrac{1}{2}\Sigma m\{(\dot{x}^2 + \dot{y}^2) + r^2\dot{\theta}^2 - 2\dot{x}\dot{\theta}r\sin\alpha + 2\dot{y}\dot{\theta}r\cos\alpha\} \\ &= \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2)\Sigma m + \tfrac{1}{2}\dot{\theta}^2\Sigma mr^2 - \dot{x}\dot{\theta}\Sigma mr\sin\alpha + \dot{y}\dot{\theta}\Sigma mr\cos\alpha. \end{aligned}$$

Hence, since $\Sigma mr\cos\alpha = \Sigma mr\sin\alpha = 0$, we have

$$T = \tfrac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}Mk^2\dot{\theta}^2,$$

where M is the mass of the body and k its radius of gyration about an axis perpendicular to the plane of motion through the centre of gravity.

Thus the kinetic energy has two parts, the kinetic energy of translation $\tfrac{1}{2}M(\dot{x}^2 + \dot{y}^2)$ and the kinetic energy of rotation $\tfrac{1}{2}Mk^2\dot{\theta}^2$.

The energy of translation is what the kinetic energy would be if the whole mass were concentrated at the centre of gravity and moved with its velocity.

The energy of rotation is what the kinetic energy would be if the centre of gravity were fixed and the body allowed to rotate about it.

Since in any displacement the change in kinetic energy of any particle is equal to the work done in the displacement by the forces, both internal and external, which act on it, and the total work done by the internal forces is zero, the change in kinetic energy of the lamina is the total work done by the *external* forces which act on it in any displacement.

If the forces are conservative, so that the work done in the displacement depends only on the initial and final positions of the body, the work done is the loss of potential energy and the total of kinetic and potential energy is constant.

5.6 Use of the Equations of Motion

In problems involving the plane motion of a rigid body the quantities to be determined in the first instance are the displacements (x, y) of the centre of gravity with respect to some fixed axes of reference and the angle θ through which some line of the body has turned in the plane of motion.

A clear figure should be drawn showing these displacements from the initial position.

The number of degrees of freedom of the motion should then be investigated.

If there is only one degree of freedom there will be two geometrical relations between the coordinates x, y and θ, and these relations will be caused by constraints, that is, forces, in general unknown, which compel the body to move in this way. The three coordinates can then be expressed in terms of one of them and the three equations of motion will determine this coordinate and the two constraints.

If there are two degrees of freedom the equations of motion will determine the two independent coordinates and the one constraint; with three degrees of freedom the three coordinates are determined by the equation of motion. Instead of cartesian coordinates (x, y) it is often convenient to fix the position of the centre of gravity by other coordinates such as polars.

Thus if (r, ϕ) be the polar coordinates of the centre of gravity and R, S the components of the external forces along and perpendicular to the radius vector the equations of motion are

$$M(\ddot{r} - r\dot{\phi}^2) = R,$$

$$M\frac{1}{r}\frac{d}{dt}(r^2\dot{\phi}) = S,$$

$$Mk^2\ddot{\theta} = G.$$

The equations of motion determine accelerations and involve second differential coefficients of the coordinates, and integration is necessary to complete the solution of the problem.

It is convenient in dealing with problems to draw two figures, each showing the body in its displaced position. On one of these can be shown the forces acting on the body, on the other the mass $\times$ acceleration of the centre of gravity and the rate of change of angular momentum about it, that is the effective forces and couple.

The equations of motion are written down by equating the forces and couples in the two figures.

Example 1. *A solid of revolution of mass M whose bounding radius is a and whose radius of gyration about its axis of symmetry is k, rolls without slipping down the line of greatest slope of a plane inclined at an angle α to the horizontal. Find its acceleration down the plane and the least coefficient of friction between the body and the plane for rolling to be possible.*

Let x be the displacement down the plane of the centre of gravity and θ the angle through which the radius to the point of contact has turned at time t (Fig. 90). The y coordinate of the centre of gravity is constant.

We have the geometrical equations

$$x = a\theta, \quad y = a,$$

and the motion has one degree of freedom.

Let R be the force normal to the plane and F the frictional force along the plane, this friction being not necessarily limiting.

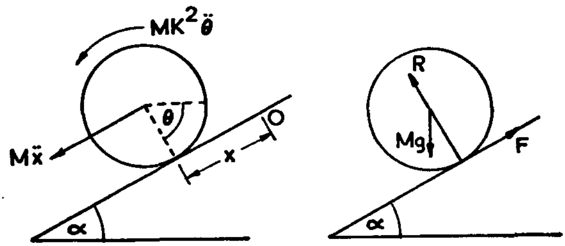

Fig. 90

The equations of motion are

$$Mg\sin\alpha - F = M\ddot{x} = Ma\ddot{\theta}, \quad (1)$$
$$R - Mg\cos\alpha = M\ddot{y} = 0, \quad (2)$$
$$Fa = Mk^2\ddot{\theta}. \quad (3)$$

Eliminating F between equations (1) and (3) we have

$$Mga\sin\alpha = M(k^2 + a^2)\ddot{\theta}$$

$$a\ddot{\theta} = \frac{a^2}{k^2 + a^2}.g\sin\alpha.$$

Hence we have the constant acceleration with which the centre of gravity moves down the plane and the constant acceleration with which the body rotates.

For a solid cylinder $\qquad k^2 = \frac{1}{2}a^2,\ a\ddot{\theta} = \frac{2}{3}g\sin\alpha.$

For a solid sphere $\qquad k^2 = \frac{2}{5}a^2,\ a\ddot{\theta} = \frac{5}{7}g\sin\alpha.$

For a spherical shell $\qquad k^2 = \frac{2}{3}a^2,\ a\ddot{\theta} = \frac{3}{5}g\sin\alpha.$

From equation (3) we have

$$F = M\frac{k^2}{a}\ddot{\theta}$$
$$= M\frac{k^2}{a^2 + k^2}g\sin\alpha.$$

Also, from (2) $\qquad R = Mg\cos\alpha,$

therefore $\qquad \dfrac{F}{R} = \dfrac{k^2}{a^2 + k^2}\tan\alpha.$

If slipping is not to occur F must be less than or equal to μR, that is we must have $\mu > \dfrac{k^2}{a^2 + k^2}\tan\alpha.$

Example 2. *A uniform rod of mass M and length 2a falls from rest in a vertical position with one end on a rough table. Find the angular velocity of the rod when it is inclined at an angle θ to the vertical and the horizontal and vertical components of the reaction of the table in this position.*

Since the rod turns about O (Fig. 91) it is clear that there is only one degree of freedom. The centre of gravity of the rod moves in a circle with angular velocity $\dot{\theta}$ and has components of acceleration $a\ddot{\theta}$ perpendicular to the rod

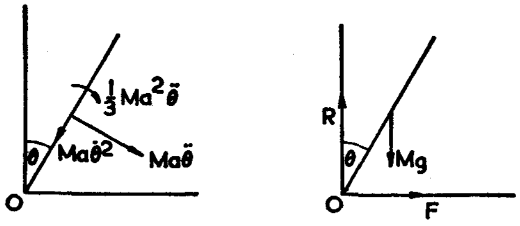

Fig. 91

and $a\dot{\theta}^2$ along the rod towards O. The components of the reaction being R and F we have the equations of motion

$$F = Ma\ddot{\theta}\cos\theta - Ma\dot{\theta}^2\sin\theta, \tag{1}$$

$$R - Mg = -Ma\ddot{\theta}\sin\theta - Ma\dot{\theta}^2\cos\theta, \tag{2}$$

$$Ra\sin\theta - Fa\cos\theta = \frac{Ma^2}{3}\ddot{\theta}. \tag{3}$$

Eliminating R and F between these equations we have

$$\frac{4}{3}Ma^2\ddot{\theta} = Mga\sin\theta. \tag{4}$$

Integrating (4) with respect to θ, remembering that $\frac{d}{d\theta}(\dot{\theta}^2) = 2\ddot{\theta}$, we have

$$\frac{2}{3}Ma^2\dot{\theta}^2 = -Mga\cos\theta + \text{constant},$$
$$= Mga(1 - \cos\theta),$$

since $\dot{\theta}$ is zero when θ is zero.

Therefore
$$\ddot{\theta} = \frac{3g}{4a}\sin\theta,$$

$$\dot{\theta}^2 = \frac{3g}{2a}(1 - \cos\theta).$$

Substituting in the equations of motion we find

$$F = \frac{3}{4}Mg\sin\theta(3\cos\theta - 2),$$

$$R = \frac{1}{4}Mg(1 - 3\cos\theta)^2.$$

Hence, R becomes zero when $3\cos\theta = 1$ and unless the end is fixed slipping will take place before θ reaches the value $\cos^{-1}\left(\frac{1}{3}\right)$.

Example 8. *A wheel of mass m and radius a is eccentrically loaded so that its centre of mass G is at a distance b from its axis C. The wheel is constrained to roll with uniform velocity V on a rough horizontal track. Prove that the horizontal force P, which must be applied at C to maintain this motion at the instant when OG has turned through an angle θ from the downward vertical, is given by*

$$P = \frac{mb}{a^2}(V^2 + ag)\sin\theta. \qquad \text{(L.U., Pt. II)}$$

The forces acting on the wheel are the weight mg, the horizontal force P, the reaction R and the frictional force F (Fig. 92).

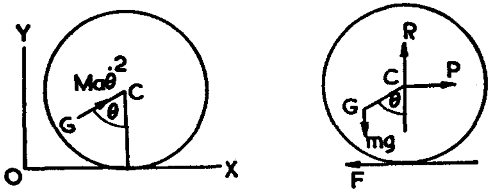

Fig. 92

The horizontal and vertical displacements of G being measured from the point of contact when GC is vertical, we have

$$x = a\theta - b\sin\theta,\ y = a - b\cos\theta.$$

The angular velocity being $\dot{\theta}$ and the horizontal velocity of C being V, a constant, we have

$$a\dot{\theta} = V,$$

$$a\ddot{\theta} = 0.$$

Therefore

$$\dot{x} = a\dot{\theta} - b\dot{\theta}\cos\theta$$

$$\ddot{x} = a\ddot{\theta} - b\ddot{\theta}\cos\theta + b\dot{\theta}^2\sin\theta$$

$$= \frac{bV^2}{a^2}\sin\theta.$$

$$\dot{y} = b\dot{\theta}\sin\theta$$

$$\ddot{y} = b\ddot{\theta}\sin\theta + b\dot{\theta}^2\cos\theta$$

$$= \frac{bV^2}{a^2}\cos\theta.$$

The equations of motion are

$$P - F = m\frac{bV^2}{a^2}\sin\theta, \quad (1$$

$$R - mg = m\frac{bV^2}{a^2}\cos\theta, \quad (2)$$

$$F(a - b\cos\theta) + Pb\cos\theta - Rb\sin\theta = mk^2\ddot{\theta} = 0. \quad (3)$$

Eliminating R and F between these equations we have

$$\left(P - \frac{mbV^2}{a^2}\sin\theta\right)(a - b\cos\theta) + Pb\cos\theta - \left(mg + \frac{mbV^2}{a^2}\cos\theta\right)b\sin\theta = 0,$$

$$P = \frac{mb}{a^2}(V^2 + ag)\sin\theta.$$

Example 4. *A solid cylinder (Fig. 93) rests on a rough horizontal plane, the coefficient of friction being μ. The cylinder is set in motion by a pull applied to the free end of a rope which passes round part of the circumference of the cylinder in the plane of symmetry and is attached to a point on it. If the direction of the pull is at an angle α above the horizontal show that the greatest acceleration that can be given to the cylinder without slipping taking place is $\mu(1 + \cos\alpha)g/(1 - \frac{1}{2}\cos\alpha + \frac{3}{2}\mu\sin\alpha)$.*

(C.U.)

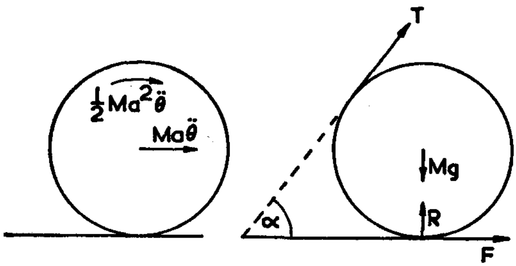

Fig. 93

If the cylinder does not slip, the horizontal acceleration is $a\ddot{\theta}$, where $\ddot{\theta}$ is the angular acceleration.

The forces being as shown in Fig. 93, the equations of motion are

$$T\cos\alpha + F = Ma\ddot{\theta}, \tag{1}$$

$$T\sin\alpha + R - Mg = 0, \tag{2}$$

$$Ta - Fa = M\frac{a^2}{2}\ddot{\theta}. \tag{3}$$

From (1) and (3)
$$F(1+\cos\alpha) = (1 - \frac{1}{2}\cos\alpha)Ma\ddot{\theta}, \tag{4}$$

$$T(1+\cos\alpha) = \frac{3}{2}Ma\ddot{\theta}. \tag{5}$$

These equations show that F must act in the direction shown in the figure. From (2) and (5)

$$R(1+\cos\alpha) = Mg(1+\cos\alpha) - \frac{3}{2}\sin\alpha . Ma\ddot{\theta}. \tag{6}$$

From (4) and (6)

$$\frac{F}{R} = \frac{\left(1 - \frac{1}{2}\cos\alpha\right)a\ddot{\theta}}{g(1+\cos\alpha) - \frac{3}{2}\sin\alpha . a\ddot{\theta}}.$$

$$a\ddot{\theta} = g(1+\cos\alpha)\left\{\frac{F/R}{\left(1-\frac{1}{2}\cos\alpha\right) + \frac{3}{2}\sin\alpha\,(F/R)}\right\}$$

$$= \frac{g(1+\cos\alpha)}{\frac{3}{2}\sin\alpha}\left\{1 - \frac{1-\frac{1}{2}\cos\alpha}{1-\frac{1}{2}\cos\alpha + \frac{3}{2}\sin\alpha\left(\frac{F}{R}\right)}\right\}.$$

Therefore $a\ddot{\theta}$ is greatest when F/R has its greatest possible value μ, and this gives the required value of $a\ddot{\theta}$.

Example 5. *The door of a railway carriage is open and perpendicular to the carriage. The train starts with an acceleration f causing the door to close. Taking the door as uniform and of width $2a$ find the velocity of the outer edge of the door as it closes.*

When the door has turned through an angle θ, let R and S be the components of the force exerted on the door at its hinges (Fig. 94). These are the only forces affecting the motion. The acceleration of the hinges is f. The acceleration of the centre of gravity is the vector sum of the acceleration of the hinges and of its acceleration relative to the hinges. This relative acceleration has components $a\ddot{\theta}$ and $a\dot{\theta}^2$ as shown in the figure. The equations of motion are

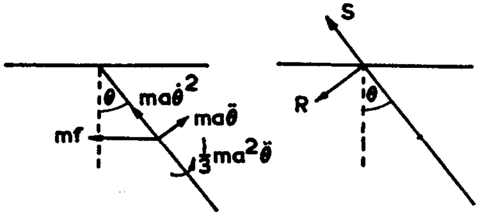

Fig. 94

$$R = m(f\cos\theta - a\ddot{\theta}), \tag{1}$$
$$S = m(f\sin\theta + a\dot{\theta}^2), \tag{2}$$
$$Ra = m\frac{a^2}{3}\ddot{\theta}. \tag{3}$$

From (1) and (3) $\quad \frac{4}{3}a\ddot{\theta} = f\cos\theta.$

Integrating with respect to θ,

$$\frac{2}{3}a\dot{\theta}^2 = f\sin\theta + \text{constant}.$$

Since $\dot{\theta} = 0$ when $\theta = 0$ the constant is zero and we have

$$a\dot{\theta}^2 = \frac{3}{2}f\sin\theta.$$

When $\theta = \frac{\pi}{2}$, $\qquad \dot{\theta} = \left(\frac{3f}{2a}\right)^{1/2}.$

The velocity of the outer edge is then $2a\dot{\theta} = (6af)^{1/2}$.

Example 6. *A uniform plank of mass M is placed symmetrically upon two equal rough uniform cylindrical rollers, each of mass m, on a rough plane inclined at an angle α to the horizontal. Assuming no slipping and that the cylinders do not touch, find the initial values of the frictional forces on each cylinder and the acceleration of the plank.* (L.U., Pt. II)

Let $a\ddot{\theta}$ be the acceleration of the centre of gravity of each cylinder down the plane. $\ddot{\theta}$ is the angular acceleration, a point of contact with the plank has acceleration $2a\ddot{\theta}$ and this is the acceleration of the plank down the plane. Let the frictional forces be F_1, F_2, S_1, S_2, as shown in Fig. 95.

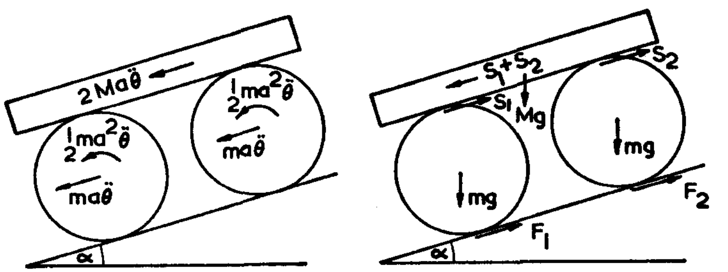

Fig. 95

Two equations of motion for each cylinder and one for the plank are

$$-S_1 - F_1 + mg\sin\alpha = ma\ddot{\theta}, \tag{1}$$

$$-aS_1 + aF_1 = m\frac{a^2}{2}\ddot{\theta}, \tag{2}$$

$$-S_2 - F_2 + mg\sin\alpha = ma\ddot{\theta}, \tag{3}$$

$$-aS_2 + aF_2 = m\frac{a^2}{2}\ddot{\theta}, \tag{4}$$

$$Mg\sin\alpha + S_1 + S_2 = M2a\ddot{\theta}. \tag{5}$$

From equations (1) to (4) $\quad S_1 = S_2,$

$$F_1 = F_2,$$

and we have $\quad S_1 = S_2 = -\frac{3}{4}ma\ddot{\theta} + \frac{1}{2}mg\sin\alpha.$

Hence, from (5)

$$Mg\sin\alpha + mg\sin\alpha = \left(2M + \frac{3}{2}m\right)a\ddot{\theta},$$

$$a\ddot{\theta} = \frac{2(M+m)g\sin\alpha}{4M+3m},$$

$$S_1 = \frac{\frac{1}{2}Mmg\sin\alpha}{4M+3m},$$

$$F_1 = \frac{\frac{1}{2}(3M+2m)mg\sin\alpha}{4M+3m}.$$

EXERCISES 5 (*a*)

1. A uniform solid sphere rolls down a rough plane inclined at 30° to the horizontal. What is the least value of the coefficient of friction that will prevent slipping? If there is no slipping find how long the sphere will take to roll 10 ft. down the plane from rest.

2. A uniform thin hollow sphere and a uniform hollow cylinder whose external radius is double its internal radius are placed side by side at the top of a plane 20 ft. long inclined at 30° to the horizontal and released. Find which will reach the bottom first and the difference in their times for the distance.

3. A light inextensible string is wound round a uniform circular cylinder of mass M and one end of the string is fixed to a point of it. The other end is held and the cylinder falls with its axis horizontal and the string vertical. Find the acceleration of the axis of the cylinder and the tension in the string.

4. A uniform circular disc 10 in. in diameter is supported on a spindle $\frac{1}{4}$ in. in diameter and of negligible weight which rolls on rails sloping at an angle $\sin^{-1} 1/30$. Find the time taken to roll 4 ft. from rest and the velocity of the centre of gravity at the end of that time.

5. Two uniform circular cylinders each of radius a and mass m but with radii of gyration k_1 and k_2 have their axes joined by two light inextensible strings so that they remain parallel and are placed on a rough plane inclined at an angle α to the horizontal with the strings

taut. If the strings remain taut throughout the motion find the acceleration down the plane and the tension in the strings.

6. A flywheel with an axle of radius r, total weight W and radius of gyration k, rolls with its axle on two parallel horizontal rails, rough enough to prevent slip. Motion is caused by a weight w attached to a light cord wrapped round the axle, with w hanging freely. Show that motion is possible in which the hanging portion of the cord makes a steady angle α with the vertical, given by,

$$\sin\alpha/(1 - \sin\alpha)^2 = wr^2/W(r^2 + k^2),$$

and that the acceleration of the axis of the flywheel is then $g \tan\alpha$.
(L.U., Pt. II)

7. A uniform solid cylinder of mass 20 lb. rolls down a plane inclined at 30° to the horizontal. A light inelastic string attached centrally to the axis of the cylinder is parallel to a line of greatest slope of the plane and passes over a smooth pulley at the top of the plane. It then hangs vertically carrying a weight of 5 lb. If the plane is rough enough to prevent slipping, find the time taken by the cylinder to roll 20 ft. down the plane from rest.

8. A horizontal force P is applied to the rim of a disc of mass M whose plane is vertical so that it rolls on a rough horizontal plane. Show that the disc will not slip if $P < 3\mu Mg$, where μ is the coefficient of friction, and find the acceleration of the centre of the disc.

9. A uniform circular cylinder of mass M and radius a is acted on by a force P through its centre, the direction of the force being at an angle α above the horizontal. The cylinder rests on a horizontal plane, the coefficient of friction being μ. Show that slipping will not occur if $P(\cos\alpha + 3\mu\sin\alpha) < 3Mg\mu$, and find the acceleration of the centre of gravity if there is no slipping.

10. A uniform rod AB of mass m and length $2a$ lies at rest on a smooth horizontal plane when a constant horizontal force P is applied to B in a direction making an angle $\alpha(< \pi/2)$ with BA. Prove that when the rod makes an angle θ with the direction of P

$$ma\dot{\theta}^2 = 6P(\cos\alpha - \cos\theta).$$

Prove also that the rod oscillates with amplitude $2(\pi - \alpha)$.
(L.U., Pt. II)

11. A uniform rod stands vertically on a rough horizontal plane and is allowed to fall. Assuming that slipping has not occurred, find the horizontal and vertical components of the reaction on the rod when the rod makes an angle θ with the vertical.

If slipping occurs when the rod is inclined at 30° to the vertical, find the coefficient of friction.

12. A uniform rod of length $2a$ swings freely under gravity about one end which is fixed at a height $h(> 2a)$ above the floor. Its greatest inclination to the vertical position is $\pi/3$. When the rod is passing through the vertical position the upper end is set free and the rod strikes the floor when it is next vertical. Prove that $h = \frac{2}{3}a(\pi^2 + 3)$.
(L.U., Pt. II)

13. Two uniform rods OA, AB, each of mass m and length $2l$, are smoothly jointed at A, and the end O is smoothly jointed to a fixed point on a horizontal table. Initially the rods are at rest and in line on the table. The rod AB is then given an angular velocity $(3g/l)^{1/2}$ about A in a vertical plane. Prove that, so long as the rod OA remains on the table, the angular velocity of the rod AB is $(2 - \sin\theta)^{1/2}(3g/2l)^{1/2}$ when it has turned through an angle θ. Find the vertical component of the reaction of OA on AB and hence find the value of θ at which the rod OA will begin to rise. (L.U., Pt. II)

14. A uniform sphere of radius b is placed on the highest point of a fixed sphere of radius a and rolls down it, the friction between the spheres being sufficient to prevent slipping. Show that as the centre of the sphere moves in a circle with angular velocity $\dot{\theta}$, the sphere turns about its centre with angular velocity $(a + b)\dot{\theta}/b$. Find the inclination to the vertical of the radius to the point of contact when the spheres separate and the velocity of the centre of moving sphere at this instant.

15. A car of weight 1 ton has two pairs of wheels; each pair of wheels and axle weighs 70 lb., the radius of gyration is 6 in. and rolling radius 9 in. The car runs freely down a slope of 30°. Show that the acceleration of the car is $18g/37$.

16. A four-wheeled railway truck of total mass M moving along a level track has an acceleration a and r is the radius and mk^2 the moment of inertia of each pair of wheels and axle. Show that the difference in the pull on the couplings at the ends of the truck is $(M + 2mk^2/r^2)a$, and find the horizontal force exerted by each axle on the truck.

17. The thrust of the brake blocks on the rim of each of the four wheels of a railway truck is R lb.wt. and μ is the coefficient of friction between the blocks and the wheels. If the truck is in motion when the brakes are applied and the force R is insufficient to lock the wheels find the retardation of the truck, M being its mass, r the radius and mk^2 the moment of inertia of each pair of wheels and axle.

18. The door of a stationary railway carriage stands open and perpendicular to the length of the train. The train starts off with acceleration f and at the same time the door is given an angular velocity Ω in the direction towards the front of the train, so as to shut the door. Show that if the door can be regarded as a smoothly hinged uniform rectangular plate of width $2a$, then Ω must be at least of magnitude $(3f/2a)^{1/2}$ in order to close the door. (L.U., Pt. II)

5.7 Use of the Energy Equation

The solutions of many of the examples of the previous section can be simplified by using the energy equation, that is, by equating the change in kinetic energy of the body to the work done by the external forces or to the loss of potential energy when the forces are conservative.

The kinetic energy involves first derivatives only of the coordinates and is a first integral of the equations of motion, that is, it can be ob-

tained by integrating some combination of the equations of motion. It does not, in general, contain the constraining forces which limit the degrees of freedom of the motion.

In the energy equation, the following types of force make no contribution to the work done:

(1) The tension of an inextensible string with one end fixed. The displacement of a particle attached to the other end must be perpendicular to the string and therefore no work is done by the tension.

(2) Internal forces of a rigid body.

These occur in pairs and the work of the forces of each pair will be equal and opposite. If a system consists of more than one body the reaction between any two bodies may be considered as an internal force when the energy of the system as a whole is considered.

(3) The normal reaction of a smooth surface with which the body is in contact.

The point of contact can only move in a direction tangential to the surface and no work is done by the force in such a displacement.

(4) The reaction at the point of contact of a rolling body.

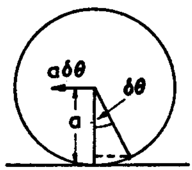

Fig. 96

If the body turns through a small angle $\delta\theta$ (Fig. 96) the centre moves forward $a\delta\theta$, where a is the radius to the point of contact. The relative displacement of the point of contact is $a \sin \delta\theta$ backwards, and $a - a \cos \delta\theta$ upwards. Its absolute displacement is, therefore, $a(\delta\theta - \sin \delta\theta)$ forwards and $a(1 - \cos \delta\theta)$ upwards. These displacements are second-order quantities, so the work done by the reaction is of the same order. If the rotation is in time δt the rate at which work is being done is a first-order quantity and is zero in the limit.

Example 7. *A solid of revolution of mass M, bounding radius a and radius of gyration k about its axis rolls without slipping down a plane inclined at an angle α to the horizontal. Find its angular velocity when it has rolled through an angle θ from rest.*

If θ be the angular displacement, the linear displacement is $a\theta$; the angular velocity is $\dot{\theta}$ and the linear velocity $a\dot{\theta}$.

When it has turned through an angle θ the kinetic energy of translation is $\frac{1}{2}M(a^2\dot{\theta}^2)$. The kinetic energy of rotation is $\frac{1}{2}Mk^2\dot{\theta}^2$. The work done by gravity is $Mg \sin \alpha \times a\theta$.

The energy equation is

$$\frac{1}{2}M(a^2 + k^2)\dot{\theta}^2 = Mg \sin \alpha \times a\theta,$$

$$\dot{\theta}^2 = \frac{2ag \sin \alpha}{a^2 + k^2}.\theta.$$

If this equation is differentiated with respect to θ we have

$$a\ddot{\theta} = \frac{a^2 g \sin \alpha}{a^2 + k^2}$$

which is the result obtained in Example 1.

Example 8. *A uniform rod of mass M and length 2a is held in a vertical position with one end in contact with a smooth horizontal surface and falls over when released. Find its velocity when it has turned through an angle θ from the vertical and the reaction of the ground in this position.*

The forces acting on the rod are its weight Mg and the reaction of the ground R (Fig. 97). There is no horizontal force, therefore the centre of gravity can have no horizontal acceleration. Since it is initially at rest it can move only in a vertical line.

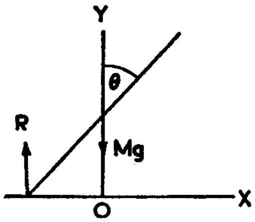

Fig. 97

Let y be the height of the centre of gravity above the ground. Then $y = a \cos \theta$.
The velocity of the centre of gravity is therefore $\dot{y}$ and

$$\dot{y} = -a \sin \theta . \dot{\theta}.$$

The kinetic energy is

$$\frac{1}{2}M(-a \sin \theta . \dot{\theta})^2 + \frac{1}{2}M\frac{a^2}{3}\dot{\theta}^2.$$

The work done is $\qquad Mg(a - a \cos \theta).$

The energy equation is

$$\frac{1}{2}M\left(\frac{a^2}{3} + a^2 \sin^2 \theta\right)\dot{\theta}^2 = Mga(1 - \cos \theta),$$

and hence the angular velocity $\dot{\theta}$ is found. The linear velocity of the centre of gravity is then $a \sin \theta . \dot{\theta}$ downwards.
To find R we may use the equation of motion for rotation about the centre of gravity

$$Ra \sin \theta = M\frac{a^2}{3}\ddot{\theta}.$$

Now
$$\dot{\theta}^2 = \frac{6g(1 - \cos \theta)}{a(1 + 3 \sin^2 \theta)}$$

$$\ddot{\theta} = \frac{1}{2}\frac{d}{d\theta}\dot{\theta}^2 = \frac{3g \sin \theta(4 - 6 \cos \theta + 3 \cos^2 \theta)}{a(1 + 3 \sin^2 \theta)^2},$$

$$R = Mg\frac{4 - 6 \cos \theta + 3 \cos^2 \theta}{(1 + 3 \sin^2 \theta)^2}.$$

Example 9. *A car of total mass M has two pairs of wheels of radius r and the moment of inertia of each pair of wheels and axle is mk^2. The car runs freely down a slope of length l inclined at an angle α to the horizontal. If u be the speed of the car at the top of the slope, find its speed at the bottom.*

Let $v = r\dot{\theta}$ be the speed at the bottom. The energy equation is

$$\frac{1}{2}Mv^2 + \frac{1}{2} \times 2mk^2\frac{v^2}{r^2} - \frac{1}{2}Mu^2 - \frac{1}{2} \times 2mk^2\frac{u^2}{r^2}$$
$$= Mgl \sin \alpha.$$

Therefore
$$v = \left\{u^2 + \frac{2Mglr^2 \sin \alpha}{Mr^2 + 2mk^2}\right\}^{1/2}.$$

Example 10. *A uniform solid sphere of mass M and radius a rests on a rough horizontal plane. A particle of mass m is fixed to the highest point of the sphere and given a small displacement. Find the angular velocity of the sphere when the radius to the particle makes an angle θ with the vertical.*

If $\dot\theta$ be the angular velocity, the linear velocity of the sphere is $a\dot\theta$ and its kinetic energy is

$$\frac{1}{2}Ma^2\dot\theta^2 + \frac{1}{2}\times\frac{2}{5}Ma^2\dot\theta^2.$$

The particle has velocity $a\dot\theta$ with respect to the centre of the sphere in a direction inclined at θ to the horizontal (Fig. 98). Its velocity is therefore the vector sum of this velocity and the velocity $a\dot\theta$ of the centre of the sphere, that is

$$\{a^2\dot\theta^2 + a^2\dot\theta^2 + 2a^2\dot\theta^2\cos\theta\}^{1/2}$$
$$= a\dot\theta(2 + 2\cos\theta)^{1/2}.$$

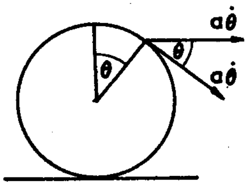

Fig. 98

The work done is $mg(a - a\cos\theta)$ and the energy equation is

$$\frac{7}{10}Ma^2\dot\theta^2 + \frac{1}{2}ma^2\dot\theta^2(2 + 2\cos\theta) = mga(1 - \cos\theta),$$

and
$$\dot\theta^2 = \frac{10mg(1-\cos\theta)}{a\{7M + 10m(1+\cos\theta)\}}.$$

EXERCISES 5 (*b*)

1. A uniform rod of length $2a$ is placed with one end against a smooth vertical wall and the other on smooth horizontal ground, the rod being in a vertical plane perpendicular to the wall and inclined at an angle α to the vertical. If the rod is then released, find its angular velocity when it is inclined at an angle β to the vertical.

2. Two similar trucks have total loaded weights M_1 and M_2 $(M_1 > M_2)$ and each has two pairs of wheels of radius r, the moment of inertia of each pair of wheels and axle being mk^2. The trucks are coupled together and run freely down a track inclined at an angle α to the horizontal with the heavier truck leading. Show that the pull on the coupling is
$$\frac{(M_1 - M_2)g\sin\alpha}{2 + r^2(M_1 + M_2)/(2mk^2)}.$$

3. A uniform hemisphere of mass M and radius a is placed on a rough horizontal plane with its plane face vertical and allowed to fall. Find its angular velocity when its plane face becomes horizontal and the vertical reaction of the plane when it is in this position.

4. A uniform rod of length $2a$ is suspended in a horizontal position by two vertical light inextensible strings each being of length l and attached to an end of the rod and to a fixed point. If the rod is given an angular velocity ω about a vertical axis through its centre, find the height to which it rises above its initial position before coming instantaneously to rest.

5. A uniform rod of length $2a$ has a light ring attached to one end and the ring can slide without friction on a horizontal wire. If the rod is held inclined at an angle α to the vertical in the vertical plane through the wire with its centre below the wire and released show that its angular velocity when the rod makes an angle θ with the vertical is

$$\{6g(\cos\theta - \cos\alpha)/a(1 + 3\sin^2\theta)\}^{1/2}.$$

Find the reaction of the wire when the rod is vertical.

6. A locomotive with six wheels, each of diameter 5 ft. and weighing 2000 lb., altogether weighs 55 tons. Find its total kinetic energy when moving at 30 m.p.h., taking into account the rotational energy of the wheels, each of which may be considered as a solid disc of uniform thickness. In calculating the distance in which a braking force of 8 tons would stop the locomotive, find by what percentage the distance would be underestimated if the rotational energy were not taken into account.

7. A uniform rod of mass m, length $2a$, has particles of the same mass m attached to its ends. It falls from the vertical with its lower end on a smooth horizontal plane. Find the velocity of the lower end when the rod makes an angle θ with the vertical. Show that when the rod becomes horizontal its angular acceleration is $9g/16a$.

(L.U., Pt. II)

8. A uniform circular disc of mass $2m$ and radius a has a particle of mass m fixed to its circumference. The disc is projected, with its plane vertical and the particle initially in its highest position, so as to roll without slipping on a horizontal rail. Prove that, when the radius to the particle makes an angle θ with the upward vertical,

$$a\dot{\theta}^2 = \{7a\Omega^2 + 2g(1 - \cos\theta)\}/(5 + 2\cos\theta)$$

where Ω is the angular speed of projection.

Hence or otherwise find the vertical reaction on the rail when the disc has turned through one right-angle. (L.U., Pt. II)

9 A thin hollow cylinder of radius a has a particle of equal mass attached symmetrically to its inner surface. If the system is disturbed from its position of stable equilibrium on a rough horizontal table and then left to itself show that, when the radius to the particle makes an angle θ with the downward vertical,

$$a\dot{\theta}^2(2 - \cos\theta) - g\cos\theta = \text{constant}.$$

(L.U., Pt. II)

10. A uniform rod AB of mass m and length $2a$ has its end B freely jointed to the centre of a uniform circular disc of radius r and mass m, where $a\sqrt{3} > r$. The end A of the rod is placed in contact with a smooth vertical wall, and the disc rests on a rough horizontal table on which it can roll without slipping. The rod and disc lie in the same vertical plane, which is at right-angles to the wall, and initially AB is inclined at an angle of 60° with the downward vertical through A. The system is held in this position and then released. Show that in the subsequent motion during which A remains in contact with the wall

$$a\dot{\theta}^2(4 + 18\cos^2\theta) = 3g(1 - 2\cos\theta),$$

where θ is the angle which AB makes with the downward vertical through A. (L.U., Pt. II)

11. A uniform circular disc of mass M and radius a rotates about a smooth fixed vertical axis through its centre perpendicular to its plane, and carries a particle of mass kM, which is free to move along a smooth radial groove. Initially the disc rotates with angular speed ω and the particle is at rest at the centre. Prove that, when the particle, after being slightly disturbed, has moved a distance r along the radius, the angular velocity of the disc is

$$a^2\omega/(a^2 + 2kr^2),$$

and that the radial velocity of the particle is $a\omega r/(a^2 + 2kr^2)^{1/2}$. (L.U., Pt. II)

5.8 Motion with Two Degrees of Freedom

When the motion has two degrees of freedom there are two independent coordinates and one unknown constraint and these are determined by the three equations of motion. The energy equation will involve the two coordinates and can be used in conjunction with the equations of motion.

An example is a body rolling and sliding on a rough plane. The displacement x of the centre parallel to the plane is now independent of the angular displacement θ, but the friction must then be limiting and the equation $F = \mu R$ reduces the number of unknown constraints to one.

Example 11. *A uniform ring of radius a is placed in a vertical position on a horizontal plane so that its centre has a velocity u parallel to the plane and it has an angular velocity ω about its axis in a sense tending to make it roll in the opposite direction to the velocity u. If the coefficient of friction between the ring and the plane is μ, find the time that elapses before the ring begins to roll.*

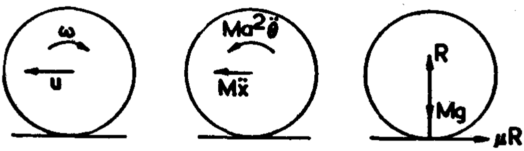

Fig. 99

Let x be the displacement of the centre and θ the angular displacement in the sense shown in Fig. 99. The normal reaction is equal to Mg and the frictional force is μMg in the direction shown.

Then
$$M\ddot{x} = -\mu Mg,$$
$$Ma^2\ddot{\theta} = a\mu Mg.$$

Integrating these equations we have

$$\dot{x} = -\mu gt + u,$$
$$a\dot{\theta} = \mu gt - a\omega.$$

Rolling begins when $\dot{x} = a\dot{\theta}$, that is after time t given by

$$\mu gt - a\omega = -\mu gt + u,$$

that is

$$t = \frac{u + a\omega}{2\mu g}.$$

After this time

$$\dot{x} = \frac{u - a\omega}{2} = a\dot{\theta}.$$

Example 12. *A rough plank rests on horizontal ground with a rough uniform solid cylinder of the same mass resting on it, the length of the plank and the axis of the cylinder being at right-angles. The plank is suddenly given a velocity v in the direction of its length. If μ is the coefficient of friction at all points of contact, show that the cylinder will cease slipping on the plank after a time $v/(6\mu g)$ and that the cylinder and plank will simultaneously come to rest after a time $v/(2\mu g)$.* (L.U., Pt. II)

Let the velocity of the plank, the velocity and the angular velocity of the cylinder after time t be $\dot{y}$, $\dot{x}$ and $\dot{\theta}$ respectively in the directions shown in Fig. 100, $\dot{y}$ being in the same direction as the initial velocity v. The reactions

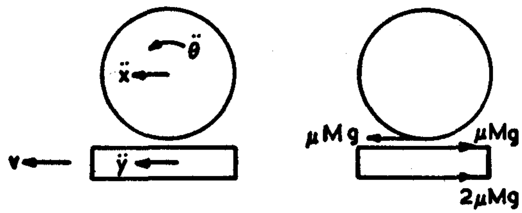

Fig. 100

between plank and ground and plank and cylinder are $2Mg$ and Mg, where M is the mass of the plank. The limiting friction is therefore $2\mu Mg$ and μMg as shown in the figure.

The equations of motion are

$$M\ddot{y} = -3\mu Mg, \qquad (1)$$

$$M\ddot{x} = \mu Mg, \qquad (2)$$

$$M\frac{a^2}{2}\ddot{\theta} = -\mu Mga. \qquad (3)$$

Hence

$$\dot{y} = -3\mu gt + v,$$

$$\dot{x} = \mu gt,$$

$$a\dot{\theta} = -2\mu gt.$$

Rolling begins when the velocity of the point of contact of the cylinder is the same as that of the plank, that is when

$$\dot{x} - a\dot{\theta} = \dot{y}.$$

This gives

$$t = \frac{v}{6\mu g},$$

$$\dot{x} = \frac{1}{6}v,$$

$$\dot{y} = \frac{1}{2}v,$$

$$= -\frac{1}{3}v$$

After this time the friction between cylinder and plank is no longer limiting and the equations of motion become

$$M\ddot{y} = -2\mu Mg - F,$$
$$M\ddot{x} = F,$$
$$M\frac{a^2}{2}\ddot{\theta} = -aF.$$

Hence
$$\ddot{x} + \ddot{y} = -2\mu g,$$
$$2\ddot{x} + a\ddot{\theta} = 0,$$

and, since rolling persists,

$$\dot{x} - a\dot{\theta} = \dot{y}.$$

Integrating and measuring the time from the instant when rolling begins, we have

$$\dot{x} + \dot{y} = -2\mu gt + \frac{2}{3}v,$$
$$2\dot{x} + a\dot{\theta} = 0.$$

Hence
$$\dot{x} = \frac{1}{3}\dot{y} = -\frac{1}{2}a\dot{\theta} = \frac{1}{6}v - \frac{1}{2}\mu gt.$$

Therefore motion ceases after time $v/(3\mu g)$, making a total time of $v/(2\mu g)$.

EXERCISES 5 (c)

1. A uniform solid cylindrical roller of radius r, rotating about its axis with angular velocity Ω radians per second, is gently placed with its axis horizontal on a rough inclined plane of slope α and begins to move up the plane. If $\tan \alpha$ is less than μ, the coefficient of friction, show that the cylinder begins to roll without slipping after a time $r\Omega/g(3\mu \cos \alpha - \sin \alpha)$. (L.U., Pt. I)

2. Find the moment of inertia about its axis of a hollow cylinder of external radius a and internal radius b.

   If the cylinder is placed on an inclined plane of angle α with its axis perpendicular to the lines of greatest slope of the plane, prove that it will roll down the plane without slipping if

   $$\mu > \frac{a^2 + b^2}{3a^2 + b^2}\tan \alpha.$$

   If μ is less than this value, show that whilst the cylinder, starting from rest, makes one revolution about its axis it will move down the plane a distance greater by

   $$\frac{\pi}{a\mu}\{(a^2 + b^2)\tan \alpha - (3a^2 + b^2)\mu\}$$

   than it does when μ is greater than this value. (L.U., Pt. II)

3. A uniform circular disc of radius a is projected in its own plane, which is vertical, along a rough horizontal table. The initial velocity of the centre is v and the initial angular velocity is ω in the direction which will make the disc return to its starting point. Prove that the disc will stop slipping when it returns to its initial position if $a\omega = 5v$. (L.U., Pt. II)

4. A uniform circular disc of mass m and radius a is at rest, with its plane vertical, on a rough horizontal table, the coefficient of friction at the point of contact being μ. A constant horizontal force P is applied to the disc in a line through its centre and in its plane. Prove that slipping will occur if $P > 3\mu mg$. If $P = 6\mu mg$ and the force is applied for a time T and then removed, prove that the disc will continue to slip for a further time T and find its velocity when slipping ceases. (L.U., Pt. II)

5. A uniform sphere of radius a is placed on a rough plane inclined at an angle α to the horizontal. The centre of gravity is given a velocity $a\omega$ down a line of greatest slope and at the same time the sphere is given an angular velocity 3ω about its horizontal diameter in the sense that would make it roll in the opposite direction. Prove that the sphere will eventually begin to roll up the plane if the coefficient of friction μ is greater than 6 tan α.

6. A uniform solid sphere of radius a and mass m rests on a uniform board of mass M which lies on a smooth horizontal plane. If the board is given a horizontal velocity u, show that the sphere will slip on the board for a time $2Mu/\{\mu g(7m + 2m)\}$, where μ is the coefficient of friction between the sphere and the board. Show also that the velocity of the board is then $7Mu/(7M + 2m)$.

7. A uniform thin hollow sphere of radius a is projected up a line of greatest slope of a rough plane inclined at an angle α to the horizontal. The initial velocity of the centre of the sphere is u and the initial angular velocity is ω, both in senses which make the sphere move upwards, and μ is the coefficient of friction between the sphere and the plane. If $a\omega > u$ and $5\mu \cos \alpha > 2 \sin \alpha$, show that the sphere will begin to roll after a time $2(a\omega - u)/\{g(5\mu \cos \alpha - 2 \sin \alpha)\}$. Find the angular velocity of the sphere when rolling begins.

5.9 Angular Momentum about any Point

Let (x, y) be the centre of gravity of a lamina, $I = Mk^2$ the moment of inertia about it and $\dot{\theta}$ the angular velocity.

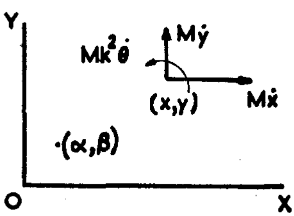

Fig. 101

The linear momentum of the lamina has components $M\dot{x}$ and $M\dot{y}$ parallel to the axes. The angular momentum, which is the moment of momentum about the centre of gravity, is $Mk^2\dot{\theta}$. Hence the whole momentum of the lamina consists of a vector with components $M\dot{x}$ and $M\dot{y}$ located at the centre of gravity and a couple $Mk^2\dot{\theta}$ which is a vector perpendicular to the lamina (Fig. 101).

If (x_1, y_1) be the coordinates of a typical particle of mass m, the angular momentum about a point (α, β)

$$= \Sigma m\{(x_1 - \alpha)\dot{y}_1 - (y_1 - \beta)\dot{x}_1\}$$
$$= \Sigma m\{(x_1 - x)\dot{y}_1 - (y_1 - y)\dot{x}_1\} + \Sigma m\{(x - \alpha)\dot{y}_1 - (y - \beta)\dot{x}_1\}$$
$$= Mk^2\dot{\theta} + (x - \alpha)\Sigma m\dot{y}_1 - (y - \beta)\Sigma m\dot{x}_1$$
$$= Mk^2\dot{\theta} + (x - \alpha)M\dot{y} - (y - \beta)M\dot{x}.$$

This expression is the moment about the point (α, β) of the momentum of the lamina shown in Fig. 101, treating the linear and angular momentum as a force and a couple respectively.

The rate of change of the angular momentum about (α, β) is

$$Mk^2\ddot{\theta} + (x - \alpha)M\ddot{y} - (y - \beta)M\ddot{x} - M(\dot{\alpha}\dot{y} - \dot{\beta}\dot{x}). \qquad (1)$$

Now if the forces acting on the lamina have components X and Y and moment G about the centre of gravity, the force system is as shown in Fig. 102, and the moment L of the force system about (α, β) is

$$L = G + (x - \alpha)Y - (y - \beta)X.$$

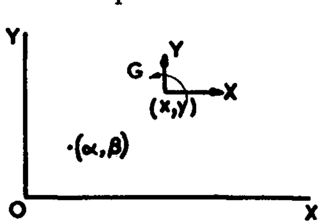

Fig. 102

Hence since

$$G = Mk^2\ddot{\theta},$$
$$X = M\ddot{x},$$
$$Y = M\ddot{y},$$

we have $L = Mk^2\ddot{\theta} + (x - \alpha)M\ddot{y} - (y - \beta)M\ddot{x}.$ (2)

The expressions (1) and (2) are identical only if $\dot{\alpha}\dot{y} - \dot{\beta}\dot{x} = 0$, and this is so if (α, β) is a fixed point. It is also true if (α, β) is the centre of gravity so that $\dot{\alpha} = \dot{x}$, $\dot{\beta} = \dot{y}$.

Therefore, the moment of the forces about any *fixed* point or about the centre of gravity is equal to the rate of change of angular momentum about the same point.

The moment of the forces about any point, whether a fixed point or not, is equal to the moment about the point of the effective forces and couple as expressed in equation (2).

Thus, in Example 5, page 139, the effective forces on the railway carriage door are $ma\ddot{\theta}$, $ma\dot{\theta}^2$, mf in the directions shown in Fig. 94, and the effective couple is $\frac{1}{3}ma^2\ddot{\theta}$. Since the force system has zero moment about the hinge the moment of the effective forces about the hinge is zero.

Therefore, $$\frac{1}{3}ma^2\ddot{\theta} + ma^2\ddot{\theta} = mf \times a\cos\theta,$$

and hence $$\frac{4}{3}a\ddot{\theta} = f\cos\theta.$$

5.10 The Instantaneous Centre

We shall show that when a lamina is moving in a plane there is at

any instant a point of the lamina which is at rest. This point is called the instantaneous centre.

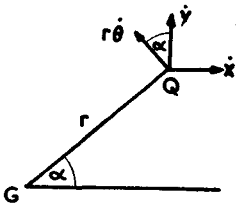

Fig. 103

Let $\dot{x}$, $\dot{y}$ be the components of the velocity of the centre of gravity and $\dot{\theta}$ the angular velocity and let the point Q which is the instantaneous centre have coordinates $(r\cos\alpha, r\sin\alpha)$ with respect to the centre of gravity. The point Q has the velocity $(\dot{x}, \dot{y})$ of the centre of gravity and a relative velocity $r\dot{\theta}$ with respect to the centre of gravity (Fig. 103). Its velocity is the vector sum of these velocities and since by hypothesis this is zero we have

$$\dot{x} = r\dot{\theta}\sin\alpha,$$
$$\dot{y} = -r\dot{\theta}\cos\alpha.$$

Therefore
$$r\cos\alpha = -\frac{\dot{y}}{\dot{\theta}},$$
$$r\sin\alpha = \frac{\dot{x}}{\dot{\theta}}.$$

Thus the point whose coordinates relative to the centre of gravity are $(-\dot{y}/\dot{\theta}, \dot{x}/\dot{\theta})$ is instantaneously at rest. These coordinates give the position of a definite point unless the motion is one of translation only with $\dot{\theta} = 0$, in which case we may say that the instantaneous centre is at infinity. The point Q whose coordinates are $(-\dot{y}/\dot{\theta}, \dot{x}/\dot{\theta})$ may be outside the lamina, but it can be thought of as rigidly connected to it.

The velocity of any point P of the lamina which is at a distance r_1 from Q is the vector sum of the velocity of Q which is zero and the velocity relative to Q, that is, its velocity is $r_1\dot{\theta}$ in a direction perpendicular to PQ.

Therefore the angular momentum of the lamina about the instantaneous centre Q is

$$\Sigma m r_1^2\dot{\theta} = Mk_1^2\dot{\theta}$$

where Mk_1^2 is the moment of inertia of the lamina about Q.

Similarly the kinetic energy of the lamina is

$$\frac{1}{2}\Sigma m r_1^2\dot{\theta}^2 = \frac{1}{2}Mk_1^2\dot{\theta}^2.$$

Therefore the kinetic energy may be quickly written down if the moment of inertia about the instantaneous centre is known.

5.11 Graphical Methods for Instantaneous Centre

The position of the instantaneous centre and the velocity of any point of a lamina at any instant may be found by drawing if the direc-

tions of the velocities of two points of the lamina are known and the magnitude of the velocity of one of them.

Let P and R be two points of the lamina which at some instant are moving in the directions PP' and RR' respectively (Fig. 104). Let v be the velocity of P. Then if Q be the instantaneous centre, P is moving in a direction perpendicular to PQ, therefore Q lies on the perpendicular to PP' through P. Similarly Q lies on the perpendicular to RR' through R. Therefore Q is found as the intersection of these perpendiculars.

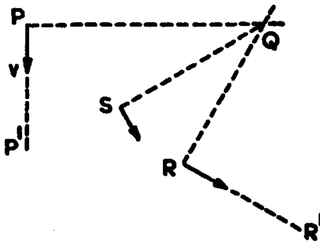

Fig. 104

Also, if $\dot{\theta}$ be the angular velocity we have

$$v = PQ \times \dot{\theta}.$$

Therefore the velocity of any other point S

$$= SQ \times \dot{\theta}$$

$$= \frac{SQ}{PQ} \times v.$$

The position of the instantaneous centre and the angular velocity may also be found by analytical methods.

Example 13. *A thin uniform rod AB moves in the plane XOY so that A slides on OX and AB touches the parabola $ky = x^2$. Prove that the locus of the instantaneous centre of the motion of AB is an arc of the parabola $ky = 4x^2 + \frac{k^2}{4}$. Find also the rate at which the instantaneous centre is describing this curve when A is at a distance a from O and is moving with velocity u.* (L.U., Pt. II)

If $y = mx + c$ be a tangent to the parabola (Fig. 105), the coordinates of the point of contact C are given by

$$x^2 - k(mx + c) = 0,$$

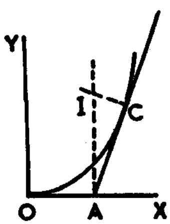

Fig. 105

and since this equation has equal roots, $k^2m^2 + 4kc = 0$. Therefore the tangent is $y = mx - \frac{1}{4}km^2$, and the coordinates of C are $\left(\frac{1}{2}km, \frac{1}{4}km^2\right)$. The tangent meets OX at the point A, $\left(\frac{1}{4}km, 0\right)$.

The point A moves along OX, therefore the instantaneous centre of the motion of the rod lies on the perpendicular to OX at A which is the line $x = \frac{1}{4}km$.

The point C can only move tangentially to the parabola. Therefore, the instantaneous centre lies on the normal to the parabola at C.

The equation of the normal at C is

$$y - \frac{1}{4}km^2 = -\frac{1}{m}\left(x - \frac{1}{2}km\right),$$

and hence the instantaneous centre is at $\left(\frac{1}{4}km, \frac{1}{4}km^2 + \frac{1}{4}k\right)$, and this point lies on the parabola $ky = 4x^2 + \frac{1}{4}k^2$.

For the displacement of the instantaneous centre we have

$$x = \frac{1}{4}km = a,$$

$$\dot{x} = \dot{a} = u.$$

$$y = \frac{1}{4}km^2 + \frac{1}{4}k,$$

$$= \frac{4a^2}{k} + \frac{1}{4}k,$$

$$\dot{y} = \frac{8a\dot{a}}{k} = \frac{8au}{k}.$$

$(\dot{x}^2 + \dot{y}^2)^{1/2} = u\left(1 + \frac{64a^2}{k^2}\right)^{1/2}$, and this is the rate at which the instantaneous centre moves along the curve.

5.12 Motion of a Connecting-rod

Let AB (Fig. 106) be the connecting-rod of a reciprocating engine and BC the crank and let AB and BC make angles ϕ and θ respectively with the line AC.

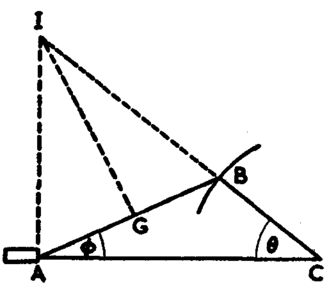

Fig. 106

Since A is moving along AC the instantaneous centre must lie on the perpendicular to AC through A.

Since B is moving in a circle its motion at any instant is perpendicular to BC and the instantaneous centre must lie on BC produced. The centre I is, therefore, at the intersection of these two lines.

Since $\dot{\theta}$ is the angular velocity of the crank and $\dot{\phi}$ that of the connecting-rod, the velocity of B considered as a point of either is

$$IB \times \dot{\phi} = BC \times \dot{\theta}.$$

Let G be the centre of gravity of the connecting-rod and Mk^2 its moment of inertia about G.

The velocity of G is $IG \times \dot{\phi}$ and hence the kinetic energy of the connecting-rod is

$$\frac{1}{2}M(IG^2 + k^2)\dot{\phi}^2 = \frac{1}{2}M(IG^2 + k^2) \times \frac{BC^2}{IB^2} \times \dot{\theta}^2.$$

The kinetic energy of the connecting-rod varies for different positions and this accounts for a variation of the crank-effort torque at C from that given by the thrust in the rod.

5.13 Moments about the Instantaneous Centre

It was shown in § 5.9 that the moment of the forces about any point (α, β) is equal to the moment of the effective forces about that point, that is,

$$L = Mk^2\ddot{\theta} + (x - \alpha)M\ddot{y} - (y - \beta)M\ddot{x}.$$

If (α, β) be the instantaneous centre,

$$\alpha - x = -\dot{y}/\dot{\theta},\ \beta - y = \dot{x}/\dot{\theta},$$

and we have
$$L = Mk^2\ddot{\theta} + \frac{1}{\dot{\theta}}M(\dot{x}\ddot{x} + \dot{y}\ddot{y}),$$
$$= \frac{1}{2\dot{\theta}}\left\{\frac{d}{dt}Mk^2\dot{\theta}^2 + \frac{d}{dt}M(\dot{x}^2 + \dot{y}^2)\right\}.$$

Let r be the distance of the instantaneous centre from the centre of gravity and k_1 the radius of gyration about it, so that $k_1^2 = k^2 + r^2$. Then the velocity of the centre of gravity is $r\dot{\theta}$ and we have

$$\begin{aligned} L &= \frac{1}{2\dot{\theta}}\frac{d}{dt}M(k^2 + r^2)\dot{\theta}^2 \\ &= \frac{1}{2\dot{\theta}}\frac{d}{dt}Mk_1^2\dot{\theta}^2 \\ &= Mk_1^2\ddot{\theta} + Mk_1\dot{k}_1\dot{\theta}. \end{aligned}$$

If k_1 is constant, that is if the instantaneous centre remains at a constant distance from the centre of gravity $\dot{k}_1 = 0$ and we have

$$L = Mk_1^2\ddot{\theta}.$$

In this case the moment of the forces about the instantaneous centre is the rate of change of angular momentum about it. If k_1 is not constant, the rate of change of angular momentum is $Mk_1^2\ddot{\theta} + 2Mk_1\dot{k}_1\dot{\theta}$ and this is not equal to the moment of the forces about the point.

We have
$$L = \frac{1}{2\dot{\theta}}\frac{d}{dt}(Mk_1^2\dot{\theta}^2)$$
$$= \frac{d}{d\theta}\left(\frac{1}{2}Mk_1^2\dot{\theta}^2\right),$$

so that
$$Ld\theta = d\left(\frac{1}{2}Mk_1^2\dot{\theta}^2\right),$$

that is, the work done by the forces in a rotation about the instantaneous centre is equal to the change of kinetic energy.

Example 14. *A solid of revolution of mass M whose bounding radius is a and whose radius of gyration about its axis of symmetry is k, rolls without slipping down the line of greatest slope of a plane inclined at an angle α to the horizontal. Find its acceleration down the plane.*

The point of contact is the instantaneous centre and the moment of inertia about it, which is constant, is $M(k^2 + a^2)$.
The moment of the forces about the point of contact is (see Example 1, page 134) $Mga \sin \alpha$.
Therefore if $\ddot{\theta}$ be the angular acceleration

$$M(k^2 + a^2)\ddot{\theta} = Mga \sin \alpha,$$

$$a\ddot{\theta} = \frac{a^2}{a^2 + k^2}\cdot g \sin \alpha.$$

Example 15. *A uniform thin hollow sphere of radius a and mass M rests on a rough horizontal plane. A particle of mass m is attached to the highest point of the sphere and displaced slightly. Find the angular velocity of the sphere when the radius to the particle makes an angle θ with the upward vertical.*

The instantaneous centre is the point of contact. The distance of the mass m from this point is $\{a^2 + a^2 + 2a^2 \cos \theta\}^{1/2} = 2a \cos \frac{1}{2}\theta$.
The moment of inertia about the point of contact is the variable quantity

$$M\left(\frac{2}{3}a^2 + a^2\right) + m\left(4a^2 \cos^2 \frac{1}{2}\theta\right).$$

The moment of the forces (Fig. 107) about the point of contact is $mga \sin \theta$ and we have

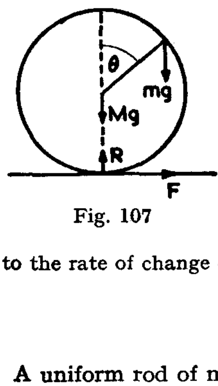

Fig. 107

$$mga \sin \theta = \frac{d}{d\theta}\frac{1}{2}\left(\frac{5}{3}M + 4m \cos^2 \frac{1}{2}\theta\right)a^2\dot{\theta}^2,$$

$$\left(\frac{5}{6}M + 2m \cos^2 \frac{1}{2}\theta\right)a\dot{\theta}^2 = mg(1 - \cos \theta),$$

$$a\dot{\theta}^2 = \frac{6mg(1 - \cos \theta)}{5M + 6m(1 + \cos \theta)}.$$

In this case it would not be correct to equate the moment of the forces about the instantaneous centre to the rate of change of angular momentum about it.

EXERCISES 5 (*d*)

1. A uniform rod of mass m and length $2a$ moves on a horizontal table so that its centre has velocity u in a direction perpendicular to the rod and its angular velocity is ω. Find its angular momentum about one end of the rod and about the instantaneous centre.

2. A uniform solid cylinder of mass M and radius b rolls on the surface of a fixed cylinder of radius a the axes of the cylinders being parallel. When the angular velocity of the moving cylinder is ω, find its angular momentum about the axis of the fixed cylinder.

3. The diagram (Fig. 108) represents three gear wheels. A rotates on a fixed shaft while C is a fixed annulus with internal teeth. A and B

have equal pitch circles of radius r. Treating B as a uniform disc of mass M, show that its angular momentum about the centre of the fixed shaft, when the latter has an angular velocity ω, is $\frac{3}{4}Mr^2\omega$.

Find the tangential forces acting on B when the shaft is given an angular acceleration $\dot{\omega}$ radians per sec. per sec. (L.U., Pt. II)

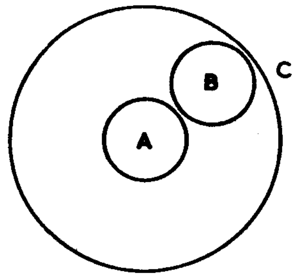

Fig. 108

4. A body consisting of two thin straight rods OX, OY which are joined at right-angles moves in a plane so that OX touches a fixed circle of radius a and OY passes through a fixed point A on the circle.

   Determine the position of the instantaneous centre of rotation of the body and prove that it lies (i) on a fixed circle of radius $\frac{1}{2}a$, (ii) on a circle fixed with respect to the body and of radius a. Find the velocity of the instantaneous centre in terms of the angular velocity of the body. (L.U., Pt. II)

5. The stroke of an engine is 24 in. and the connecting-rod is 4 ft. long. If the crank is rotating at 300 r.p.m. find the velocity of the piston at the mid-point of the stroke.

   The mass of the connecting-rod is 50 lb., and its radius of gyration about its centre of gravity, which is central, is 18 in. Find its kinetic energy at the mid-point of the stroke.

6. A uniform rod of mass M and length $2a$ is placed with one end against a smooth wall and the other end on a smooth horizontal floor, the rod being in a vertical plane perpendicular to the wall and initially inclined at an angle α to the wall. Show that as the rod slides down the instantaneous centre is at a constant distance from its centre and find the angular velocity of the rod when the rod leaves the wall.

7. A uniform solid sphere of radius b is placed on the highest point of a fixed sphere of radius a and rolls down it. Show that if θ be the angle which the common radius makes with the vertical at any instant the angular velocity of the rolling sphere is $(1 + a/b)\dot{\theta}$.

   Prove that $7(a + b)\dot{\theta}^2 = 10g(1 - \cos\theta)$ and show that if the spheres are rough enough to prevent slipping they will separate when $\cos\theta = 10/17$.

8. A uniform hemisphere of mass M and radius a is placed with its plane face vertical in contact with a horizontal table rough enough to prevent slipping and released. Show that when the plane face of the hemisphere is inclined at an angle θ to the vertical its moment of inertia about the point of contact with the table is

$$Ma^2(28 - 15\sin\theta)/20.$$

Show that the angular velocity in this position is

$$\{15g\sin\theta/a(28 - 15\sin\theta)\}^{1/2}.$$

9. A uniform rectangular block has edges of length $2a$, $2b$, $2c$ and $ABCD$ is one of its faces with $AB = CD = 2c$ and $BC = AD = 2a$.

   The block is placed with AB in contact with a smooth vertical wall and CD in contact with a smooth horizontal floor, the face $ABCD$ being inclined at an angle α to the vertical, and released.

   Prove that if $\alpha > \tan^{-1}(b/a)$ the angular velocity of the block when the face $ABCD$ reaches the floor will be

$$\left\{\frac{3}{2}g(a\cos\alpha + b\sin\alpha - b)/(a^2+b^2)\right\}^{1/2}.$$

10. A uniform solid circular cylinder of radius b rolls without slipping inside a fixed hollow cylinder of radius a. Show that if the centre of gravity of the moving cylinder has velocity v in its lowest position it will make a complete revolution around the inside surface of the fixed cylinder if $3v^2 > 11g(a - b)$.

5.14 Initial Accelerations

When a constraining force which keeps a body in equilibrium is suddenly removed we can find the initial values of the acceleration and of the other forces by writing down the equations of motion. These equations are considerably simplified for the following reasons:

(1) At the initial instant all velocities are zero, therefore, for example the polar components of acceleration are $\ddot{r}$ and $r\ddot{\theta}$.

(2) The forces may be taken as acting in their initial directions.

(3) The total moment L of the forces about the instantaneous centre is (§ 5.13) equal to

$$Mk_1^2\ddot{\theta} + Mk_1\dot{k}_1\dot{\theta},$$

where $\dot{\theta}$ is the angular velocity and k_1 the radius of gyration about the instantaneous centre. Since $\dot{k}_1$ is initially zero we have for the initial motion

$$L = Mk_1^2\ddot{\theta}.$$

Therefore in finding the initial acceleration we may equate the moment of the forces about the instantaneous centre to the rate of change of angular momentum about it, treating k_1 as a constant.

Example 16. *A uniform rod AB of mass M and length $2a$ has a light inextensible string of length l attached to each end. The other ends of the strings are attached to a fixed point O so that the rod is suspended in a horizontal position. If one of the strings is suddenly cut, find the initial angular acceleration of the rod and the tension in the other string.*

The motion of the rod has two degrees of freedom and is specified by coordinates θ and ϕ, where θ is the inclination of the string to the vertical and ϕ the inclination of the rod to the horizontal (Fig. 109).
Initially $\theta = \alpha$, where $\sin\alpha = a/l$, $\phi = 0$ and $\dot{\theta} = \dot{\phi} = 0$.
The point A moves in a circle of radius l and has initially acceleration $l\ddot{\theta}$ perpendicular to OA.

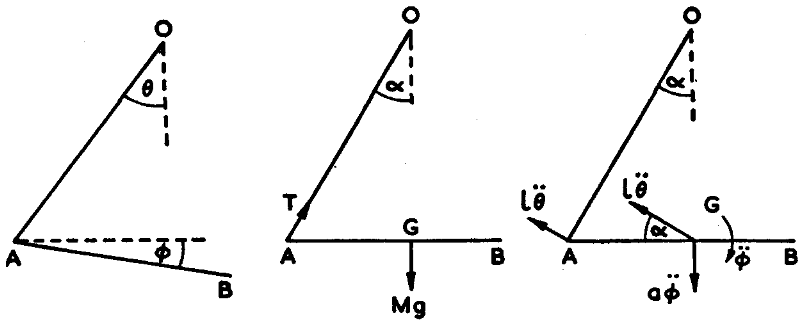

Fig. 109

The centre of gravity G of the rod moves in a circle of radius a about A, therefore its acceleration relative to A is $a\ddot{\phi}$ perpendicular to AB.
Therefore G has components of acceleration $a\ddot{\phi}$ and $l\ddot{\theta}$ perpendicular to AB and OA respectively.
Taking the forces in their initial positions the equations of motion are

$$T\cos\alpha - Mg = M(l\ddot{\theta}\sin\alpha - a\ddot{\phi}), \tag{1}$$

$$T\sin\alpha = -M(l\ddot{\theta}\cos\alpha), \tag{2}$$

$$T\cos\alpha \times a = M\frac{a^2}{3}\ddot{\phi}. \tag{3}$$

Hence

$$T = \frac{Mg\cos\alpha}{4 - 3\sin^2\alpha} = Mg\frac{l(l^2 - a^2)^{1/2}}{4l^2 - 3a^2},$$

$$l\ddot{\theta} = -\frac{g\sin\alpha}{4 - 3\sin^2\alpha} = -\frac{gal}{4l^2 - 3a^2},$$

$$a\ddot{\phi} = \frac{3g\cos^2\alpha}{4 - 3\sin^2\alpha} = \frac{3g(l^2 - a^2)}{4l^2 - 3a^2}.$$

Example 17. *A uniform solid sphere of mass M and radius a rests on a rough horizontal table, the coefficient of friction being μ. If a particle of mass m is attached to the surface of the sphere at the level of its centre and the system is released, find the condition that the sphere should begin to roll without slipping and in this case find the angular acceleration of the sphere.*

If the sphere rolls, its instantaneous centre is the point of contact (Fig. 110) and the moment of inertia about it is

$$M\left(\frac{2}{5}a^2 + a^2\right) + m(2a^2).$$

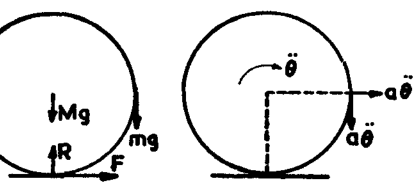

Fig. 110

Taking moments about the instantaneous centre for the initial motion we have

$$\left(\frac{7}{5}Ma^2 + 2ma^2\right)\ddot{\theta} = mga$$

$$\ddot{\theta} = \frac{5mg}{a(7M + 10m)}.$$

The friction F and the reaction R are given by the equations of motion

$$F = (M + m)a\ddot{\theta},$$

$$(M + m)g - R = ma\ddot{\theta}.$$

Hence

$$F = \frac{5m(M + m)g}{7M + 10m},$$

$$R = (M + m)g - \frac{5m^2g}{7M + 10m},$$

$$= \frac{(7M^2 + 17Mm + 5m^2)g}{7M + 10m}.$$

The condition for rolling without slipping is therefore

$$\mu > \frac{5m(M + m)}{7M^2 + 17Mm + 5m^2}.$$

5.15 Calculation of Internal Forces

When the plane motion of a rigid body is fully known, that is, the linear acceleration of its centre of gravity, its angular acceleration and the forces acting on it are known, the internal forces acting at any section of the body can be found in the following manner.

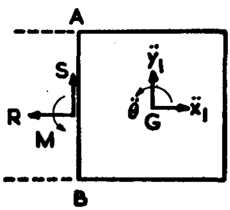

Fig. 111

Let G be the centre of gravity of a portion of the body of mass M_1 cut off by a plane section AB (Fig. 111). Then the components $\ddot{x}_1$, $\ddot{y}_1$, of the linear acceleration of G may be found from a knowledge of the acceleration of the centre of gravity of the body as a whole. The angular acceleration $\ddot{\theta}$ is the same as for the whole body.

The internal forces exerted by the remainder of the body on M_1 may be reduced to a force and a couple. Let the force have components R perpendicular to AB and S parallel to AB, and let M be the couple. Then by writing down the three equations of motion for M_1 the values of R, S and M may be found.

S is the shearing force at the section and M is the bending moment. Thus the tendency of a body to break at some section during its motion may be investigated.

Example 18. *A uniform rod AB of mass m and length $2a$ rests on a smooth horizontal plane. A horizontal force P is applied to the end B of the rod; P is constant in magnitude and acts in a fixed direction which is initially perpendicular to the length of the rod. Find the bending moment and shearing force in the rod at a distance $2c$ from A when the rod has turned through an angle θ.*

Let x be the displacement of the centre of the rod in the direction of P at time t and θ the angular displacement from its initial position (Fig. 112). The equations of motion are

$$P = m\ddot{x},$$

$$Pa\cos\theta = m\frac{a^2}{3}\ddot{\theta}.$$

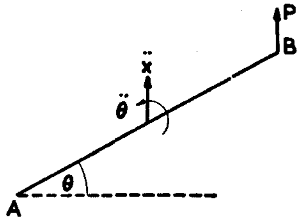

Fig. 112

Therefore
$$\ddot{x} = \frac{P}{m},$$

$$\ddot{\theta} = \frac{3P\cos\theta}{am},$$

and by integration
$$\dot{\theta}^2 = \frac{6P\sin\theta}{am}.$$

Now consider the portion AC of the rod (Fig. 113) of length $2c$ and mass (mc/a).

The centre of gravity of AC has the acceleration $\ddot{x}$ of the centre of gravity of the rod and components of acceleration relative to the centre of gravity $(a - c)\dot{\theta}^2$ and $(a - c)\ddot{\theta}$ along and perpendicular to the rod.

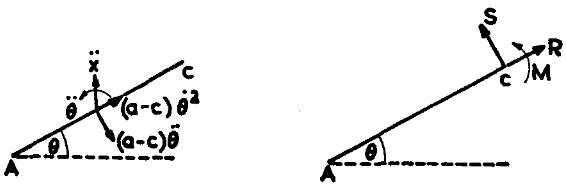

Fig. 113

If R and S be the components of the internal force at C and M the moment there, we have

$$R = \left(\frac{mc}{a}\right)\{\ddot{x}\sin\theta + (a - c)\dot{\theta}^2\},$$

$$S = \left(\frac{mc}{a}\right)\{\ddot{x}\cos\theta - (a - c)\ddot{\theta}\},$$

$$M + Sc = \left(\frac{mc}{a}\right)\frac{c^2}{3}\ddot{\theta}.$$

Substituting the values obtained for $\ddot{x}$, $\ddot{\theta}$, and $\dot{\theta}^2$, we have

$$R = \frac{c(7a - 6c)}{a^2}.P\sin\theta,$$

$$S = -\frac{c(2a - 3c)}{a^2}.P\cos\theta,$$

$$M = \frac{2c^2(a - c)}{a^2}.P\cos\theta.$$

For a given angle θ the bending moment is greatest when $c^2(a - c)$ is a maximum, that is when $c = \frac{2a}{3}$.

EXERCISES 5 (e)

1. A uniform rod of length $2a$ is suspended in a horizontal position by two light inextensible vertical strings each attached to one end of the rod and to a fixed point. If one of the strings is cut, show that the tension in the other is instantly halved.

2. A uniform thin bar whose centre of gravity is G is of length $2a$ and rests horizontally on two props at points distant b and c from G. If either prop is suddenly removed, show that the load on the other is instantaneously increased or decreased accordingly as $a^2 > 3bc$ or $a^2 < 3bc$. (L.U., Pt. II)

3. A uniform bar AB of length $2a$ is suspended in equilibrium by two light inextensible strings OA, OB, each of length $2a$. Find the tension in each string and show that if one of them is cut the tension in the other is instantly reduced to 6/13 of its equilibrium value. (L.U., Pt. II)

4. A solid uniform hemisphere of weight W and radius a is held at rest on a horizontal plane with its plane face vertical. If it is released from rest, find the initial values of the vertical component of the reactions of the plane and the angular acceleration (*a*) if the plane is perfectly smooth, (*b*) if it is rough enough to prevent slipping. (L.U., Pt. II)

5. A uniform thin rod of length $3l$ is supported horizontally on two small rough pegs each at a distance $l/2$ from its mid-point. If one peg be suddenly removed, show that the reaction at the other is instantaneously increased by half its original value. If in the subsequent motion the rod begins to slip on the second peg when its inclination to the horizontal is $\tan^{-1} 0{\cdot}1$, find the coefficient of friction. (L.U., Pt. II)

6. A uniform circular solid cylinder of mass $2m$ and radius a has a particle of mass m attached to a point on the circumference of its central section. The cylinder is placed on a horizontal plane, coefficient of friction μ, with the radius to the particle inclined at an angle of 60° to the upward vertical through the centre. Show that the cylinder will begin to roll without slipping if $\mu \geqslant 7\sqrt{3}/69$.

7. A uniform bar of length a and weight W is freely pivoted at one end, and is let fall from a horizontal position. Determine the angular velocity when the rod has fallen through an angle θ.

   Show that in this position, the tension in the rod at a distance x from the pivot is $\frac{1}{2}W \sin \theta(5 - 2x/a - 3x^2/a^2)$. (L.U., Pt. II)

8. A uniform rod of mass m and length l is freely suspended from one end and swings through an angle α either side of the vertical. Find the bending moment and shearing force in the rod at a distance a from the pivot when the rod is in the position of instantaneous rest, and show that the bending moment is greatest at a point of trisection of the rod.

9. Two uniform circular cylinders each of mass m and radius a are fastened one to each end of another uniform cylinder of mass m and

radius $\frac{1}{2}a$ so that the axes of the three cylinders are in line. The body rolls without slipping down a plane inclined at an angle α to the horizontal with the axis perpendicular to a line of greatest slope. Find the shearing force and the torsional moment at the join of two of the cylinders as the body rolls.

5.16 Equations of Impulse

The equations of motion for the plane motion of a rigid body are:

$$M\ddot{x} = X,$$
$$M\ddot{y} = Y,$$
$$Mk^2\ddot{\theta} = G.$$

Here $\ddot{x}$ and $\ddot{y}$ are the components of acceleration of the centre of gravity, $\ddot{\theta}$ the angular acceleration, M the mass and k the radius of gyration about the centre of gravity. X and Y are the components of the external forces and G their moment about the centre of gravity.

The impulses, which are the time integrals of the forces, are found by integrating these equations between $t = 0$ and $t = \tau$, giving

$$\Big[M\dot{x}\Big]_0^\tau = \int_0^\tau X dt = J, \tag{1}$$

$$\Big[M\dot{y}\Big]_0^\tau = \int_0^\tau Y dt = K, \tag{2}$$

$$\Big[Mk^2\dot{\theta}\Big]_0^\tau = \int_0^\tau G dt = L. \tag{3}$$

If very large forces act on the body for a very short time τ we may regard the integrals in these equations as limits and the impulses as being instantaneous. Then the effect of the other forces such as gravity which act continuously on the body is negligible during the period of the impulse.

J is therefore the sum of the components parallel to the x-axis of the impulses which act on the body and K is the sum of the components parallel to the y-axis.

$\Big[M\dot{x}\Big]_0^\tau$ and $\Big[M\dot{y}\Big]_0^\tau$ are the changes in the linear momentum of the body parallel to the x and y-axis respectively and are the changes in the respective components of velocity of the centre of gravity multiplied by the mass. L is the sum of the moments of impulse about the centre of gravity and $\Big[Mk^2\dot{\theta}\Big]_0^\tau$ is the change of angular momentum about the centre of gravity.

It is evident that if the component of impulse in any direction is zero there is no change in momentum in that direction.

Example 19. *A uniform rod of mass m and length 2a on a smooth horizontal table has angular velocity ω and its centre has velocity u in a direction perpendicular to the rod. It receives a blow J at one end in a direction perpendicular to the rod and opposing the motion of that end. Find the velocity of the centre of gravity and the angular velocity immediately after the impulse, and the change of kinetic energy.*

Let u_1 be the velocity of the centre of gravity after the impulse and ω_1 the angular velocity. The velocities before and after the impulse and the impulse are shown in three separate diagrams (Fig. 114).

Fig. 114

The impulse J reduces the linear momentum of the centre of gravity from mu to mu_1, therefore

$$J = mu - mu_1. \tag{1}$$

The moment of impulse reduces the angular momentum about the centre of gravity from $\frac{1}{3}ma^2\omega$ to $\frac{1}{3}ma^2\omega_1$, therefore

$$Ja = \frac{1}{3}ma^2\omega - \frac{1}{3}ma^2\omega_1. \tag{2}$$

Hence

$$u_1 = u - \frac{J}{m},$$

$$a\omega_1 = a\omega - \frac{3J}{m}.$$

The loss of kinetic energy is

$$\frac{1}{2}m(u^2 - u_1^2) + \frac{1}{2} \times \frac{1}{3}ma^2(\omega^2 - \omega_1^2).$$

Substituting for u_1 and ω_1 this is

$$J(u + a\omega) - 2\frac{J^2}{m}.$$

5.17 Induced Impulse

When an impulse is applied to a body there may be one or more constraints which prevent the body from moving in the manner dictated by the impulse. These constraints are increased during the time the impulse lasts and become additional impulses which with the applied impulse determine the changes in linear and angular momentum. Each induced impulse involves a geometrical relation between the coordinates and therefore between the components of linear and angular momentum.

Thus if a rod at rest on a smooth table is freely hinged at one end and receives a blow J perpendicular to the rod through the other end we must assume an induced impulse X at the hinge (Fig. 115) opposing the motion which that end would otherwise have.

Taking m the mass and $2a$ the length of the rod, u and ω the velocities we have

Fig. 115

$$J + X = mu,$$

$$Ja - Xa = m\frac{a^2}{3}\omega.$$

We have also the geometrical relation that the hinged end is stationary, that is,

$$u - a\omega = 0.$$

From these equations $u = a\omega = \dfrac{3J}{2m}$, $X = \dfrac{J}{2}$.

There is an induced impulse when a point of a body is suddenly brought to rest; the corresponding geometrical equations are found by equating to zero the velocity of the point after the impulse.

Example 20. *Two uniform rods AB and BC, each of mass m and length $2a$ are freely jointed at B and rest in a straight line on a smooth horizontal plane with the extremity A freely hinged to a fixed point. An impulse J is applied at C perpendicular to the rod BC. Find the kinetic energy imparted to the rods.*

Let the linear and angular velocities of AB and BC be u_1, ω_1 and u_2, ω_2 as shown in Fig. 116.

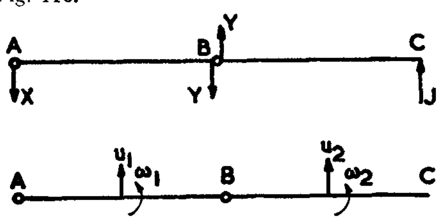

Fig. 116

Induced impulses X and Y must be assumed at A and B and the corresponding geometrical equations are,

for A, $\qquad u_1 - a\omega_1 = 0, \qquad$ (1)

for B, $\qquad u_1 + a\omega_1 = u_2 - a\omega_2. \qquad$ (2)

The equations of impulse for the two rods are

$$J + Y = mu_2, \tag{3}$$

$$X + Y = -mu_1, \tag{4}$$

$$Ja - Ya = m\frac{a^2}{3}\omega_2, \tag{5}$$

$$Xa - Ya = m\frac{a^2}{3}\omega_1. \tag{6}$$

These six equations determine the six unknown quantities u_1, u_2, ω_1, ω_2, X, Y. Substituting for u_1, u_2, ω_1, ω_2 in equations (1) and (2) from equations (3) to (6) we have

$$-X - Y - 3X + 3Y = 0,$$
$$-X - Y + 3X - 3Y = J + Y - 3J + 3Y.$$

Therefore
$$X = \frac{1}{7}J,\ Y = \frac{2}{7}J,$$
$$u_1 = -\frac{3J}{7m},\ u_2 = \frac{9J}{7m},$$
$$a\omega_1 = -\frac{3J}{7m},\ a\omega_2 = \frac{15J}{7m}.$$

The kinetic energy imparted is
$$\frac{1}{2}mu_1^2 + \frac{1}{2}mu_2^2 + \frac{1}{2}m\frac{a^2}{3}\omega_1^2 + \frac{1}{2}m\frac{a^2}{3}\omega_2^2 = \frac{12J^2}{7m}.$$

EXERCISES 5 (*f*)

1. A straight uniform rigid rod AB lies at rest on a smooth horizontal table. The rod is struck at B by a blow whose direction is horizontal and perpendicular to AB. Prove that the centre of the rod will begin to move with a speed equal to one-quarter of the initial speed of B, and that the rod will begin to rotate about the point P of its length, such that $AP = (1/3)AB$. (L.U., Pt. I)

2. A uniform rod of mass m and length $2l$ is rotating about its centre G with angular velocity ω, on a smooth horizontal table, when a point of trisection A hits a small inelastic stop so that the rod begins to turn about A. Find the new angular velocity and show that one quarter of the energy disappears in the impact. (L.U., Pt. I)

3. A lamina is moving in its own plane. Determine the changes in the motion of the lamina if it is acted upon by an impulse whose line of action is in the plane of the lamina.

   A uniform thin rod AB, of mass M, has a particle of mass m fixed to it at B. The rod is spinning on a smooth horizontal table with angular speed ω about a smooth fixed pivot at A. Suddenly A is released and the mid-point C of AB is fixed. Determine the new angular speed of the rod and the impulsive reaction at C. (L.U., Pt. II)

4. Two uniform thin rigid rods AB, BC of equal length and mass are freely hinged at B and the extremity A is freely hinged in a fixed point. The rods are at rest in a straight-line when an impulse is applied to the mid-point of BC perpendicular to the rods. Show that the initial angular velocity of the rod BC is double that of rod AB.

   Find also the ratio of the initial K.E.'s of the rods. (L.U., Pt. II)

5. Two equal uniform rods freely hinged at a common end are lying in a straight line on a smooth horizontal table and one rod is struck at its free end by a horizontal blow at right-angles to its length. Prove that the kinetic energy generated is greater in the ratio 7 : 5 than it would be if the rods were rigidly fastened together. (L.U., Pt. II)

6. Two equal uniform thin rods AB, BC, each of mass m, length l are freely joined at B and are in line, moving perpendicular to their length with velocity u. The mass-centre of AB is suddenly brought to rest. Find the angular velocity of each rod immediately after im-

pact and prove that four-sevenths of the original kinetic energy is lost by the impact. (L.U., Pt. II)

7. A uniform rod of mass M and length $2a$ moving parallel to itself with velocity v strikes a stationary particle of mass m which adheres to the rod at a distance x from the centre. Show that the magnitude of the impulse is $Mma^2v/\{Ma^2 + m(a^2 + 3x^2)\}$ and find the loss of kinetic energy. (L.U., Pt. II)

5.18 Moments about any Point

Let u be the change in velocity parallel to the x-axis of the centre of gravity of a lamina due to an impulse, v the change parallel to the y-axis and ω the change in angular velocity. If M is the mass and k the radius of gyration about the centre of gravity, the changes in linear and angular momentum are Mu, Mv and $Mk^2\omega$.

Let X and Y be the components of impulse parallel to the x- and y-axis and G the moment of impulse about the centre of gravity.

Then
$$X = Mu,$$
$$Y = Mv,$$
$$G = Mk^2\omega.$$

We thus have complete equivalence of the system of momentum and impulse of the lamina as shown in Fig. 117.

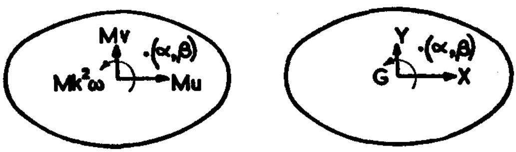

Fig. 117

The change in angular momentum about any point whose coordinates with reference to the centre of gravity are (α, β) is

$$Mk^2\omega + \beta Mu - \alpha Mv = G + \beta X - \alpha Y.$$

Hence we may equate the change in angular momentum about any point of the lamina to the moment of the impulses about that point.

It is often convenient to take moments about a point at which an induced impulse acts to find the change in momentum without evaluating the induced impulse.

Example 21. *In Example* 20 *let us find the velocities without finding the induced impulses.*

Taking moments about B for the rod BC we have

$$m\frac{a^2}{3}\omega_2 + mau_2 = J \times 2a. \qquad (1)$$

Taking moments about A for the two rods together we have

$$m\frac{a^2}{3}\omega_1 + mau_1 + m\frac{a^2}{3}\omega_2 + 3mau_2 = J \times 4a. \tag{2}$$

The geometrical equations are as before

$$u_1 - a\omega_1 = 0, \tag{3}$$

$$u_1 + a\omega_1 = u_2 - a\omega_2. \tag{4}$$

From these four equations the values of u_1, u_2, ω_1, ω_2 can be found as before.

Example 22. *A uniform sphere of radius a rolling with angular velocity ω on a horizontal plane meets a step of height $h(< a)$ at right-angles to its path. If the step is inelastic and rough enough to prevent slipping, find the angular velocity of the sphere after impact and show that it will mount the step if $\omega^2 > 70gh/(7a - 5h)^2$.*

Let ω' be the angular velocity immediately after impact. The sphere will then be turning about the point of contact with the step and will have velocity $a\omega'$ perpendicular to the radius to the point of contact. The impulse

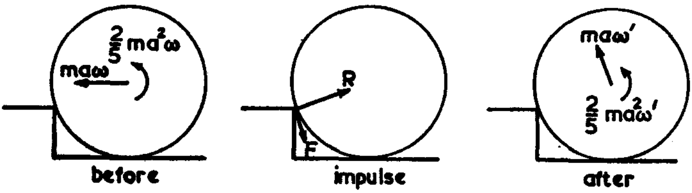

Fig. 118

will have components R and F as shown in Fig. 118, and will have no moment about the point of contact with the step.

Therefore $$\frac{2}{5}ma^2\omega + ma\omega(a - h) = \frac{2}{5}ma^2\omega' + ma\omega' \times a,$$

$$\omega' = \left(1 - \frac{5h}{7a}\right)\omega.$$

The kinetic energy after impact is

$$\frac{1}{2}ma^2(\omega')^2 + \frac{1}{2}\frac{2}{5}ma^2(\omega')^2$$

$$= \frac{7}{10}ma^2\left(1 - \frac{5h}{7a}\right)^2\omega^2,$$

$$= \frac{1}{70}m(7a - 5h)^2\omega^2.$$

The potential energy lost when the sphere has mounted the step, so that its centre has risen through a height h, is mgh.

The kinetic energy after impact must be greater than this if the sphere is to mount the step, and we must have

$$\frac{1}{70}m(7a - 5h)^2\omega^2 > mgh,$$

that is $$\omega^2 > \frac{70gh}{(7a - 5h)^2}.$$

EXERCISES 5 (*g*)

1. A uniform rod AB of mass m and length $2a$ is free to move on a smooth horizontal table about a pivot at A. Initially the rod is at rest and a particle of mass m is attached by a light string to the end B and is at rest at B. If the particle is projected with a velocity V along the table at right-angles to AB, show that the angular velocity with which the rod begins to move is $3mV/2a(M + 3m)$ and find the impulse on the pivot and the impulsive tension in the string. (L.U., Pt. II)

2. A uniform straight rod of mass m and length $2l$, standing upright on a table, is slightly disturbed and allowed to fall, no slipping occurring during the motion. When it reaches an inclination of 60° to the vertical it strikes a small fixed peg at a distance $\frac{1}{2}l$ from the lower end and begins to turn about it. Show that the angular velocity immediately after impact is $\frac{5}{7}\sqrt{(3g/l)}$, and find the impulse on the peg. (L.U., Pt. I)

3. Equal uniform bars PQ, QR, each of mass m freely jointed at Q lie at rest on a smooth horizontal table inclined to each other at an obtuse angle $\pi - \alpha$. A horizontal blow of impulse I is applied to P in direction perpendicular to PQ. Show that the velocity given to P has a component along PQ of magnitude
$$\frac{6I \sin \alpha \cos \alpha}{m(16 + 9 \sin^2 \alpha)}.$$ (L.U., Pt. II)

4. A uniform rectangular block of length $2b$ and square section of side $2a$ stands on one of its square ends on a smooth horizontal floor. It receives a horizontal blow J at a height $b/2$ above the floor normal to and in the centre line of one face. Determine the initial motion of the block, and show that if
$$J^2 > 8m^2g(4a^2 + b^2)\{(a^2 + b^2)^{1/2} - b\}/3b^2$$
the block will topple over. (L.U., Pt. II)

5. A uniform solid sphere of mass M and radius a, resting on a table, is given a horizontal blow J in a vertical plane containing the centre, at a height $3a/4$ above the table. Calculate the linear velocity of the centre and the angular velocity, immediately after the blow, and also the amount of kinetic energy which it generates. (L.U., Pt. II)

6. A cubical block of edge $2a$ stands on a horizontal plane rough enough to prevent sliding. If the plane is suddenly given a horizontal velocity v parallel to two vertical faces of the block, determine the initial motion of the block, and prove that the block will upset if
$$v^2 > \frac{16}{3}ag(\sqrt{2} - 1).$$ (L.U., Pt. II)

7. A uniform inelastic sphere of radius a, rolling without slipping along a horizontal plane with constant velocity v, comes in contact with a step of height $\frac{2}{5}a$ perpendicular to its plane of motion. Assuming that

the step is sufficiently rough to prevent slipping, prove that the sphere will surmount the step if $v^2 > \frac{420}{121}ag$. (L.U., Pt. II)

8. In an impact shear test a heavy pendulum carrying a hammer at its lower end is released from rest at an inclination of 60° to the vertical. At the bottom of its swing the hammer meets the test-piece at a point 3 ft. below the pivot and, after shearing through it, rises to an inclination of 30° to the vertical. If the mass of the pendulum and hammer is 50 lb., the distance of its centre of gravity from the pivot 2·25 ft., and its moment of inertia about the pivot 400 lb.ft.^{2}, find (*a*) the energy dissipated during the impulse and (*b*) the total impulse of test-piece on hammer. (L.U., Pt. II)

9. A uniform heavy circular cylinder is rolling along a horizontal plane with speed V when it meets a plane inclined to the horizontal at an angle $\alpha = \cos^{-1}(3/4)$, the line of intersection of the two planes being parallel to the axis of the cylinder. Assuming the impact to be inelastic and no slipping to occur, find with what speed the cylinder will begin to roll up the plane, and show that the magnitude of the impulse on the cylinder is $2MV/3$, where M is the mass of the cylinder. (L.U., Pt. II)

10. A uniform solid cube of mass M and edge of length $2a$ rests on one face on a smooth horizontal table. It is given a horizontal impulse I at the mid-point of one edge of its top face and perpendicular to that edge. Show that the impulsive reaction at the table is $3I/5$, and find the initial angular velocity of the cube.

    Show also that the cube will overturn in the subsequent motion if $I^2 > 10M^2ga(\sqrt{2} - 1)/3$. (L.U., Pt. II)

11. A thin uniform rod of mass m and length $2a$ falls freely with its length vertical. When the rod is moving with speed v the lower end strikes a smooth inelastic plane fixed at an angle of 30° to the horizontal. Prove that the magnitude of the impulsive reaction of the plane is $\frac{2\sqrt{3}}{7}mv$, and find the speed after impact of the end striking the plane. (L.U., Pt. II)

12. A uniform circular disc of radius a rolls without slipping, with its plane vertical on a rough horizontal table, the speed of the centre being v. If the disc strikes a rough inelastic step of height $h(< a)$ show that it will begin to turn about the top of the step with angular speed $(3a - 2h)v/(3a^2)$ assuming that it maintains contact with the step.

    Show also, that the disc will surmount the step if
    $$v^2 > 12a^2gh/(3a - 2h)^2.$$ (L.U., Pt. II)

5.19 Centre of Percussion

When a body which is turning about a fixed horizontal axis receives at some point a blow which is horizontal and perpendicular to the axis, there is in general an induced impulse at the axis. The point at which

the blow must be struck in order that the induced impulse shall be zero is called the *centre of percussion.*

Let M be the mass, Mk^2 the moment of inertia about the axis and h the distance of the centre of gravity G from the axis at O (Fig. 119).

Let a blow J whose line of action is at a distance c from the axis cause a change of angular velocity ω and let X be the induced impulse at the axis.

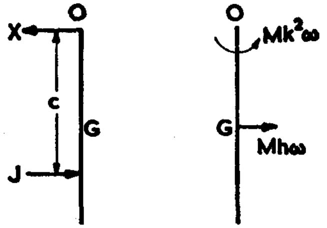

Fig. 119

Taking moments about O we have

$$J \times c = Mk^2\omega.$$

Also, since G has velocity $h\omega$, we have

$$J - X = Mh\omega.$$

If $X = 0$ we have from these two equations on eliminating J,

$$c = \frac{k^2}{h}.$$

Therefore the distance of the centre of percussion from the axis is equal to the length of the equivalent simple pendulum.

5.20 Impulsive Bending Moment and Shearing Force

When the change of momentum of a body and the external impulses which act on it are known, it is possible to find the internal impulses which act at any section of the body by considering the change of momentum of one of the two parts into which the body is divided by the section.

Let XX' (Fig. 120) be a section of the body and let u and v be the changes in velocity of the centre of gravity G of one part and ω the change in angular velocity. These quantities are known if the changes in velocity of the whole body are known.

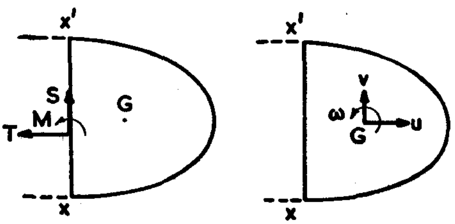

Fig. 120

Let the components of impulse at XX' be T and S perpendicular to and parallel to XX' and let M be the moment of impulse at the section. Then the system of impulses S, T and M together with the external impulses which act on this portion of the body causes the changes of momentum corresponding to u, v, and ω, and can be found from the equations of impulse.

T is an impulsive tension or compression, S is a shearing impulse and

is called the impulsive shearing force at the section. M is called the impulsive bending moment at the section.

Example 23. *A uniform rod of mass m and length 2a is at rest on a smooth horizontal table. It receives a horizontal blow J in a direction perpendicular to the rod through one end. Find the impulsive bending moment and shearing force at a distance 2c from the end which is struck.*

If u be the velocity imparted to the centre of gravity and ω the angular velocity we have

$$J = mu,$$

$$Ja = m\frac{a^2}{3}\omega.$$

Let S be the impulsive shearing force and M the impulsive bending moment at the section which cuts off a length $2c$ (Fig. 121). The mass of the portion cut off is mc/a, and the velocity given to its centre of gravity is $u + (a - c)\omega$.

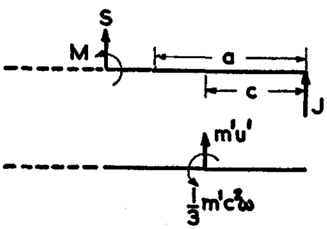

Fig. 121

Therefore we have

$$S + J = \frac{mc}{a}\{u + (a - c)\omega\},$$

$$M + Jc - Sc = \frac{mc}{a} \times \frac{c^2}{3}\omega.$$

Substituting the values obtained for u and ω we have

$$S = -\frac{J}{a^2}(a - c)(a - 3c),$$

$$M = -\frac{2Jc}{a^2}(a - c)^2.$$

EXERCISES 5 (*h*)

1. A uniform rod of mass M and length $2l$ is rotating with angular velocity ω, about an axis through its mid-point perpendicular to its length when it is brought to rest by one end striking a fixed stop. Find the impulse on the stop and that on the pivot, and show that the impulsive bending moment at the mid-point of the rod is $\frac{1}{6}Ml^2\omega$. (L.U., Pt. I)

2. Two equal uniform rods AB, BC each of length l and mass m are connected at B by a pin joint having sufficient friction to maintain the rods in a straight line as they turn, with spin ω, on a smooth horizontal plane, about a fixed vertical axis at A. The rods are brought to rest by meeting a fixed inelastic pin P which strikes BC so that no impulsive bending moment is produced at B. Find the distance of this pin from B, and the impulsive reactions at A, B and P. (L.U., Pt. II)

3. Two equal straight uniform rods AB, BC, each of mass m and length a, are jointed together at B. When the rods rest in line on a smooth

horizontal table C is jerked into motion with velocity v perpendicular to BC. If ABC remains a straight line after motion begins prove that the friction at the joint B must supply an impulsive couple of magnitude $\frac{1}{8}mva$. (L.U., Pt. II)

4. A uniform rod OA of mass m and length $2a$ turns freely in a horizontal plane about a fixed vertical axis through O. A similar rod AB is hinged to the first at A so that it turns in a horizontal plane. At the moment when OA, AB are in line and rotating with respective angular velocities ω, ω' in opposite senses, the hinge at A is suddenly locked. If the rods come to rest prove that $5\omega' = 11\omega$, and find the impulsive reaction on the axis through O in terms of ω. (L.U., Pt. II)

5. A rectangular trapdoor, of uniform thickness and mass M, is pivoted so as to turn about a line in its plane parallel to two sides, the length of either of the other sides being a. When closed it is in a horizontal plane and the side farthest from the pivot axis rests on a stop. What must be the position of the pivots in order that they may be subjected to no impulsive force when the door is allowed to fall into the closed position. If the door falls from the vertical position what will be the magnitude of the impulse on the stop? (C.U.)

CHAPTER 6

SMALL OSCILLATIONS AND VIBRATIONS

6.1 Oscillation with One Degree of Freedom

When a body is in a position of equilibrium and is slightly displaced it will, if the equilibrium be stable, tend to return to the equilibrium position and will in fact oscillate about it. We shall show that if the motion has only one degree of freedom and the amplitude of the oscillations is small the motion is approximately simple harmonic, and, if θ be a coordinate in terms of which the displacement can be expressed, the equation of motion is of the form

$$l\ddot{\theta} + g\theta = 0.$$

The solution of this equation is readily obtained and the period of the motion is $2\pi\sqrt{(l/g)}$. The length l is, therefore, the length of the equivalent simple pendulum.

This equation of motion is easily obtained by differentiating the energy equation and approximating to the equation obtained by supposing that the displacement, velocity and acceleration are all small quantities whose squares and products may be neglected.

Example 1. *A uniform hemisphere of mass M and radius a rests with its curved surface in contact with a rough horizontal plane. Find the period of its small oscillations about its equilibrium position.*

The centre of gravity is distant $\frac{3}{8}a$ from the centre and the moment of inertia about it is

$$M\left\{\frac{2}{5}a^2 - \left(\frac{3}{8}a\right)^2\right\}.$$

When the hemisphere has turned through an angle θ (Fig. 122) the distance of the point of contact, which is the instantaneous centre, from the centre of gravity, is

$$\left\{a^2 + \left(\frac{3}{8}a\right)^2 - \frac{3}{4}a^2\cos\theta\right\}^{1/2},$$

and the moment of inertia about the point of contact is

$$\frac{1}{20}Ma^2(28 - 15\cos\theta).$$

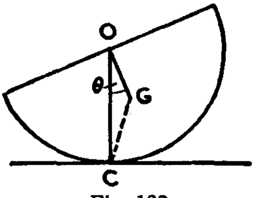

Fig. 122

The kinetic energy is therefore

$$\frac{1}{40}Ma^2(28 - 15\cos\theta)\dot{\theta}^2.$$

The potential energy in this position is $Mg\left(a - \frac{3}{8}a\cos\theta\right)$, and the energy equation is

$$\frac{1}{40}Ma^2(28 - 15\cos\theta)\dot{\theta}^2 + Mg\left(a - \frac{3}{8}a\cos\theta\right) = \text{constant.}$$

Differentiating this equation with respect to θ, and dividing by Ma, we have

$$\frac{1}{20}a(28 - 15\cos\theta)\ddot{\theta} + \frac{15}{40}a\sin\theta\,\dot{\theta}^2 + \frac{3}{8}g\sin\theta = 0.$$

In this equation we neglect θ^2 and $\dot{\theta}^2$, and replacing $\sin\theta$ by θ and $\cos\theta$ by 1, we have

$$\frac{13}{20}a\ddot{\theta} + \frac{3}{8}g\theta = 0,$$

that is

$$\frac{26a}{15}\ddot{\theta} + g\theta = 0.$$

Then the length of the equivalent simple pendulum is $26a/15$ and the period of the oscillation is $2\pi\sqrt{(26a/15g)}$.

6.2 Use of the Instantaneous Centre

We have seen (§ 5.13) that if $Mk_1{}^2\dot{\theta}$ be the angular momentum about the instantaneous centre the moment of the forces about the instantaneous centre is

$$Mk_1{}^2\ddot{\theta} + Mk_1\dot{k}_1\dot{\theta}.$$

Now k_1 is in general a function of θ and for small values of θ we may write

$$\begin{aligned} k_1 &= a + b\theta + c\theta^2 + \dots \\ &= a + b\theta, \text{ approximately,} \\ \dot{k}_1 &= b\dot{\theta}, \text{ approximately.} \end{aligned}$$

Therefore $\quad Mk_1\dot{k}_1\dot{\theta} = M(a + b\theta)b\dot{\theta}^2$

and this quantity may be neglected.

Also $\quad \begin{aligned} Mk_1{}^2\ddot{\theta} &= M(a^2 + 2ab\theta + \dots)\ddot{\theta} \\ &= Ma^2\ddot{\theta}, \text{ approximately.} \end{aligned}$

Therefore, when dealing with small oscillations we may equate the moment of the forces about the instantaneous centre to the change of angular momentum about it giving to the moment of inertia its value in the equilibrium position.

Thus for the hemisphere considered in Example 1, the moment of inertia about the instantaneous centre in the equilibrium position is $\frac{13}{20}Ma^2$. The moment of the weight about the instantaneous centre is

$Mg \times \frac{3}{8}a\sin\theta$ and we have

$$\frac{13}{20}Ma^2\ddot{\theta} = -Mg \times \frac{3}{8}a \sin\theta$$

$$= -\frac{3}{8}Mga\theta, \text{ approximately,}$$

that is $$\frac{26}{15}a\ddot{\theta} + g\theta = 0.$$

6.3 Small Oscillations and Stability

When a body moves with one degree of freedom so that its position may be specified by one coordinate θ, the expression for its kinetic energy will be of the form

$$f(\theta)\dot{\theta}^2,$$

where $f(\theta)$ may vary with θ but is positive for all values of θ.

If the forces are conservative the potential energy is also a function of θ which we may write as $V(\theta)$.

The energy equation is, therefore,

$$f(\theta)\dot{\theta}^2 + V(\theta) = \text{constant.}$$

Differentiating this equation with respect to θ we have

$$2f(\theta)\ddot{\theta} + \dot{\theta}^2\left\{\frac{d}{d\theta}f(\theta)\right\} + \frac{d}{d\theta}V(\theta) = 0. \tag{1}$$

In a position of equilibrium we must have $\frac{d}{d\theta}V(\theta) = 0$, since this is the condition that if $\dot{\theta}$ is zero $\ddot{\theta}$ is also zero and the body will remain in this position.

Therefore, if $\theta = \alpha$ be a position of equilibrium we have

$$\left(\frac{dV}{d\theta}\right)_{\theta=\alpha} = 0.$$

Now suppose that $\theta - \alpha$ is small so that we can expand $f(\theta)$ and $V(\theta)$ in a Taylor series in powers of $(\theta - \alpha)$.

We have

$$f(\theta) = f(\alpha) + (\theta - \alpha)\left(\frac{df}{d\theta}\right)_{\theta=\alpha} + \ldots,$$

$$V(\theta) = V(\alpha) + (\theta - \alpha)\left(\frac{dV}{d\theta}\right)_{\theta=\alpha} + \frac{(\theta-\alpha)^2}{2!}\left(\frac{d^2V}{d\theta^2}\right)_{\theta=\alpha} + \ldots,$$

$$\frac{dV}{d\theta} = \left(\frac{dV}{d\theta}\right)_{\theta=\alpha} + (\theta - \alpha)\left(\frac{d^2V}{d\theta^2}\right)_{\theta=\alpha} + \ldots,$$

$$= (\theta - \alpha)\left(\frac{d^2V}{d\theta^2}\right)_{\theta=\alpha}, \text{ approximately.}$$

Substituting in equation (1) and neglecting $\dot{\theta}^2$ and $(\theta - \alpha)\ddot{\theta}$ we have

$$2f(\alpha)\ddot{\theta} + (\theta - \alpha)\left(\frac{d^2V}{d\theta^2}\right)_{\theta=\alpha} = 0.$$

Since $f(\alpha)$ is positive this is an equation of simple harmonic motion, the variable being $\theta - \alpha$, if

$$\left(\frac{d^2V}{d\theta^2}\right)_{\theta=\alpha} > 0.$$

In this case the equilibrium is stable and the period of the oscillatory motion about the position $\theta = \alpha$ is

$$2\pi\left\{2f(\alpha) \div \left(\frac{d^2V}{d\theta^2}\right)_{\theta=\alpha}\right\}^{1/2}.$$

Example 2. *A body rests in equilibrium on another fixed rough body, the portions of the two bodies near the point of contact being spherical with radii r and R respectively and the centre of gravity of the upper body distant h from the point of contact. Find the condition that the equilibrium be stable and if it is stable find the period of small oscillations about the point of contact, the moment of inertia of the upper body about the point of contact being Mk_1^2.*

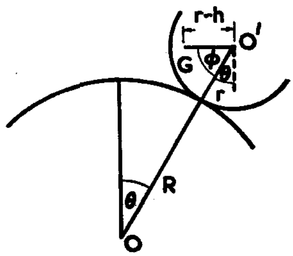

Fig. 123

When the radius to the point of contact of the lower body is inclined at an angle θ to the vertical let the upper body have turned through an angle $\theta + \phi$ by rolling (Fig. 123). Then since the arcs of rolling are equal

$$r\phi = R\theta,$$

$$\theta + \phi = \frac{R + r}{r}\theta.$$

The potential energy in this position is proportional to the height of the centre of gravity G above the centre O of the fixed body.

We have

$$V = Mg\left\{(R + r)\cos\theta - (r - h)\cos\frac{R + r}{r}\theta\right\},$$

$$\frac{dV}{d\theta} = (R + r)Mg\left\{-\sin\theta + \frac{r - h}{r}\sin\frac{R + r}{r}\theta\right\},$$

$$\frac{d^2V}{d\theta^2} = (R + r)Mg\left\{-\cos\theta + \frac{(r - h)(R + r)}{r^2}\cos\frac{R + r}{r}\theta\right\}.$$

In the position of equilibrium $\theta = 0$ and we have

$$\left(\frac{d^2V}{d\theta^2}\right)_0 = (R + r)Mg\left\{-1 + \frac{(r - h)(R + r)}{r^2}\right\}$$

$$= (R + r)Mg\frac{Rr - hR - hr}{r^2}$$

$$= \frac{(R + r)Rh}{r}Mg\left(\frac{1}{h} - \frac{1}{R} - \frac{1}{r}\right).$$

The equilibrium will be stable if this quantity is positive, that is if

$$\frac{1}{h} > \frac{1}{R} + \frac{1}{r}.$$

The angular velocity being $\frac{R+r}{r}\dot{\theta}$ we have for the kinetic energy

$$\frac{1}{2}Mk_1^2\left(\frac{R+r}{r}\right)^2\dot{\theta}^2.$$

If the equilibrium is stable we have for the small oscillations

$$Mk_1^2\left(\frac{R+r}{r}\right)^2\ddot{\theta} + \frac{(R+r)Rh}{r}Mg\left(\frac{1}{h} - \frac{1}{R} - \frac{1}{r}\right)\theta = 0,$$

and the period is

$$2\pi\left\{k_1^2(R+r) \div Rrhg\left(\frac{1}{h} - \frac{1}{R} - \frac{1}{r}\right)\right\}^{1/2}.$$

If $\frac{1}{h} = \frac{1}{R} + \frac{1}{r}$, $\left(\frac{d^2V}{d\theta^2}\right)_0 = 0$. It is easily seen that in this case

$$\left(\frac{d^3V}{d\theta^3}\right)_0 = 0,$$

$$\left(\frac{d^4V}{d\theta^4}\right)_0 = (R+r)Mg\left\{1 - \left(\frac{R+r}{r}\right)^2\right\}.$$

Hence $\frac{d^4V}{d\theta^4}$ is negative and for a small displacement $\ddot{\theta}$ is positive so that the equilibrium is unstable.

EXERCISES 6 (*a*)

1. A uniform thin rod of length $2l$ is balanced on a rough circular cylinder of radius a, the cylinder being fixed with its axis horizontal and the rod being perpendicular to the axis. If the rod be slightly displaced by rolling on the cylinder, find the period of its small oscillations.

2. A uniform hemisphere of radius a rests with its plane face horizontal and its curved surface in contact with a smooth horizontal plane. Find the period of its small oscillations about this position.

3. A uniform bar of length $2a$ has a small ring attached to one end, free to slide along a smooth horizontal wire. If the bar makes small oscillations under gravity, show that the period is $2\pi(a/3g)^{1/2}$. (L.U., Pt. II)

4. A thin hollow cylinder of radius a has a particle of equal mass attached symmetrically to its inner surface. If the system is disturbed from its position of stable equilibrium on a rough horizontal table and then left to itself show that, when the radius to the particle makes an angle θ with the downward vertical,

$$a\dot{\theta}^2(2 - \cos\theta) - g\cos\theta = \text{constant}.$$

Hence or otherwise, prove that the period of small oscillations is $2\pi\sqrt{(2a/g)}$. (L.U., Pt. II)

5. A uniform thin rod of length $2a$ is suspended from two fixed points at the same level by vertical strings each of length b, attached to its ends. It is turned through an angle α about a vertical axis through its centre, and let go, the strings being kept taut. Construct the energy equation for the subsequent motion. If α is small determine the period of the oscillation. (L.U., Pt. II)

6. A pulley of mass M is fixed to a shaft which can turn with negligible friction in horizontal bearings. The centre of mass, G, of the system is distant a from the axis of the shaft and the radius of gyration is k about that axis. A mass m hangs from a light cord coiled round the shaft and attached to a point on it. In equilibrium the line joining G to the axis is at an angle α to the vertical. Show that the period of small oscillations about the equilibrium position is

$$2\pi\{(Mk^2 + Ma^2 \sin^2 \alpha)/mag \cos \alpha\}^{1/2}$$

when m moves vertically. (L.U., Pt. I)

7. A uniform rod AB, of mass m and length $2a$, can turn freely about a horizontal axis at A. An elastic string of unstretched length a and modulus λ is attached to B and to a point C vertically over A, where $AC = 2a$. If $3\lambda > mg$, prove that there is a stable position of equilibrium given by $\sin \frac{1}{2}\alpha = \lambda/(4\lambda - mg)$, where α is the angle the rod makes with the upward vertical.

If the rod is slightly displaced from this position, show that the period of a small oscillation is

$$2\pi\{(4\lambda - mg)\, am/3(5\lambda - mg)(3\lambda - mg)\}^{1/2}.$$

(L.U., Pt. II)

8. A uniform circular cylinder of radius a and height h has a plane face joined to the plane face of a uniform hemisphere of radius a. The body rests with the curved surface of the hemisphere in contact with a rough horizontal plane. Show that the equilibrium is stable if $2h^2 < a^2$. If $h = \frac{1}{2}a$, find the length of the equivalent simple pendulum for small oscillations about the position of equilibrium.

9. A uniform rod AB of mass M lb. and length a ft. is hinged at A and supported by a spring attached to B and to a point C, which is a ft. vertically above A. When the rod is horizontal the spring is just unstretched and the strength of the spring is such that a pull of M lb. elongates it a length of e ft. Show that a position of stable equilibrium is attained when ϕ the inclination of AB to the vertical is given by $\cos \frac{1}{2}\phi = a\sqrt{2}/(2a - e)$.

Show that the time of a small oscillation about this position is given by $t = 2\pi\{(e\sqrt{2} \cos \frac{1}{2}\phi)/(3g \sin^2 \frac{1}{2}\phi)\}^{1/2}$.

(C.U.)

10. A pair of wheels and axle are free to roll without resistance along a horizontal track. The radius of the wheels is 3 ft., the moment of inertia of the system about the centre line of the axle is 9000 lb.ft.^{2}, and its mass is 1500 lb.

Each wheel is then fitted with a mass of 200 lb., which may be regarded as concentrated at a point distant 2 ft. from the wheel centre, the two masses being in the same radial direction. The wheels are

placed so that the added masses are vertically above the axle and then slightly displaced. Calculate the linear velocity of the wheels at the instant when the masses have descended to their lowest position.

If this system is mounted so that the axle turns freely in fixed horizontal bearings, find the time of a small oscillation about the position of statical equilibrium. (C.U.)

6.4 Static Displacement of Springs

Let a weight W be suspended from a light helical spring of stiffness s and let δ be the static displacement of W, that is the extension of the spring in the equilibrium position (Fig. 124).

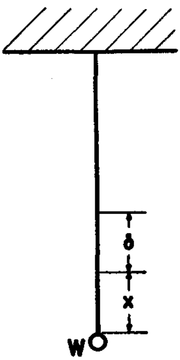

Fig. 124

Then $$W = s.\delta.$$

When W is displaced a further distance x downwards the additional energy stored in the spring is

$$\frac{1}{2}\{s\delta + s(\delta + x)\}x,$$

and the loss of potential energy by W is $Wx = xs\delta$.

Hence the energy equation is

$$\frac{W}{2g}\dot{x}^2 + \frac{1}{2}(2s\delta + sx)x - xs\delta = \text{constant},$$

that is $$\frac{W}{2g}\dot{x}^2 + \frac{1}{2}sx^2 = \text{constant}.$$

Differentiating with respect to x we have

$$\frac{W}{g}\ddot{x} + sx = 0,$$

that is $$\delta\ddot{x} + gx = 0.$$

Therefore δ is the length of the equivalent simple pendulum and the period of the oscillation is $2\pi\sqrt{(\delta/g)}$.

6.5 Vibrations of Light Beams

The concept of the static displacement may be used to write down the period of oscillation of a light beam.

For example, if a light cantilever of length l and flexural rigidity EI carries a load W at its free end the deflexion is

$$\frac{Wl^3}{3EI}.$$

The deflexion being proportional to the weight, the motion for small oscillations is equivalent to that of a spring and the length δ of the equivalent simple pendulum is the static displacement.

We have, therefore, $$\delta = \frac{Wl^3}{3EI}.$$

Example 3. *A light beam of length l and flexural rigidity EI is end supported and carries a weight W at a distance c from one end. Find the period of the small oscillations of the weight.*

The deflexion equation is (Fig. 125),

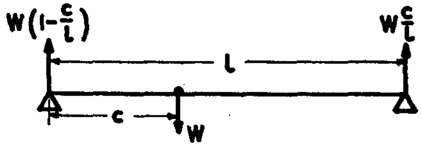

Fig. 125

$$EI\frac{d^2y}{dx^2} = -\frac{W(l-c)}{l}x + \Big[W(x-c)\Big]_{x>c},$$

$$EI\frac{dy}{dx} = -\frac{W(l-c)}{l}\cdot\frac{x^2}{2} + \left[\frac{W(x-c)^2}{2}\right]_{x>c} + C,$$

$$EIy = -\frac{W(l-c)}{l}\cdot\frac{x^3}{6} + \left[W\frac{(x-c)^3}{6}\right]_{x>c} + Cx.$$

Since $y = 0$ when $x = l$, $C = \dfrac{Wc(l-c)(2l-c)}{6l}$,

$$\Big(EIy\Big)_{x=c} = -\frac{Wc^3(l-c)}{6l} + \frac{Wc^2(l-c)(2l-c)}{6l}$$

$$= \frac{Wc^2(l-c)^2}{3l}.$$

The static displacement is therefore

$$\delta = \frac{Wc^2(l-c)^2}{3lEI},$$

and the period of small oscillations is $2\pi\sqrt{(\delta/g)}$.

Example 4. *A light uniform flexible lath of length l is clamped vertically at its upper end, and carries a small mass m at its lower end. Show that the time period of its small flexural oscillations is equal to that of a simple pendulum of length $(nl - \tanh nl)/n$, where EI is the flexural rigidity of the lath and $EIn^2 = mg$.*

(L.U., Pt. II)

We require to find the force P that will give a static deflexion Y to the lower end (Fig. 126).

If y be the deflexion at distance x from the clamp the bending moment at this point is $-mg(Y - y) + P(l - x)$.

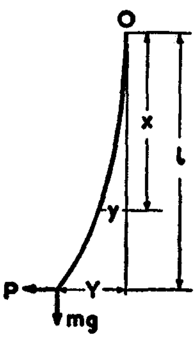

Fig. 126

Therefore $$EI\frac{d^2y}{dx^2} = -mg(Y-y) + P(l-x),$$

that is $$\frac{d^2y}{dx^2} - n^2y = -n^2Y + \frac{Pn^2}{mg}(l-x).$$

The particular integral is easily found and the complete solution of this equation is

$$y = A \cosh nx + B \sinh nx + Y - \frac{P}{mg}(l - x).$$

When $x = 0$, we have $y = \dfrac{dy}{dx} = 0$, and hence

$$0 = A + Y - \frac{Pl}{mg},$$

$$0 = Bn + \frac{P}{mg}.$$

Also $y = Y$ when $x = l$, therefore

$$Y = A \cosh nl + B \sinh nl + Y.$$

Substituting for A and B we find a relation between P and Y, namely,

$$P = \frac{mng}{nl - \tanh nl} . Y.$$

The restoring force when the displacement of the mass is Y is therefore P and the equation of motion is

$$m\ddot{Y} = -\frac{mng}{nl - \tanh nl} Y.$$

The length of the equivalent simple pendulum is therefore

$$(nl - \tanh nl)/n.$$

6.6 Forced Oscillation of a Compound Pendulum

Let the pendulum have mass M, moment of inertia Mk^2 about its axis and let its centre of gravity be distant h from the axis.

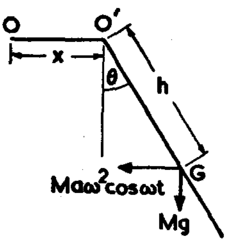

Fig. 127

Suppose the axis is made to oscillate horizontally about O (Fig. 127) so that its distance from O at time t is $x = a \cos \omega t$.

Then the axis has an acceleration $a\omega^2 \cos \omega t$ towards O. Since every particle of the body has the acceleration of the axis together with its acceleration relative to the axis the centre of gravity has in addition to the other effective forces an effective force $Ma\omega^2 \cos \omega t$ parallel to $O'O$.

Therefore, the equation of motion is

$$Mk^2\ddot{\theta} - Ma\omega^2 \cos \omega t \times h \cos \theta = -Mgh \sin \theta,$$

that is, for small values of θ

$$\ddot{\theta} + \frac{gh}{k^2}\theta = \frac{ah}{k^2}\omega^2 \cos \omega t.$$

Differential equations of this type were considered in Chapter 2, and the solution was seen to be

$$\theta = A\cos\left[\sqrt{\left(\frac{gh}{k^2}\right)}t + \varepsilon\right] + \frac{ah\omega^2}{gh - k^2\omega^2}\cos\omega t,$$

where A and ε are constants determined by the initial conditions.

Resonance will be set up if $\omega^2 = \dfrac{gh}{k^2}$.

EXERCISES 6 (*b*)

1. When a weight is suspended from a light helical spring and oscillates the length of the equivalent simple pendulum is a. When the same weight is suspended from a second spring the length of the equivalent simple pendulum is b. Find the length of the equivalent simple pendulum if the weight is suspended from the two springs coupled together (i) in series, (ii) in parallel.

2. A uniform light beam is freely supported at its ends in a horizontal position. Show that if a load W is placed on the beam at a distance equal to one-quarter of its length from one support the period of its oscillation is the same as that of a load $9W/16$ placed at the centre of the beam.

3. A uniform light beam of length l and constant flexural rigidity EI is end supported and its ends are clamped horizontally. If a mass W is placed at its centre and oscillates vertically, find the length of the equivalent simple pendulum.

4. A uniform rod of length l is supported by its upper end and is free to swing in a vertical plane. If the upper end is given a horizontal reciprocal motion, the displacement x from the mean position being given by $x = a\sin 2\pi nt$, obtain an expression for the angular movement of the rod after the motion has become steady, on the assumption that the oscillations generated are small in amplitude. (C.U.)

5. A horizontal beam is mounted on a vertical axis passing through its centre of gravity and its moment of inertia about the axis is I. The beam is made to perform angular oscillations under the influence of an alternating couple $L\cos 2\pi nt$ acting in a horizontal plane. If the friction at the pivot introduces a couple resisting motion of magnitude $\mu\dfrac{d\theta}{dt}$, where θ is the angular displacement of the beam, show that when the motion has become steady

$$\theta = -\frac{L\cos(2\pi nt + \phi)}{2\pi n(\mu^2 + 4\pi^2n^2I^2)^{1/2}}, \text{ where } \tan\phi = \frac{\mu}{2\pi nI}.$$

Show that if $\theta = 0$ and $\dfrac{d\theta}{dt} = 0$, when $t = 0$, the earlier stage of the motion is given by

$$\theta = L\left\{\frac{I_e^{-\mu t/I}}{\mu^2 + 4\pi^2n^2I^2} - \frac{\cos(2\pi nt + \phi)}{2\pi n(\mu^2 + 4\pi^2n^2I^2)^{1/2}}\right\}. \qquad \text{(C.U.)}$$

6. A light beam of length l and constant flexural rigidity EI is clamped at its lower end so that it is vertical and a small mass m is fixed to its upper end. If the mass is slightly displaced, find the length of the equivalent simple pendulum in the subsequent oscillation.

7. The lower end of a uniform light cantilever of length l and flexural rigidity EI is clamped at an angle α to the vertical and a mass W is attached to the upper end. Show that the mean deflexion of the end of the beam is $\tan\alpha(\tan n - nl)/n$, where $EIn^2 = W\cos\alpha$, and find the length of the equivalent simple pendulum for oscillations about this position.

6.7 Deflexion of a Loaded String

Let the string be of length l fixed between two points at the same level. Let x be the horizontal distance measured from O (Fig. 128), y the deflexion and w the intensity of load at any point. Let T_0 and S_0 be the horizontal and vertical components of tension at O, T the tension at a point $P(x, y)$ at which the angle of slope is θ.

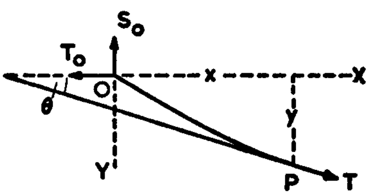

Fig. 128

Then from the equilibrium of the portion OP

$$T\cos\theta = T_0,$$

$$T\sin\theta = S_0 - \int_0^x w\,dx.$$

Therefore
$$T_0\tan\theta = S_0 - \int_0^x w\,dx,$$

that is
$$T_0\frac{dy}{dx} = S_0 - \int_0^x w\,dx,$$

$$T_0\frac{d^2y}{dx^2} = -w.$$

If w is constant this leads to the parabolic equation

$$y = \frac{w}{2T_0}x(l - x).$$

If w_0 is the weight of string which is uniform we have

$$w\,dx = w_0\,ds,$$

so that
$$T_0\frac{d^2y}{dx^2} = -w_0\frac{ds}{dx},$$

and this leads to the equation of a catenary.

6.8 Transverse Vibrations of a Taut String

Suppose that the string is vibrating in a horizontal plane so that an element at a distance x from O (Fig. 129) has a horizontal displacement

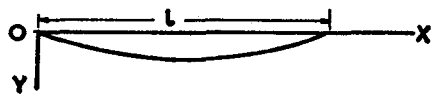

Fig. 129

y. Since the string is taut the mass of an element is $m\delta x$, where m is the mass per unit length of the string, and its acceleration is $\ddot{y}$, therefore the effective force on the element is $\left(\frac{m\delta x}{g}\right)\frac{d^2y}{dt^2}$.

The element may be considered as being in equilibrium under the forces acting on it together with the reversed effective force, and the displacement at time t will be that of an element of a loaded string where the load w is given by

$$w = -\frac{m}{g}\frac{d^2y}{dt^2}.$$

Hence, if T be the tension in the string, the differential equation satisfied by the displacement of the string is

$$T\frac{d^2y}{dx^2} = -w,$$

that is
$$T\frac{\partial^2 y}{\partial x^2} - \frac{m}{g}\frac{\partial^2 y}{\partial t^2} = 0. \tag{1}$$

The use of partial derivatives is necessary to distinguish the variation of y with x and with t.

A solution of this partial differential equation which is consistent with harmonic vibration of the string is obtained by writing

$$y = z(x)\cos\omega t,$$

where z depends on x and not on t and ω is a constant. It is evident that z is the maximum displacement of the string at any point.

Substituting in (1) we have

$$T\frac{d^2z}{dx^2} + \frac{m}{g}\omega^2 z = 0.$$

Hence
$$z = A\cos\sqrt{\left(\frac{m}{gT}\right)}\omega x + B\sin\sqrt{\left(\frac{m}{gT}\right)}\omega x.$$

Since $z = 0$ when $x = 0$, $A = 0$, and since also $z = 0$ when $x = l$ we have

$$0 = B\sin\sqrt{\left(\frac{m}{gT}\right)}\omega l.$$

This equation can only be satisfied with B not equal to zero if

$$\sqrt{\left(\frac{m}{gT}\right)}\omega l = n\pi,$$

where n is an integer.

Hence we have solutions

$$z_n = B_n \sin \frac{n\pi x}{l},$$

$$y_n = B_n \sin \frac{n\pi x}{l} \cos \sqrt{\left(\frac{gT}{m}\right)} \frac{n\pi t}{l},$$

for $n = 1, 2, 3, \ldots$, the constants B_n being as yet undetermined.

The complete solution is the sum of a number of the partial solutions y_n, the constants B_n being determined by the initial conditions.

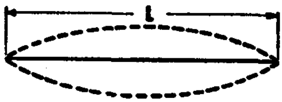

Fig. 130

The vibration with the lowest frequency y_1 is called the fundamental vibration of the string. If this vibration only were present the string would move between the two extreme positions shown in Fig. 130, with period $2l\sqrt{(m/gT)}$.

The vibrations with higher frequencies are called harmonics. Thus for $n = 2$ we have a vibration whose extreme positions are shown in Fig. 131, with a node at $x = \frac{1}{2}l$ and period $l\sqrt{(m/gT)}$.

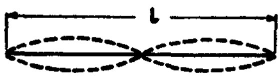

Fig. 131

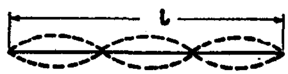

Fig. 132

For $n = 3$ the extreme positions are as shown in Fig. 132 and the period is $\frac{2}{3}l\sqrt{(m/gT)}$.

The complete picture of the vibration of the string is, therefore, that of the fundamental mode combined with harmonics of higher frequencies and diminishing amplitudes.

6.9 Vibration of a Uniform Beam

Consider a uniform beam of length l and mass m per unit length simply supported at its ends (Fig. 133).

If an element $m\delta x$ at x from O has displacement y the effective force

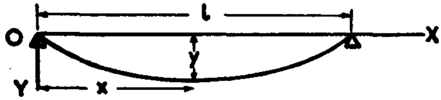

Fig. 133

on the element is $\frac{(m\delta x)}{g}\frac{d^2y}{dt^2}$ and its displacement at time t will be that due to a load w where

$$w = -\frac{m}{g}\frac{d^2y}{dt^2}.$$

Hence the deflexion equation for the beam is

$$EI\frac{d^4y}{dx^4} = w,$$

that is
$$EI\frac{\partial^4 y}{\partial x^4} + \frac{m}{g}\frac{\partial^2 y}{\partial t^2} = 0. \tag{1}$$

Writing $y = z\cos\omega t$, where z depends on x alone and is, therefore, the maximum deflexion we have,

$$\frac{d^4z}{dx^4} - \frac{m\omega^2}{gEI}z = 0,$$

that is
$$\frac{d^4z}{dx^4} - a^4z = 0, \tag{2}$$

where
$$a = \left(\frac{m\omega^2}{gEI}\right)^{1/4}.$$

The auxiliary equation for (2) obtained by substituting $z = e^{\lambda x}$ is

$$\lambda^4 - a^4 = 0,$$

and has solutions $\lambda = \pm a$ and $\pm ia$.

Hence the solution of (2) is of the form

$$\begin{aligned} z &= ae^{ax} + be^{-ax} + ce^{iax} + de^{-iax} \\ &= A\cosh ax + B\sinh ax + C\cos ax + D\sin ax \end{aligned} \tag{3}$$

where A, B, C, D are constants.

When $x = 0$, $z = 0$ and, since the bending moment is zero, $\frac{d^2z}{dx^2} = 0$,

therefore
$$\begin{aligned} 0 &= A + C, \\ 0 &= a^2A - a^2C, \end{aligned}$$

and we have
$$A = C = 0.$$

When $x = l$, $z = 0$ and $\frac{d^2z}{dx^2} = 0$, therefore,

$$\begin{aligned} B\sinh al + D\sin al &= 0, \\ a^2B\sinh al - a^2D\sin al &= 0. \end{aligned}$$

Hence, we must have $B = 0$, and if D is not zero

$$\begin{aligned} \sin al &= 0, \\ al &= n\pi, \end{aligned}$$

where n is an integer.

That is
$$\frac{m\omega^2}{gEI} = \left(\frac{n\pi}{l}\right)^4$$
$$\omega = \sqrt{\left(\frac{gEI}{m}\right)}.\frac{n^2\pi^2}{l^2}.$$

Hence we have solutions
$$y_n = D_n \sin\frac{n\pi x}{l} \cos\sqrt{\left(\frac{gEI}{m}\right)}.\frac{n^2\pi^2}{l^2}t.$$

The fundamental mode of vibrations has, therefore, a period $\sqrt{\left(\frac{m}{gEI}\right)}.\frac{2l^2}{\pi}$ and the period corresponding to the nth harmonic is equal to this period divided by n^2.

6.10 Beam with Clamped Ends

If a beam of length l has its ends clamped so as to be horizontal the fundamental equation is the same but the boundary conditions are

$$\text{when } x = 0,\ z = 0,\ \frac{dz}{dx} = 0,$$
$$\text{when } x = l,\ z = 0,\ \frac{dz}{dx} = 0.$$

Substituting in Equation (3) of § 6.9 we have from the conditions when $x = 0$,
$$0 = A + C$$
$$0 = aB + aD.$$

Hence $z = A(\cosh ax - \cos ax) + B(\sinh ax - \sin ax).$

From the conditions when $x = l$ we have
$$0 = A(\cosh al - \cos al) + B(\sinh al - \sin al)$$
$$0 = A(\sinh al + \sin al) + B(\cosh al - \cos al).$$

If these equations are to give consistent values for the ratio A/B we must have
$$\frac{\cosh al - \cos al}{\sinh al + \sin al} = \frac{\sinh al - \sin al}{\cosh al - \cos al},$$
$$\cosh^2 al + \cos^2 al - 2\cosh al \cos al = \sinh^2 al - \sin^2 al,$$
that is
$$\cosh al \cos al = 1.$$

This transcendental equation has to be solved by graphical or numerical methods to obtain the critical values of al.

The least value of a which satisfies this equation is
$$al = 1{\cdot}506\pi,$$
and this gives
$$\omega = \sqrt{\left(\frac{gEI}{m}\right)}\left(\frac{1{\cdot}506\pi}{l}\right)^2$$

so that the fundamental period is

$$\sqrt{\left(\frac{m}{gEI}\right)}\frac{l^2}{1\cdot 135\pi}.$$

Since cosh al increases rapidly the higher frequencies are given approximately by taking cos $al = 0$, or

$$al = \left(n + \frac{1}{2}\right)\pi,$$

for $n = 2, 3, \ldots$, and

$$\omega = \sqrt{\left(\frac{gEI}{m}\right)}\left\{\frac{(n + 1/2)\pi}{l}\right\}^2.$$

6.11 Vibration of a Cantilever

For the vibration of a cantilever of length l the end conditions are

$$\text{when } x = 0, \quad z = 0, \quad \frac{dz}{dx} = 0,$$

$$\text{when } x = l, \quad \frac{d^2z}{dx^2} = 0, \quad \frac{d^3z}{dx^3} = 0,$$

that is, there is no bending moment or shearing force at the free end.

Thus we have as in § 6.10

$$z = A(\cosh ax - \cos ax) + B(\sinh ax - \sin ax).$$

From the conditions when $x = l$ we have

$$0 = A(\cosh al + \cos al) + B(\sinh al + \sin al)$$
$$0 = A(\sinh al - \sin al) + B(\cosh al + \cos al).$$

For consistent values of the ratio A/B we have

$$\frac{\sinh al + \sin al}{\cosh al + \cos al} = \frac{\cosh al + \cos al}{\sinh al - \sin al},$$

that is $\cosh al \cos al = -1.$

The least value of al which satisfies this transcendental equation is

$$al = 0\cdot 60\pi,$$

and this gives $\omega = \sqrt{\left(\frac{gEI}{m}\right)}\left(\frac{0\cdot 60\pi}{l}\right)^2,$

so that the fundamental period is $\sqrt{\left(\frac{m}{gEI}\right)}\cdot\frac{l^2}{0\cdot 18\pi}.$

Since cosh al increases rapidly the roots corresponding to higher frequencies are given approximately by taking cos $al = 0$, or

$$al = \left(n - \frac{1}{2}\right)\pi,$$

for $n = 2, 3, \ldots$, and $\omega = \sqrt{\left(\frac{gEI}{m}\right)}\left\{\frac{(n - 1/2)\pi}{l}\right\}^2.$

6.12 Whirling of a Disc

Consider a disc of mass M rotating with angular velocity Ω on a light shaft through its centre and let its centre of gravity G be at a distance e from the centre of symmetry C (Fig. 134).

If the point C is deflected a distance y from the line of the bearings there is a restoring force λy due to the elasticity of the shaft. For example, if the disc is at the centre of a uniform hinged shaft of length l and flexural rigidity EI we have

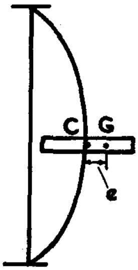

Fig. 134

$$\lambda = \frac{48EIg}{l^3}.$$

The natural frequency of vibration of the mass M on the shaft is given by the equation

$$M\ddot{y} = -\lambda y,$$

and the period is $\dfrac{2\pi}{\omega}$, where $\omega^2 = \dfrac{\lambda}{M}$.

If the centre of gravity G is moving in a circle of radius $y + e$ and the motion is steady the centrifugal force is balanced by the restoring force on the shaft and we have

$$\begin{aligned} M(y+e)\Omega^2 &= \lambda y \\ &= M\omega^2 y. \end{aligned}$$

Hence
$$y = \frac{e\Omega^2}{\omega^2 - \Omega^2}.$$

It follows that the deflection increases rapidly as Ω approaches the value ω and is theoretically infinite when $\Omega = \omega$.

If Ω increases beyond this speed y becomes negative and decreases as Ω increases until in the limit when ω/Ω is negligible $y = -e$. In this position the centre of gravity of the disc is in the line of the bearings.

6.13 Whirling of a Shaft

Consider a uniform shaft of length l, mass m per unit length and flexural rigidity EI running in bearings which do not restrain the direction of its ends. If the shaft is rotating with angular velocity ω and an element which is distant x from one end O (Fig. 135) has deflexion z from the line of the bearings the centrifugal force on the element is

$$\left(\frac{m\delta x}{g}\right)z\omega^2.$$

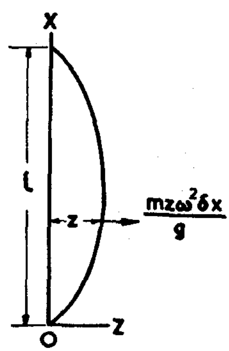

Fig. 135

Hence the deflexion is the same as for a load w per unit length, where $w = mz\omega^2/g$ and we have

$$EI\frac{d^4z}{dx^4} - \frac{m\omega^2}{g}z = 0.$$

This is the same equation that was found in § 6.9 for the modes of vibration of a uniform beam and, the end conditions being the same, we have solutions of the form

$$z = D \sin ax,$$

where $a^4 = m\omega^2/(gEI)$, which satisfy the end conditions only if $a = \dfrac{n\pi}{l}$.

Hence the critical speeds of rotation of the shaft at which such solutions are possible are given by

$$\frac{m\omega^2}{gEI} = \left(\frac{n\pi}{l}\right)^4$$

$$\omega = \frac{n^2\pi^2}{l^2}\sqrt{\frac{gEI}{m}},$$

for $n = 1, 2, 3 \ldots$.

When the speed of rotation of the shaft reaches one of these critical speeds the phenomenon of *whirling* occurs and the shaft may become violently unstable. This is explained by considering the centrifugal force as having components $(mz\omega^2/g)\cos \omega t$ and $(mz\omega^2/g)\sin \omega t$ in fixed directions. We thus have alternating disturbing forces in two directions at right-angles and (as was seen in Chapter 2) when the frequency of the disturbing force coincides with the natural frequency of the beam there is resonance. At speeds between the critical speeds stability returns.

EXERCISES 6 (c)

1. A uniform beam of flexural rigidity EI, mass m per unit length and length l is clamped horizontally at one end and freely supported at the other end. Show that the modes of vibration of the beam are given by the equation $\tan al = \tanh al$, where $gEIa^4 = m\omega^2$ and that the frequency of the fundamental mode of vibration is

$$\sqrt{\left(\frac{gEI}{m}\right)}\frac{(3 \cdot 93)^2}{2l^2\pi}.$$

2. The second moments of area of an R.S.J. about principal axes through the centroid of a section are 35 in.^{4} and 7·93 in.^{4} and the weight is 24 lb. per ft. If the joist is 10 ft. long and the ends are fixed but direction free, find the frequencies of the fundamental modes of vibration in the directions of the principal axes.

3. A disc 1 ft. in diameter and of mass 5 lb. is fitted at the centre of a light shaft $\frac{1}{2}$ in. in diameter and of length 1 ft. which runs freely in bearings. Taking $E = 30 \times 10^6$ lb./in.^{2}, find the whirling speed if the disc is slightly eccentric.

4. A steel shaft $\frac{3}{4}$ in. in diameter and 2 ft. long runs freely in bearings so that its ends are direction free. Taking $E = 30 \times 10^6$ lb./in.^{2} and the steel as weighing 480 lb./ft.^{3}, find the least speed of rotation at which whirling will occur.

5. If the shaft in Question 4 had been direction fixed at its ends show that the whirling speed would have been about 125 per cent greater.

6. A steel shaft 3 in. in diameter, weighing 480 lb. per cu. ft. is supported by bearings 6 ft. apart. The shaft is unloaded but transmits a torque. Calculate the whirling speed. $E = 3 \times 10^7$ lb./in.^{2}
(L.U., Pt. II)

6.14 Fourier Series

There are many problems in engineering which can be solved only by expressing some function of a variable x (such as the loading on a beam) as a series of trigonometrical functions known as a Fourier series.

An example of such a series is

$$\frac{1}{4}x^2 = \frac{1}{12}\pi^2 - \cos x + \frac{1}{4}\cos 2x - \frac{1}{9}\cos 3x + \frac{1}{16}\cos 4x \ldots.$$

We shall see that the sum of this series is equal to $\frac{1}{4}x^2$ for all values of x between $x = -\pi$ and $x = \pi$.

If x_1 be a value of x in this range, that is $-\pi < x_1 < \pi$, the sum of the series is unchanged if $x_1 + 2\pi$, $x_1 + 4\pi$, or $x_1 + 2n\pi$ is substituted for x_1. The values of $\frac{1}{4}x_1^2$ and $\frac{1}{4}(x_1 + 2\pi)^2$ are not, however, the same. Hence the trigonometrical series being periodic can only be equal to $\frac{1}{4}x^2$ within the range $-\pi < x < \pi$.

The graphs of $\frac{1}{4}x^2$ and of the series are shown in Fig. 136 and it is seen that they coincide only between $-\pi$ and π.

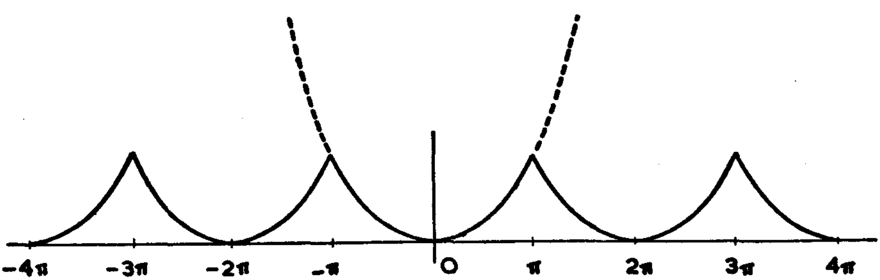

Fig. 136

The general form for a Fourier series is

$$f(x) = \frac{1}{2}a_0 + a_1 \cos x + a_2 \cos 2x + \dots$$
$$+ b_1 \sin x + b_2 \sin 2x + \dots$$
$$= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

The coefficients a_n and b_n are determined by formulae and the series is equal to $f(x)$ only in the range $-\pi < x < \pi$. The fundamental range is often taken as $0 < x < 2\pi$ with a slight alteration in the formulae for the coefficients.

The term $(a_n \cos nx + b_n \sin nx)$ is called the nth harmonic of the series. The numerical values of the harmonics diminish as n increases and for a good approximation to the function it is often sufficient to take only the first two or three harmonics.

Fourier series can be found for any functions which remain finite and have only a finite number of discontinuities in the fundamental interval. They are particularly useful in dealing with functions which have discontinuities. A function of this kind may be defined by formulae such as the following:

$$f(x) = 0, \; -\pi < x < -\frac{\pi}{2},$$
$$f(x) = 1, \; -\frac{\pi}{2} < x < 0,$$
$$f(x) = 2, \quad 0 < x < \frac{\pi}{2},$$
$$f(x) = 3, \quad \frac{\pi}{2} < x < \pi.$$

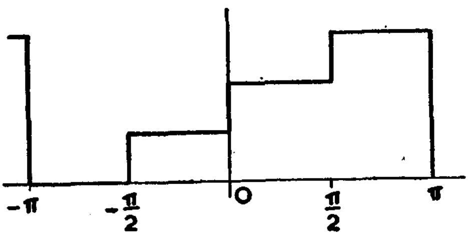

Fig. 137

The graph of this function is shown in Fig. 137.

6.15 Trigonometrical Integrals

To determine the coefficients a_n and b_n we use the following relations:

$$\int_{-\pi}^{\pi} \sin nx \cos mx dx = 0,$$
$$\int_{-\pi}^{\pi} \sin nx \sin mx dx = 0 \text{ if } n \neq m,$$
$$= \pi \text{ if } n = m,$$
$$\int_{-\pi}^{\pi} \cos nx \cos mx dx = 0 \text{ if } n \neq m,$$
$$= \pi \text{ if } n = m.$$

In these equations n and m are integers.

The above relations are all proved by similar methods.

Thus if $n \neq m$,

$$\int_{-\pi}^{\pi} \sin nx \cos mx dx = \frac{1}{2}\int_{-\pi}^{\pi} \{\sin (n+m)x + \sin (n-m)x\} dx$$
$$= \frac{1}{2}\left[-\frac{\cos (n+m)x}{n+m} - \frac{\cos (n-m)x}{n-m}\right]_{-\pi}^{\pi} = 0,$$

$$\int_{-\pi}^{\pi} \sin nx \sin mx dx = \frac{1}{2}\left[-\frac{\sin (n+m)x}{n+m} + \frac{\sin (n-m)x}{n-m}\right]_{-\pi}^{\pi} = 0,$$

$$\int_{-\pi}^{\pi} \cos nx \cos mx dx = \frac{1}{2}\left[\frac{\sin (n+m)x}{n+m} + \frac{\sin (n-m)x}{n-m}\right]_{-\pi}^{\pi} = 0.$$

If $n = m$,

$$\int_{-\pi}^{\pi} \sin nx \cos nx dx = \frac{1}{2}\left[-\frac{\cos 2nx}{2n}\right]_{-\pi}^{\pi} = 0,$$

$$\int_{-\pi}^{\pi} \sin nx \sin nx dx = \frac{1}{2}\int_{-\pi}^{\pi} (1 - \cos 2nx) dx$$
$$= \frac{1}{2}\left[x - \frac{\sin 2nx}{2n}\right]_{-\pi}^{\pi} = \pi,$$

$$\int_{-\pi}^{\pi} \cos nx \cos nx dx = \frac{1}{2}\int_{-\pi}^{\pi} (1 + \cos 2nx) dx$$
$$= \frac{1}{2}\left[x + \frac{\sin 2nx}{2n}\right]_{-\pi}^{\pi} = \pi.$$

6.16 Formulae for Coefficients of a Fourier Series

Let the series be, for $-\pi < x < \pi$,

$$f(x) = \frac{1}{2}a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx).$$

Then

$$f(x) \cos nx = \frac{1}{2}a_0 \cos nx + \sum_{m=1}^{\infty} (a_m \cos mx \cos nx + b_m \sin mx \sin nx).$$

$$\int_{-\pi}^{\pi} f(x) \cos nx dx = \frac{1}{2}a_0 \int_{-\pi}^{\pi} \cos nx dx + \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \cos mx \cos nx dx$$
$$+ \sum_{m=1}^{\infty} b_m \int_{-\pi}^{\pi} \sin mx \cos nx dx.$$

Now the integral multiplying a_m is zero except for the term where $m = n$, and the integral multiplying b_m is zero for all values of m.

Therefore we have $(n \neq 0)$

$$\int_{-\pi}^{\pi} f(x) \cos nx dx = \pi a_n.$$

If $n = 0$ we have

$$\int_{-\pi}^{\pi} f(x) dx = \pi a_0.$$

Similarly, multiplying throughout by $\sin nx$ we have

$$\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{2}a_0\int_{-\pi}^{\pi}\sin nx dx + \sum_{m=1}^{\infty} a_m\int_{-\pi}^{\pi}\cos mx\sin nx dx + \sum_{m=1}^{\infty} b_m\int_{-\pi}^{\pi}\sin mx\sin nx dx.$$

On the right-hand side every term is zero except that containing b_n and we have

$$\int_{-\pi}^{\pi} f(x)\sin nx dx = \pi b_n.$$

We thus have the formulae for the Fourier coefficients

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx,$$

for $n = 0, 1, 2$, etc.

It should be noted that the first term of the series is taken as $\frac{1}{2}a_0$, so that a_0 may be found from the general formula for a_n. It is usually necessary, however, to calculate a_0 separately, and $\frac{1}{2}a_0$ is the mean value of $f(x)$ over the range $-\pi < x < \pi$.

Example 5. *Find a Fourier series expansion for $f(x) = x$, $-\pi < x < \pi$.*

We have
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x dx = \frac{1}{\pi}\left[\frac{1}{2}x^2\right]_{-\pi}^{\pi} = 0.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos nx dx$$

$$= \frac{1}{\pi}\left[\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right]_{-\pi}^{\pi} = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx dx$$

$$= \frac{1}{\pi}\left[-\frac{x}{n}\cos nx + \frac{1}{n^2}\sin nx\right]_{-\pi}^{\pi}$$

$$= -\frac{2}{n}\cos n\pi.$$

Hence
$$x = -2\sum_{n=1}^{\infty}\frac{1}{n}\cos n\pi\sin nx$$

$$= 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x\ldots\right).$$

Example 6. *Find a Fourier series expansion for a function of x defined by* (Fig. 138),

$$f(x) = 0, \quad -\pi < x < 0,$$
$$f(x) = 1, \quad 0 < x < \pi.$$

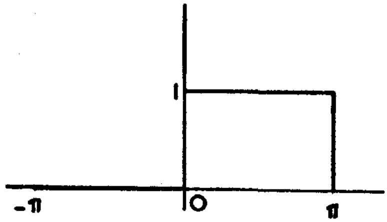

Fig. 138

In calculating the coefficients the integrals taken between $x = -\pi$ and $x = 0$ yield nothing and we have

$$a_0 = \frac{1}{\pi}\int_0^\pi 1.dx = 1.$$

$$a_n = \frac{1}{\pi}\int_0^\pi \cos nx dx = \frac{1}{n\pi}\Big[\sin nx\Big]_0^\pi = 0.$$

$$b_n = \frac{1}{\pi}\int_0^\pi \sin nx dx = \frac{1}{n\pi}\Big[-\cos nx\Big]_0^\pi$$
$$= \frac{1}{n\pi}(1 - \cos n\pi).$$

Therefore
$$f(x) = \frac{1}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos n\pi}{n}\sin nx$$
$$= \frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right).$$

6.17 Approximation by Successive Harmonics

In Example 6 we found a Fourier series for what is called the square-wave function defined by $f(x) = 0, -\pi < x < 0, f(x) = 1, 0 < x < \pi$.

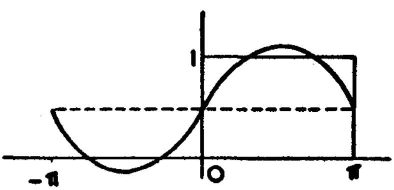

Fig. 139

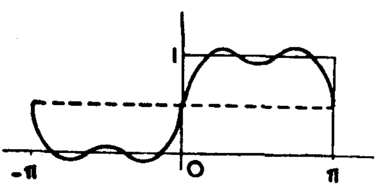

Fig. 140

In Fig. 139 the value of the constant term and the first harmonic, that is $\frac{1}{2} + \frac{2}{\pi}\sin x$ is plotted on the same graph as $f(x)$.

In Fig. 140 another harmonic is added and the graph of $\frac{1}{2} + \frac{2}{\pi}\left(\sin x + \frac{1}{3}\sin 3x\right)$ is drawn.

In Fig. 141 two more harmonics are added and the approximation begins to be evident.

It will be seen that at $x = 0$ where the function is discontinuous the sum of the series is $\frac{1}{2}$ which is the mean value of the function at either side of the discontinuity.

The sum of a large number of terms of the series in the neighbourhood of $x = 0$ is of interest and is shown in Fig. 142.

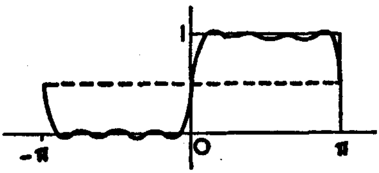

Fig. 141

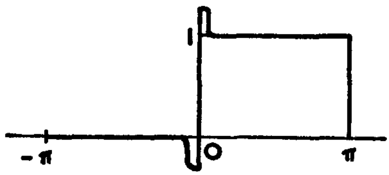

Fig. 142

The sum of the series differs from the value of $f(x)$ by a small amount near the discontinuity and the approximation to $f(x)$ by the Fourier series in this neighbourhood is not good. This is known as *Gibb's phenomenon.*

6.18 Use of Leibnitz Theorem

When $f(x)$ is a polynomial in x it is necessary to use the method of integration by parts when calculating the coefficients. This may be done by using Leibnitz theorem for the nth differential coefficient of a product of two functions with $n = -1$. Thus if u and v be two functions of x we have

$$D^n(u.v) = uD^nv + nDuD^{n-1}v + \frac{n(n-1)}{2!}D^2uD^{n-2}v + \ldots$$

and with $n = -1$,

$$D^{-1}(u.v) = uD^{-1}v - Du.D^{-2}v + D^2u.D^{-3}v - D^3u.D^{-4}v + \ldots.$$

Thus if $u = x^3$ and $v = \cos nx$,

$$D^{-1}(x^3 \cos nx) = x^3\left(\frac{1}{n}\sin nx\right) - 3x^2\left(-\frac{1}{n^2}\cos nx\right) + 6x\left(-\frac{1}{n^3}\sin nx\right) - 6\left(\frac{1}{n^4}\cos nx\right).$$

That is

$$\int x^3 \cos nx dx = \frac{x^3}{n}\sin nx + \frac{3x^2}{n^2}\cos nx - \frac{6x}{n^3}\sin nx - \frac{6}{n^4}\cos nx.$$

Example 7. *Find a Fourier series expansion for* $f(x) = x^3 + \pi x^2$, $-\pi < x < \pi$.

We have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^3 + \pi x^2)dx$$

$$= \frac{1}{\pi}\left[\frac{x^4}{4} + \pi\frac{x^3}{3}\right]_{-\pi}^{\pi} = \frac{2\pi^3}{3}.$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^3 + \pi x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x^3 + \pi x^2)\left(\frac{1}{n}\sin nx\right) - (3x^2 + 2\pi x)\left(-\frac{1}{n^2}\cos nx\right) + (6x + 2\pi)\left(-\frac{1}{n^3}\sin nx\right) - 6\left(\frac{1}{n^4}\cos nx\right)\right]_{-\pi}^{\pi}$$

$$= \frac{4\pi}{n^2}\cos n\pi.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi}(x^3 + \pi x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x^3 + \pi x^2)\left(-\frac{1}{n}\cos nx\right) - (3x^2 + 2\pi x)\left(-\frac{1}{n^2}\sin nx\right) + (6x + 2\pi)\left(\frac{1}{n^3}\cos nx\right) - 6\left(\frac{1}{n^4}\sin nx\right)\right]_{-\pi}^{\pi}$$

$$= -\frac{2\pi^2}{n}\cos n\pi + \frac{12}{n^3}\cos n\pi.$$

Hence

$$f(x) = \frac{\pi^3}{3} + \sum_{n=1}^{\infty}\left\{\left(\frac{4\pi}{n^2}\right)\cos n\pi\cos nx + \left(-\frac{2\pi^2}{n} + \frac{12}{n^3}\right)\cos n\pi\sin nx\right\}.$$

6.19 Functions Defined in an Arbitrary Range

If a function $f(x)$ is defined for $-l < x < l$ we may obtain an expansion of the function for this range of values of x by a simple change of variable. Let $z = \frac{\pi x}{l}$, then the function $f(x) = f\left(\frac{lz}{\pi}\right)$ is a function of z defined for $-\pi < z < \pi$ and can be expanded in the Fourier series

$$f\left(\frac{lz}{\pi}\right) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nz + b_n\sin nz).$$

Substituting for z we have

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right).$$

Example 8. *A function of x is equal to x^2 for $-l < x < l$. Find an expansion of the function in a trigonometrical series valid for this range.*

Let
$$z = \frac{\pi x}{l},$$

$$x^2 = \frac{l^2}{\pi^2}z^2.$$

The coefficients of the Fourier series for l^2z^2/π^2 are

$$a_0 = \frac{l^2}{\pi^3}\int_{-\pi}^{\pi} z^2dz = \frac{2}{3}l^2.$$

$$a_n = \frac{l^2}{\pi^3}\int_{-\pi}^{\pi} z^2 \cos nz dz$$

$$= \frac{l^2}{\pi^3}\left[z^2\left(\frac{1}{n}\sin nz\right) - 2z\left(-\frac{1}{n^2}\cos nz\right) + 2\left(-\frac{1}{n^3}\sin nz\right)\right]_{-\pi}^{\pi}$$

$$= \frac{4l^2}{n^2\pi^2}\cos n\pi.$$

$$b_n = \frac{l^2}{\pi^3}\int_{-\pi}^{\pi} z^2 \sin nz dz$$

$$= \frac{l^2}{\pi^3}\left[z^2\left(-\frac{1}{n}\cos nz\right) - 2z\left(-\frac{1}{n^2}\sin nz\right) + 2\left(\frac{1}{n^3}\cos nz\right)\right]_{-\pi}^{\pi}$$

$$= 0.$$

Therefore
$$\frac{l^2}{\pi^2}z^2 = \frac{1}{3}l^2 + \sum_{n=1}^{\infty}\frac{4l^2}{n^2\pi^2}\cos n\pi \cos nz,$$

that is
$$x^2 = \frac{1}{3}l^2 + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos n\pi}{n^2}\cos\frac{n\pi x}{l}.$$

Example 9. *If the function $f(x)$ can be expanded in the Fourier series $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$, show that*

$$a_n + ib_n = \frac{1}{\pi}\int_0^{2\pi} f(x)e^{inx}dx.$$

Hence or otherwise, expand the function $x(x-1)(x-2)$ in a full range Fourier series valid in the interval (0, 2) *giving the general term.*

(L.U., Pt. II)

The fundamental range is here taken as 0 to 2π and we have for the coefficients of the Fourier series

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx dx,$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx dx,$$

and
$$a_n + ib_n = \frac{1}{\pi}\int_0^{2\pi} f(x)(\cos nx + i\sin nx)dx,$$

$$= \frac{1}{\pi}\int_0^{2\pi} f(x)e^{inx}dx.$$

To change the interval from (0, 2) to (0, 2π) we write $z = \pi x$ so that the function of z to be expanded is $z(z-\pi)(z-2\pi)\ \pi^3$.

$$a_n + ib_n = \frac{1}{\pi^4}\int_0^{2\pi}(z^3 - 3\pi z^2 + 2\pi^2 z)e^{inz}dz$$

$$= \frac{1}{\pi^4}\left[\frac{1}{ni}(z^3 - 3\pi z^2 + 2\pi^2 z)e^{inz} - \frac{1}{(ni)^2}(3z^2 - 6\pi z + 2\pi^2)e^{inz} + \frac{1}{(ni)^3}(6z - 6\pi)e^{inz} - \frac{6}{(ni)^4}e^{inz}\right]_0^{2\pi}$$

$$= \frac{1}{\pi^4}\cdot\frac{1}{(ni)^3}\cdot 12\pi$$

$$= \frac{12i}{n^3\pi^3}.$$

Hence, $a_n = 0$, $b_n = 12/n^3\pi^3$ and we have

$$\frac{z(z-\pi)(z-2\pi)}{\pi^3} = \frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{\sin nz}{n^3},$$

that is
$$x(x-1)(x-2) = \frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\sin n\pi x.$$

EXERCISES 6 (*d*)

1. Find a Fourier series expansion for $f(x) = x^2 + 2x$ for $-\pi < x < \pi$.

2. A function $f(x)$ has period 2π and $f(x) = 1$, $-\pi < x < -\frac{1}{2}\pi$, $f(x) = -1$, $-\frac{1}{2}\pi < x < \frac{\pi}{2}$, $f(x) = 1$, $\frac{\pi}{2} < x < \pi$. Find a Fourier series for the function.

3. Find a Fourier series expansion for e^x for $-\pi < x < \pi$.

4. A function $f(x)$ repeats itself every time x is increased or diminished by a multiple of $2a$ and $f(-x) = -f(x)$. From $x = 0$ to $x = \frac{1}{2}a$, $f(x) = x$, and from $x = \frac{1}{2}a$ to $x = a$, $f(x) = a - x$.

   Show that
$$f(x) = \frac{4a}{\pi^2}\left\{\sin\frac{\pi x}{a} - \frac{1}{3^2}\sin\frac{3\pi x}{a} + \frac{1}{5^2}\sin\frac{5\pi x}{a}\cdots\right\}.$$

5. Express as a Fourier series the following repeating function, giving all terms up to and including the third harmonic:
$$\theta = 0 \text{ to } \theta = \frac{2\pi}{3},\quad y = a,$$
$$\theta = \frac{2\pi}{3} \text{ to } \theta = 2\pi,\quad y = 0,$$
$$\theta = 2\pi \text{ to } \theta = \frac{8\pi}{3},\quad y = a, \text{ etc.}$$
(C.U.)

6. Given that $f(x) = 0$, $-\pi < x < 0$ and $f(x) = \sin x$, $0 < x < \pi$, find a Fourier series expansion for the function for $-\pi < x < \pi$.

6.20 Symmetry about the Origin

If $f(x) = f(-x)$ the function $f(x)$ is symmetrical about $x = 0$ and is called an *even* function of x. For example, x^2, $x \sin x$, $x^2 \cos x$, are all even functions of x.

If $f(x) = -f(-x)$ the function $f(x)$ is skew symmetrical about $x = 0$ and is called an *odd* function of x. For example x^3, $x \cos x$, $x^2 \sin x$, are all odd functions of x.

If $f(x)$ is an even function of x the Fourier coefficients are a_n and b_n. We have

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx dx, \\ &= \frac{1}{\pi}\int_{-\pi}^{0} f(x) \cos nx dx + \frac{1}{\pi}\int_{0}^{\pi} f(x) \cos nx dx. \end{aligned}$$

Substituting $x = -z$ in the integral between $-\pi$ and 0 we have

$$\begin{aligned} dx &= -dz \\ f(x) &= f(-z) = f(z) \\ \cos nx &= \cos nz \\ \int_{-\pi}^{0} &= -\int_{0}^{-\pi} \end{aligned}$$

and hence $$\frac{1}{\pi}\int_{-\pi}^{0} f(x) \cos nx dx = \frac{1}{\pi}\int_{0}^{\pi} f(z) \cos nz dz.$$

Since z is only a variable of integration the two parts of the integral for a_n are equal and we have

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(x) \cos nx dx.$$

Similarly $$b_n = \frac{1}{\pi}\int_{-\pi}^{0} f(x) \sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} f(x) \sin nx dx.$$

Since $\sin nx = -\sin nz$ there is a further change of sign when $-z$ is substituted for x, the two parts of the integral are of opposite signs and we have

$$b_n = 0.$$

If $f(x)$ is an odd function of x we have $f(x) = f(-z) = -f(z)$, there is a further change of sign in the transformation of the first part of the integrals for a_n and b_n and we have

$$\begin{aligned} a_n &= 0, \\ b_n &= \frac{2}{\pi}\int_{0}^{\pi} f(x) \sin nx dx. \end{aligned}$$

Thus the Fourier series for an even function of x contains cosine terms only and the coefficients are

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx dx.$$

The Fourier series of an odd function of x contains only sine terms and the coefficients are

$$b_n = \frac{2}{\pi}\int_0^\pi f(x)\sin nx dx.$$

Example 10. *A function $f(x)$ is defined as*

$$f(x) = -x, \quad -\pi < x < 0,$$
$$f(x) = x, \quad 0 < x < \pi.$$

Find the Fourier series expansion of the function.

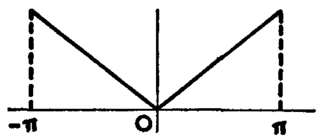

Fig. 143

The function $f(x)$ is even and its graph is as shown in Fig. 143. It contains only cosine terms and we have

$$a_0 = \frac{2}{\pi}\int_0^\pi x dx = \pi.$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi x\cos nx dx \\ &= \frac{2}{\pi}\left[\frac{x}{n}\sin nx + \frac{1}{n^2}\cos nx\right]_0^\pi \\ &= \frac{2}{\pi n^2}(\cos n\pi - 1). \end{aligned}$$

$$\begin{aligned} f(x) &= \frac{1}{2}\pi - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos n\pi}{n^2}\cos nx \\ &= \frac{1}{2}\pi - \frac{4}{\pi}\left(\cos x + \frac{1}{9}\cos 3x + \frac{1}{25}\cos 5x + \dots\right). \end{aligned}$$

6.21 Cosine and Sine Series

If a function $f(x)$ is given only between $x = 0$ and $x = \pi$ we may treat it as if it were an even function of x or as if it were an odd function of x.

In the former case we may write

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx,$$

where

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx dx.$$

In the latter case we may write

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where
$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx dx.$$

Thus for a function defined only between 0 and π we may write down either a cosine series or a sine series. Either series represents the function between 0 and π.

Example 11. *Find Fourier cosine and sine series for a function $f(x)$ given by*

$$f(x) = \cos \frac{1}{2}x, \ 0 < x < \pi.$$

For the cosine series

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \cos \frac{1}{2}x dx = \frac{4}{\pi},$$

$$\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^{\pi} \cos \frac{1}{2}x \cos nx dx,\\
&= \frac{1}{\pi}\int_0^{\pi} \left\{ \cos \left(n + \frac{1}{2}\right)x + \cos \left(n - \frac{1}{2}\right)x \right\} dx,\\
&= \frac{1}{\pi}\left[\frac{\sin (n + 1/2)x}{n + 1/2} + \frac{\sin (n - 1/2)x}{n - 1/2}\right]_0^{\pi},\\
&= \frac{1}{\pi}\left\{\frac{\cos n\pi}{n + 1/2} - \frac{\cos n\pi}{n - 1/2}\right\}\\
&= -\frac{4}{\pi}\frac{\cos n\pi}{4n^2 - 1}.
\end{aligned}$$

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\cos n\pi}{4n^2 - 1} \cos nx.$$

For the sine series

$$\begin{aligned}
b_n &= \frac{2}{\pi}\int_0^{\pi} \cos \frac{1}{2}x \sin nx dx\\
&= \frac{1}{\pi}\int_0^{\pi} \left\{ \sin \left(n + \frac{1}{2}\right)x + \sin \left(n - \frac{1}{2}\right)x \right\} dx\\
&= \frac{1}{\pi}\left[-\frac{\cos (n + 1/2)x}{n + 1/2} - \frac{\cos (n - 1/2)x}{n - 1/2}\right]_0^{\pi}\\
&= \frac{1}{\pi}\left(\frac{1}{n + 1/2} + \frac{1}{n - 1/2}\right)\\
&= \frac{8}{\pi}\frac{n}{4n^2 - 1}.
\end{aligned}$$

$$f(x) = \frac{8}{\pi}\sum_{n=1}^{\infty} \frac{n}{4n^2 - 1} \sin nx.$$

Example 12. *The loading on a beam of length l is $wx(l - x)$ at a distance x from one end. Find a Fourier sine series for the loading.*

We have
$$f(x) = wx(l - x), \quad 0 < x < l.$$
Let
$$z = \frac{\pi x}{l},$$
$$f(x) = \frac{wl^2}{\pi^2} z(\pi - z), \quad 0 < z < \pi.$$

For the sine series in z
$$b_n = \frac{2}{\pi}\int_0^\pi \frac{wl^2}{\pi^2} z(\pi - z) \sin nz\,dz$$
$$= \frac{2wl^2}{\pi^3}\left[z(\pi - z)\left(-\frac{1}{n}\cos nz\right) - (\pi - 2z)\left(-\frac{1}{n^2}\sin nz\right) + (-2)\left(\frac{1}{n^3}\cos nz\right)\right]_0^\pi$$
$$= \frac{4wl^2}{\pi^3 n^3}(1 - \cos n\pi).$$
$$\frac{wl^2}{\pi^2} z(\pi - z) = \frac{4wl^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1 - \cos n\pi}{n^3}\sin nz,$$
that is
$$wx(l - x) = \frac{4wl^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1 - \cos n\pi}{n^3}\sin\frac{n\pi x}{l}$$
$$= \frac{8wl^2}{\pi^3}\left(\sin\frac{\pi x}{l} + \frac{1}{3^3}\sin\frac{3\pi x}{l} + \frac{1}{5^3}\sin\frac{5\pi x}{l}\ldots\right)$$
$$= \frac{8wl^2}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n + 1)^3}\sin\frac{(2n + 1)\pi x}{l}.$$

6.22 Symmetry about $\frac{\pi}{2}$

The graphs of $\cos 2x$, $\cos 4x$, ..., $\cos 2nx$, $\sin x$, $\sin 3x$, ..., $\sin (2n + 1)x$ are all symmetrical about $x = \frac{1}{2}\pi$ (Fig. 144).

The graphs of $\cos x$, $\cos 3x$, ..., $\cos (2n + 1)x$, $\sin 2x$, $\sin 4x$, ..., $\sin 2nx$ are all skew symmetrical about $x = \frac{\pi}{2}$ (Fig. 145).

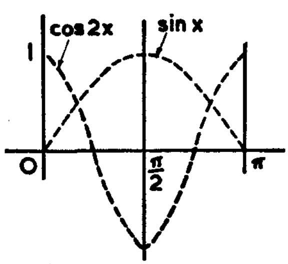

Fig. 144

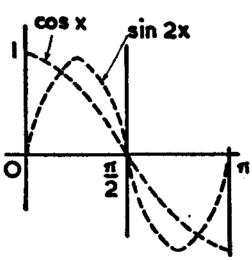

Fig. 145

That is
$$\begin{aligned}\cos 2nx &= \cos 2n(\pi - x)\\ \sin(2n+1)x &= \sin(2n+1)(\pi - x)\\ \cos(2n+1)x &= -\cos(2n+1)(\pi - x)\\ \sin 2nx &= -\sin 2n(\pi - x).\end{aligned}$$

It follows that if a function of x, $f(x)$ defined for $0 < x < \pi$ is symmetrical about $x = \frac{1}{2}\pi$, so that $f(x) = f(\pi - x)$, the cosine series for $f(x)$ will contain only cosines of *even* multiples of x and the sine series will continue only sines of *odd* multiples of x.

If $f(x)$ is skew symmetrical about $x = \frac{1}{2}\pi$, so that $f(x) = -f(\pi - x)$, the cosine series for $f(x)$ will contain only cosines of *odd* multiples of x and the sine series only sines of *even* multiples of x.

Suppose that $f(x) = f(\pi - x)$ and that we require the cosine series.

$$\begin{aligned}a_n &= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx\\ &= \frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(x)\cos nx dx + \frac{2}{\pi}\int_{\frac{\pi}{2}}^{\pi} f(x)\cos nx dx.\end{aligned}$$

In the integral from $\frac{1}{2}\pi$ to π let $x = \pi - z$.

$$\begin{aligned}f(x) &= f(\pi - z) = f(z)\\ dx &= -dz\\ \cos nx &= \cos n(\pi - z) = \cos n\pi \cos nz,\end{aligned}$$

$$\int_{\frac{\pi}{2}}^{\pi} = -\int_{\pi}^{\frac{\pi}{2}}.$$

Hence
$$\begin{aligned}\int_{\frac{\pi}{2}}^{\pi} f(x)\cos nx dx &= \cos n\pi \int_0^{\frac{\pi}{2}} f(z)\cos nz dz\\ &= \cos n\pi \int_0^{\frac{\pi}{2}} f(x)\cos nx dx.\end{aligned}$$

Therefore
$$a_n = \frac{2}{\pi}(1 + \cos n\pi)\int_0^{\frac{\pi}{2}} f(x)\cos nx dx.$$

Hence we have
$$a_{2n+1} = 0,$$

$$a_{2n} = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} f(x)\cos 2nx dx.$$

A similar proof applies to the other cases of symmetry and skew symmetry about $\frac{1}{2}\pi$.

The results may be summed up as follows:

$f(x) = f(\pi - x)$, cosine series, $a_{2n+1} = 0$

$$a_{2n} = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} f(x) \cos 2nx dx,$$

sine series, $b_{2n} = 0$

$$b_{2n+1} = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} f(x) \sin (2n + 1)x dx.$$

$f(x) = -f(\pi - x)$, cosine series, $a_{2n} = 0$

$$a_{2n+1} = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} f(x) \cos (2n + 1)x dx,$$

sine series, $b_{2n+1} = 0$

$$b_{2n} = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} f(x) \sin 2nx dx.$$

Example 18. *A function $f(x)$ is defined for $0 < x < \pi$ so that* (Fig. 146),

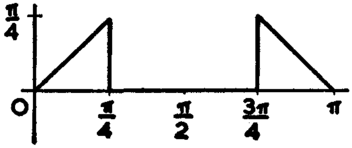

Fig. 146

$$f(x) = x, \quad 0 < x < \frac{\pi}{4},$$

$$f(x) = 0, \quad \frac{\pi}{4} < x < \frac{\pi}{2},$$

$$f(\pi - x) = f(x).$$

Find a cosine series for $f(x)$.

$$a_0 = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} f(x)dx$$

$$= \frac{4}{\pi}\int_0^{\frac{\pi}{4}} x dx = \frac{\pi}{8}.$$

$$a_{2n} = \frac{4}{\pi}\int_0^{\frac{\pi}{4}} x \cos 2nx dx$$

$$= \frac{4}{\pi}\left[\frac{x}{2n} \sin 2nx + \frac{1}{4n^2} \cos 2nx\right]_0^{\frac{\pi}{4}}$$

$$= \frac{4}{\pi}\left\{\frac{\pi}{8n} \sin n\frac{\pi}{2} + \frac{1}{4n^2}\left(\cos n\frac{\pi}{2} - 1\right)\right\}.$$

If $n = 2m - 1$,

$$a_{2n} = -\frac{\cos m\pi}{4m-2} - \frac{1}{\pi(2m-1)^2}.$$

If $n = 4m$,

$$a_{2n} = \frac{1}{16m^2\pi}(\cos 2m\pi - 1) = 0.$$

If $n = 4m - 2$,

$$a_{2n} = \frac{-2}{(4m-2)^2\pi}.$$

$$f(x) = \frac{\pi}{16} - \sum_{m=1}^{\infty}\left\{\frac{\cos m\pi}{4m-2} + \frac{4}{\pi(4m-2)^2}\right\}\cos(4m-2)x$$

$$- \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{1}{(4m-2)^2}\cos(8m-4)x$$

$$= \frac{\pi}{16} + \left(\frac{1}{2}\cos 2x - \frac{1}{6}\cos 6x + \frac{1}{10}\cos 10x - \ldots\right)$$

$$- \frac{4}{\pi}\left(\frac{1}{2^2}\cos 2x + \frac{1}{6^2}\cos 6x + \frac{1}{10^2}\cos 10x + \ldots\right)$$

$$- \frac{2}{\pi}\left(\frac{1}{4}\cos 4x + \frac{1}{36}\cos 12x + \frac{1}{100}\cos 20x + \ldots\right).$$

6.23 Conditions for a Fourier Expansion

Any function of x can be expanded in a Fourier series if it satisfies what are called Dirichlet's conditions.

The first of these is that it should be *sectionally continuous* in the fundamental interval, that is the fundamental interval can be divided into a finite number of intervals in each of which the function is continuous and has finite limits as the variable approaches either end point from within the interval.

At a discontinuity we can distinguish between the limit of $f(x)$ at any point x_0 according to the direction in which the variable approaches x_0.

Thus if $\varepsilon > 0$

$$\lim_{\varepsilon \to 0} f(x_0 + \varepsilon)$$

is written $f(x_0 + 0)$ and called the right-hand limit of $f(x)$ at $x = x_0$. Similarly, $f(x_0 - 0)$ is the limit of $f(x_0 - \varepsilon)$ and is called the left-hand limit.

In the same way the right-hand derivative of $f(x)$ at $x = x_0$

$$= \lim_{\varepsilon \to 0} \frac{f(x_0 + \varepsilon) - f(x_0)}{\varepsilon}$$

and the left-hand derivative is

$$= \lim_{\varepsilon \to 0} \frac{f(x_0 - \varepsilon) - f(x_0)}{-\varepsilon}.$$

Dirichlet's conditions are

(1) $f(x + 2\pi) = f(x)$, for all x,
(2) $f(x)$ is sectionally continuous in the fundamental interval.

If these conditions are satisfied the Fourier series for $f(x)$ converges to the value $\frac{1}{2}\{f(x + 0) + f(x - 0)\}$ at every point at which $f(x)$ has a right-hand and a left-hand derivative.

6.24 Differentiation and Integration of Fourier Series

Consider the expansion obtained in Example 5

$$x = 2\left(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x \ldots\right).$$

The series obtained by differentiating the series term by term is

$$2(\cos x - \cos 2x + \cos 3x \ldots).$$

Now this series is not convergent since the value of the nth term does not tend to zero as n tends to infinity.

A Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where $f(x)$ is continuous in the interval $-\pi$ to π with $f(-\pi) = f(\pi)$, may be differentiated term by term if its derivative $f'(x)$ is sectionally continuous in the same interval, and at each point at which $f'(x)$ has a derivative

$$f'(x) = \sum_{n=1}^{\infty} n(-a_n \sin nx + b_n \cos nx).$$

An example of this is

$$\frac{x^2}{4} = \frac{\pi^2}{12} - \cos x + \frac{1}{4} \cos 2x - \frac{1}{9} \cos 3x + \ldots.$$

Differentiating we have

$$\frac{x}{2} = \sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \ldots.$$

Integration of a Fourier series gives a series which is more strongly convergent than the original series and hence any Fourier series can be integrated term by term.

Thus if

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

$$\int f(x)dx = \frac{1}{2}a_0 x + \sum_{n=1}^{\infty} \frac{1}{n}(a_n \sin nx - b_n \cos nx).$$

The integrated series is not, however, a Fourier series unless $a_0 = 0$.

6.25 Harmonic Analysis

The term harmonic analysis is usually applied to mechanical or numerical methods of obtaining the coefficients of a Fourier series for a function whose graph is known. The periodicity of the function will be evident from the graph and the fundamental interval may be taken as equal to 2π.

The fundamental interval is divided into a number of smaller intervals and the ordinates of the graph measured at the beginning of each interval. It is convenient to take 12 sub-intervals and thus we have the ordinates

$$y_0, y_1, y_2, \ldots, y_{11}, y_{12}(= y_0).$$

We find the coefficients a_n and b_n by using the mid-ordinate rule to approximate to the values of the integrals.

Thus
$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_0^{2\pi} y\,dx \\
&= \frac{1}{\pi} \times \frac{\pi}{6}\left\{\frac{y_0 + y_1}{2} + \frac{y_1 + y_2}{2} + \ldots + \frac{y_{11} + y_{12}}{2}\right\} \\
&= \frac{1}{6}(y_1 + y_2 + \ldots + y_{12}) \\
&= \frac{1}{6}\sum_{r=1}^{12} y_r. \\
a_n &= \frac{1}{\pi}\int_0^{2\pi} y \cos nx\,dx \\
&= \frac{1}{6}\sum_{r=1}^{12} y_r \cos\frac{nr\pi}{6}. \\
b_n &= \frac{1}{\pi}\int_0^{2\pi} y \sin nx\,dx \\
&= \frac{1}{6}\sum_{r=1}^{12} y_r \sin\frac{nr\pi}{6}.
\end{aligned}$$

Thus a_n and b_n are taken as twice the mean values of the numbers $y_r \cos nr\pi/6$ and $y_r \sin nr\pi/6$ respectively.

Example 14. *Find the first three harmonics of the function whose period is 2π and whose values at equal intervals of* 30° *are* (Fig. 147),

x (*degrees*)	30	60	90	120	150	180	210	240	270	300	330	360
y	2·41	3·30	3·00	2·75	2·40	2·60	2·10	2·00	1·50	1·35	1·20	1·00

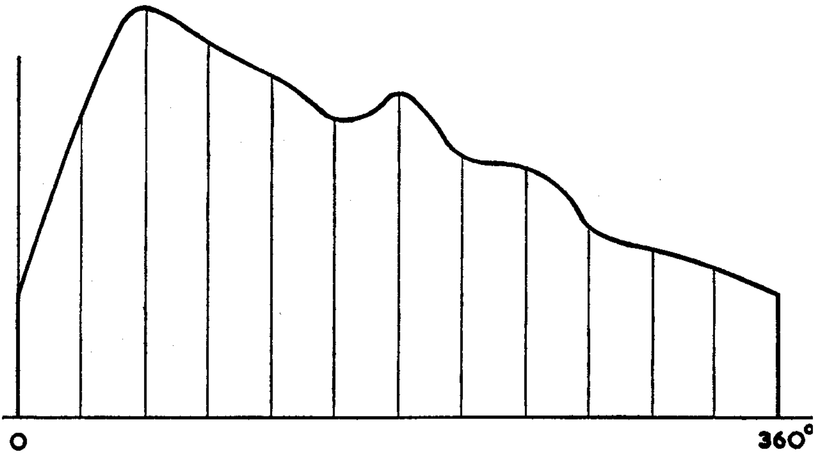

Fig. 147

The working may be tabulated as follows:

x	y	$\cos x$	$y \cos x$	$\sin x$	$y \sin x$	$\cos 2x$	$y \cos 2x$
30	2·41	0·866	2·09	0·5	1·21	0·5	1·21
60	3·30	0·5	1·65	0·866	2·86	−0·5	−1·15
90	3·00	0	0	1·0	3·00	−1·0	−3·00
120	2·75	−0·5	−1·38	0·866	2·38	−0·5	−1·38
150	2·40	−0·866	−2·08	0·5	1·20	0·5	1·20
180	2·60	−1·0	−2·60	0	0	1·0	2·60
210	2·10	−0·866	−1·82	−0·5	−1·05	0·5	1·05
240	2·00	−0·5	−1·00	−0·866	−1·73	−0·5	−1·00
270	1·50	0	0	−1·0	−1·50	−1·0	−1·50
300	1·35	0·5	0·68	−0·866	−1·17	−0·5	−0·68
330	1·20	0·866	1·04	−0·5	−0·60	0·5	0·60
360	1·00	1·0	1·00	0	0	1·0	1·00
Total	25·61		−2·42		4·60		−1·05
2 Mean	4·27		−0·40		0·77		−0·18

y	$\sin 2x$	$y \sin 2x$	$\cos 3x$	$y \cos 3x$	$\sin 3x$	$y \sin 3x$
2·41	0·866	2·09	0	0	1	2·41
3·30	0·866	2·86	−1	−3·30	0	0
3·00	0	0	0	0	−1	−3·00
2·75	−0·866	−2·38	1	2·75	0	0
2·40	−0·866	−2·08	0	0	1	2·40
2·60	0	0	−1	−2·60	0	0
2·10	0·866	1·82	0	0	−1	−2·10
2·00	0·866	1·73	1	2·00	0	0
1·50	0	0	0	0	1	1·50
1·35	−0·866	−1·17	−1	−1·35	0	0
1·20	−0·866	−1·04	0	0	−1	−1·20
1·00	0	0	1	1·00	0	0
Total		1·83		−1·50		0·01
2 Mean		0·31		−0·25		0·00

Thus we have

$$f(x) = 2{\cdot}14 - 0{\cdot}40 \cos x - 0{\cdot}18 \cos 2x - 0{\cdot}25 \cos 3x + \ldots$$
$$+ 0{\cdot}77 \sin x + 0{\cdot}31 \sin 2x + 0 \sin 3x + \ldots.$$

EXERCISES 6 (*e*)

1. Show that if the Fourier series of period 2π for a function $f(x)$ contains sine terms only, then the coefficient of $\sin nx$ is given by

$$\frac{2}{\pi}\int_0^\pi f(x) \sin nx dx.$$

If $f(x)$ is zero when x is between 0 and $\pi/2$, and unity when x is between $\pi/2$ and π, find the coefficients of the Fourier sine series, and draw the graph of $f(x)$ from $x = 0$ to $x = 2\pi$. (L.U., Pt. II)

2. Show that, in the range $0 < x < \pi$, $\sin x$ can be represented by the cosine Fourier series

$$\frac{4}{\pi}\left\{\frac{1}{2} - \frac{\cos 2x}{3} - \frac{\cos 4x}{15} - \frac{\cos 6x}{35} - \ldots - \frac{\cos 2px}{4p^2 - 1} - \ldots\right\}.$$

Show by a sketch the function represented by the sum of the series for values of x from $-\pi$ to 3π. (L.U., Pt. II)

3. Show that the function of x which is equal to $x(\pi - x)$ in the range $0 < x < \pi$ can be represented by a Fourier series containing sines of odd multiples of x only. Sketch the graph of the function represented by the series for the range $-2\pi < x < 4\pi$. Determine the coefficients in the series, including that of $\sin (2n - 1)x$. (L.U., Pt. II)

4. If $f(x) = -f(-x)$ in the range $-\pi < x < \pi$, show that the Fourier series for $f(x)$ in this range contains no cosine terms.

If the above function is defined by $f(x) = x$ when $0 < x < \frac{\pi}{2}$, and $f(x) = \pi - x$ when $\frac{\pi}{2} < x < \pi$, show that in the range $-\pi < x < \pi$, $f(x) = \frac{4}{\pi}\sum_{n=0}^{\infty}(-1)^n\frac{\sin(2n+1)x}{(2n+1)^2}$. (L.U., Pt. II)

5. What conditions must be satisfied by a periodic function of x of period 2π in order that its Fourier series may contain (i) cosine terms only, (ii) cosines of odd multiples of x only?

If the function $f(x)$ is defined by $f(x) = \cos\frac{\pi x}{2a}$, $0 < x < a$, $f(x) = 0$, $a < x < \frac{1}{2}\pi$, and satisfies the conditions for case (ii) above, draw a sketch showing the form of the function throughout one period. Determine the Fourier series. (L.U., Pt. II)

6. Show how to obtain the coefficients in the Fourier expansion of a function $f(x)$ which is given in the range $(-\pi, \pi)$.

Obtain the harmonic analysis of the function defined by $f(x) = c$ in the range $(0, a)$, $f(x) = 0$ in the range (a, π) and $f(x) = f(-x)$.

Describe briefly the effect of (i) integrating, (ii) differentiating a Fourier series. (L.U., Pt. II)

7. The function $f(x)$ is defined for $0 < x < \pi$ by $f(x) = \frac{\pi}{3}$, $0 < x < \frac{\pi}{3}$, $f(x) = 0$, $\frac{\pi}{3} < x < \frac{2}{3}\pi$, $f(x) = -\frac{\pi}{3}$, $\frac{2\pi}{3} < x < \pi$. Obtain the expansion in a series of cosines of multiples of x. Show that the terms $\cos nx$ are absent when the remainder on dividing n by 6 is 0, 2, 3 or 4. (L.U., Pt. II)

8. A function $f(x) = \frac{1}{4}x^2$ in the interval $-\pi < x < \pi$ and is repeated with this period. Prove that

$$f(x) = \frac{\pi^2}{12} - \cos x + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \ldots,$$

and find the value of the term containing $\cos 2px$ when $x = \pi/2$. (L.U., Pt. II)

9. State the conditions as to symmetry and periodicity which must be satisfied by a function if it can be expanded in a Fourier series of (i) sines of multiples of x, (ii) cosines of odd multiples of x.

The function $f(x)$ is defined in the range $0 < x < \pi/2$ by

$$f(x) = \pi/2 - x.$$

If it satisfies conditions (ii) above, draw a graph of the function throughout a period and find its Fourier series. (L.U., Pt. II)

10. A function of x, of period 2π, is zero for $0 < x < \pi/2$ and $\cos x$ for $\pi/2 \leqslant x \leqslant \pi$. Expand the function in a series of cosines of multiples of x, valid when $0 \leqslant x \leqslant \pi$. Sketch the graph of the sum of the series for values of x between 0 and 3π. (L.U., Pt. II)

11. A function of x, of period 2π, is equal to $-x^2$ for $-\pi \leqslant x \leqslant 0$ and is equal to x^2 for $0 \leqslant x \leqslant \pi$. Express the function as a Fourier series, and sketch the graph of the sum of the series for values of x between -3π and 5π. (L.U., Pt. II)

12. A function is defined by $f(x) = \pi x - x^2$ for $0 < x < \pi$. Expand the function in (i) a series of sines only, (ii) a series of cosines only of integral multiples of x, each of the series being valid for $0 < x < \pi$. Sketch the graph of the sum of each series for $-\pi < x < \pi$. (L.U., Pt. II)

13. A function $f(x)$ of period 2π in x, is such that $f(x) = 3x$ for $0 < x < \dfrac{\pi}{3}$, $f(x) = \pi$ for $\dfrac{\pi}{3} < x < \dfrac{\pi}{2}$, and the corresponding Fourier series contains sines of odd multiples of x only.

Sketch the graph of $f(x)$ from $x = 0$ to 2π and obtain the first five non-vanishing terms of the Fourier series.

By considering the value $f(\pi/4)$ obtain a series expression for π^2. (L.U., Pt. II)

14. A function $f(x)$ is such that

$$f(x) = f(x + 2\pi) \text{ and } f(x) = \frac{1}{4}x^2 \text{ for } -\pi < x < \pi.$$

Sketch the graph of $f(x)$ from $x = -2\pi$ to 2π and state what terms are absent from the Fourier expansion of $f(x)$ in the range $-\pi$ to π of x.

Obtain the general term in this series and deduce the value of $\sum\limits_{n=1}^{\infty} n^{-2}$. (L.U., Pt. II)

15. Assuming that for values of x between 0 and c it is possible to expand a function $f(x)$ in the form $\Sigma b_n \sin \dfrac{n\pi x}{c}$, show that

$$b_n = \frac{2}{c}\int_0^c f(x) \sin \frac{n\pi x}{c} dx.$$

If $0 < \lambda < 1$, and $f(x) = x$ when $0 \leqslant x \leqslant \lambda\pi$,

$$f(x) = \frac{\lambda}{1-\lambda}(\pi - x) \text{ when } \lambda\pi < x < \pi,$$

show that

$$f(x) = \frac{2}{\pi(1-\lambda)} \sum_{n=1}^{\infty} \frac{\sin n\lambda\pi}{n^2} \sin nx, \text{ when } 0 < x < \pi.$$

(L.U., Pt. II)

16. The following are 12 values of a periodic function of period 2π in x for $x = 0$, $2\pi/12$, $4\pi/12$, etc. Find the first three harmonics of the Fourier series which represents the function.

2·34,	3·73,	4·42,	3·72,	2·47,	1·58,
1·10,	0·99,	1·32,	1·93,	1·91,	1·70.

17. The following 12 values of a periodic function of period 2π in x are for $x = 0$, $x = 2\pi/12$, etc. Find the first three harmonics of the Fourier series for the function.

8·40,	7·22,	5·20,	3·05,	1·57,	0·95,
1·04,	1·73,	2·94,	4·70,	6·54,	8·00.

CHAPTER 7

VECTOR ANALYSIS

7.1 Definitions and Notation

A vector is defined as a quantity which has both magnitude and direction and which is such that the sum of two of these quantities may be obtained by the parallelogram law.

In Chapter 1 it was shown that displacements, velocities, accelerations and forces are vectors, and it was seen that vectors can be resolved into components and differentiated with respect to the time.

We now consider some more general theorems on vectors using some of the methods of three-dimensional coordinate geometry, and it will be seen that there is an algebra for vectors with rules similar to those of ordinary algebra and a calculus for vectors with rules similar to those of the ordinary calculus.

A vector whose direction is from A to B and whose magnitude is proportional to the length AB may be denoted by **AB** (in clarendon type), or by a single letter such as **a**. For written work it is convenient to use one of the symbols $\overrightarrow{AB}$, $\underline{AB}$, $\vec{a}$ or $\underline{a}$. The modulus of a vector is its magnitude. Thus a vector **AB** has modulus proportional to the length AB; the modulus of a vector **a** is denoted by a.

The vector whose direction is that of a given vector **a**, and whose modulus is unity is denoted by $\hat{\mathbf{a}}$, and called the unit vector in the direction of **a**. When Cartesian coordinates are used unit vectors parallel to the axes of reference OX, OY and OZ are denoted by $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ respectively (in clarendon type) or, more usually, by i, j and k (in italics).

A vector as such is not considered as having any definite location. Thus equal and parallel vectors are equivalent and the statement $\mathbf{a} = \mathbf{b}$ means that the vectors **a** and **b** are equal in magnitude and in parallel directions. Now, a force is localized in a straight line, a velocity is localized at a point, but we may treat these quantities as vectors without reference to their position and the theory of vectors will give results concerning their magnitude and direction.

The term scalar is used to denote a quantity which has magnitude but not direction; thus any number is a scalar if it is not associated with a direction. In mechanics such quantities as mass, temperature and energy are scalar quantities.

7.2 Addition of Vectors

If vectors $\mathbf{a}$ and $\mathbf{b}$ are placed end to end so that (Fig. 148) $\mathbf{AB} = \mathbf{a}$ and $\mathbf{BC} = \mathbf{b}$ then $\mathbf{AC} = \mathbf{c}$ is the vector sum of $\mathbf{AB}$ and $\mathbf{BC}$ and we write

$$\mathbf{c} = \mathbf{a} + \mathbf{b}.$$

Completing the parallelogram $ABCD$, since the opposite sides are equal and parallel we have

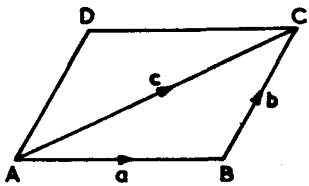

Fig. 148

$$\mathbf{AD} = \mathbf{BC} = \mathbf{b},$$
$$\mathbf{DC} = \mathbf{AB} = \mathbf{a},$$

and since

$$\mathbf{AC} = \mathbf{AD} + \mathbf{DC},$$
$$\mathbf{c} = \mathbf{b} + \mathbf{a}.$$

Thus the order in which the vectors are taken is immaterial and the so-called addition of vectors obeys the commutative law of algebra, since

$$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}.$$

From a diagram, such as Fig. 149, it is easily seen that when three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are added we have

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$$
$$= \mathbf{b} + (\mathbf{a} + \mathbf{c}).$$

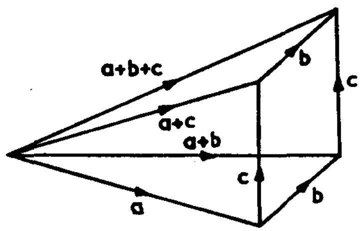

Fig. 149

In these expressions the quantities in brackets are added first and these equations show that vector addition obeys the associative law of algebra.

The vector $-\mathbf{b}$ being defined as a vector of the same modulus as $\mathbf{b}$ but in the opposite direction, the subtraction of $\mathbf{b}$ from $\mathbf{a}$ is equivalent to the addition of $\mathbf{a}$ and $-\mathbf{b}$.

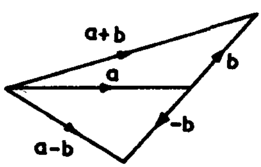

Fig. 150

Thus the equation

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$$

defines the operation of subtraction (Fig. 150).

7.3 Multiplication of a Vector by a Scalar

If m be a real positive number and $\mathbf{a}$ a vector, the product $m\mathbf{a}$ is defined as a vector whose direction is the same as that of $\mathbf{a}$ and whose modulus is ma. Then $-m\mathbf{a}$ is a vector of modulus ma in the opposite direction to $\mathbf{a}$.

It follows that any vector can be expressed as the product of its modulus and the unit vector in the same direction thus

$$\mathbf{a} = a\hat{\mathbf{a}}.$$

It is easily seen that the associative and distributive laws of algebraic products hold for products of scalars and vectors.

Thus
$$m(n\mathbf{a}) = n(m\mathbf{a}) = mn\mathbf{a},$$
$$(m+n)\mathbf{a} = m\mathbf{a} + n\mathbf{a},$$
$$n(\mathbf{a}+\mathbf{b}) = n\mathbf{a} + n\mathbf{b}.$$

Example 1. *Prove that if λ and μ are scalar quantities*
$$\lambda\mathbf{AB} + \mu\mathbf{AC} = (\lambda+\mu)\mathbf{AD},$$
where
$$\lambda\mathbf{DB} + \mu\mathbf{DC} = 0.$$

Since $\lambda\mathbf{DB} + \mu\mathbf{DC} = 0$ the vectors **DB** and **DC** must be parallel and hence D must lie on BC (Fig. 151), and divide BC in the ratio $\mu : \lambda$.

Now
$$\mathbf{AB} = \mathbf{AD} + \mathbf{DB},$$
$$\mathbf{AC} = \mathbf{AD} + \mathbf{DC},$$
therefore
$$\lambda\mathbf{AB} + \mu\mathbf{AC} = (\lambda+\mu)\mathbf{AD} + (\lambda\mathbf{DB} + \mu\mathbf{DC})$$
$$= (\lambda+\mu)\mathbf{AD}.$$

7.4 Components of a Vector

Let a vector $\mathbf{r} = \mathbf{OP}$ join the origin of coordinates to a point P whose coordinates are (x, y, z) (Fig. 152); $\mathbf{r}$ is called the position vector of P.

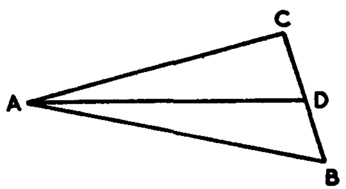

Fig. 151

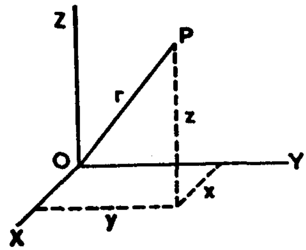

Fig. 152

The projections of OP on the coordinate axes are x, y and z respectively and **OP** is the sum of three vectors whose moduli are x, y and z, and whose directions are parallel to OX, OY and OZ respectively.

These three vectors may be written as products of their moduli and the unit vectors in the direction of the axes, that is xi, yj, zk respectively.

We have therefore
$$\mathbf{r} = xi + yj + zk.$$
The vectors xi, yj, zk are called the components of the vector $\mathbf{r}$ in the directions OX, OY, OZ respectively.

The sum of two vectors is easily obtained as the sums of the components of the vectors.

Thus if
$$\mathbf{r}_1 = x_1i + y_1j + z_1k,$$
$$\mathbf{r} + \mathbf{r}_1 = (x+x_1)i + (y+y_1)j + (z+z_1)k,$$
and any number of vectors can be added in this way.

If the direction cosines of a vector $\mathbf{r}$ are given as (l, m, n), where $l^2 + m^2 + n^2 = 1$, the projections of $\mathbf{r}$ on the axes are rl, rm and rn, and we have

$$\begin{aligned}\mathbf{r} &= rli + rmj + rnk \\ &= r(li + mj + nk).\end{aligned}$$

The vector $li + mj + nk$ is a unit vector in the direction of $\mathbf{r}$, since its modulus is $(l^2 + m^2 + n^2)^{1/2} = 1$.

Thus if a vector is $2i - 3j + 4k$, we may write this as a unit vector multiplied by the modulus.

We have

$$2^2 + 3^2 + 4^2 = 29,$$

and
$$2i - 3j + 4k = \sqrt{(29)}\left\{\frac{2}{\sqrt{(29)}}i - \frac{3}{\sqrt{(29)}}j + \frac{4}{\sqrt{(29)}}k\right\}.$$

Then $\sqrt{(29)}$ is the modulus and the unit vector is

$$(2i - 3j + 4k)/\sqrt{(29)}.$$

The direction cosines give the direction of the vector and its sense since they are the cosines of the angles made by vector with the positive directions of the axes. Thus the vector $-li - mj - nk$ has the same direction as the vector $li + mj + nk$ but is in the opposite sense.

Example 2. *A force of* 12 *lb. acts in a direction whose direction cosines are* $\frac{1}{3}, -\frac{2}{3}, -\frac{2}{3}$, *and a force of* 21 *lb. acts in a direction whose direction cosines are* $-\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$. *Find the magnitude and direction of the resultant of these forces.*

The forces expressed as vectors are

$$12\left(\frac{1}{3}i - \frac{2}{3}j - \frac{2}{3}k\right) = 4i - 8j - 8k,$$

$$21\left(-\frac{2}{7}i + \frac{3}{7}j + \frac{6}{7}k\right) = -6i + 9j + 18k.$$

The vector sum is $\quad -2i + j + 10k,$

$$= \sqrt{(105)}\left\{-\frac{2}{\sqrt{(105)}}i + \frac{1}{\sqrt{(105)}}j + \frac{10}{\sqrt{(105)}}k\right\}.$$

Hence the magnitude of the resultant force is $\sqrt{(105)}$ lb. and its direction cosines are

$$-\frac{2}{\sqrt{(105)}}, \quad \frac{1}{\sqrt{(105)}}, \quad \frac{10}{\sqrt{(105)}}.$$

Example 3. *In a tetrahedron OABC, the angles BOC, COA, AOB, and the sides OA, OB, OC, are denoted by* α, β, γ *and* a, b, c *respectively. If G is the centroid of ABC, prove that* $3\mathbf{OG} = \mathbf{OA} + \mathbf{OB} + \mathbf{OC}$ *and hence or otherwise prove that* $(3OG)^2 = a^2 + b^2 + c^2 + 2bc\cos\alpha + 2ca\cos\beta + 2ab\cos\gamma$. (L.U., Pt. II)

If D be the mid-point of AB (Fig. 153),

$$\mathbf{OA} = \mathbf{OD} + \mathbf{DA},$$
$$\mathbf{OB} = \mathbf{OD} + \mathbf{DB},$$

and since $\mathbf{DA} + \mathbf{DB} = 0$,

$$\mathbf{OA} + \mathbf{OB} = 2\mathbf{OD}.$$

Now CD is a median of the triangle ABC and the centroid G lies on CD, so that $2\mathbf{GD} + \mathbf{GC} = 0$.

Therefore

$$\mathbf{OC} = \mathbf{OG} + \mathbf{GC},$$
$$2\mathbf{OD} = 2\mathbf{OG} + 2\mathbf{GD},$$
$$\mathbf{OC} + 2\mathbf{OD} = 3\mathbf{OG},$$

that is

$$\mathbf{OA} + \mathbf{OB} + \mathbf{OC} = 3\mathbf{OG}.$$

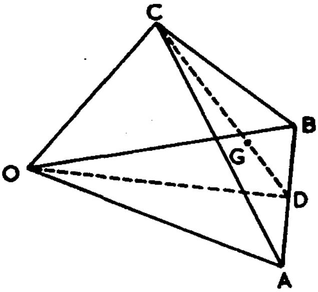

Fig. 153

Let (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) be the direction cosines of OA, OB, OC respectively.

Then

$$\mathbf{OA} = a(l_1 i + m_1 j + n_1 k)$$
$$\mathbf{OB} = b(l_2 i + m_2 j + n_2 k)$$
$$\mathbf{OC} = c(l_3 i + m_3 j + n_3 k).$$

$$\mathbf{OA} + \mathbf{OB} + \mathbf{OC} = (al_1 + bl_2 + cl_3)i + (am_1 + bm_2 + cm_3)j + (an_1 + bn_2 + cn_3)k$$
$$= 3\mathbf{OG}.$$

Therefore the modulus squared is

$$(3OG)^2 = (al_1 + bl_2 + cl_3)^2 + (am_1 + bm_2 + cm_3)^2 + (an_1 + bn_2 + cn_3)^2.$$

In this expression the coefficient of a^2 is $l_1^2 + m_1^2 + n_1^2 = 1$, and the coefficient of $2bc$ is $l_2l_3 + m_2m_3 + n_2n_3$. This is the cosine of the angle between the directions (l_2, m_2, n_2) and (l_3, m_3, n_3) and is $\cos\alpha$.

Hence by symmetry we have

$$(3OG)^2 = a^2 + b^2 + c^2 + 2bc\cos\alpha + 2ca\cos\beta + 2ab\cos\gamma.$$

7.5 Angular Velocity as a Vector

Angular velocity about an axis is a vector whose magnitude is the magnitude of the angular velocity and whose direction is that of the axis of rotation. The sense of this vector is taken as the direction in which the angular velocity would drive a right-handed screw along the axis of rotation. Thus the minute hand of a clock has an angular velocity of 1/60 r.p.m. and this angular velocity can be represented by a vector of magnitude 1/60 perpendicular to the clock face through its centre towards the interior of the clock.

To establish the fact that angular velocity is a vector we must show that two angular velocities can be added vectorially.

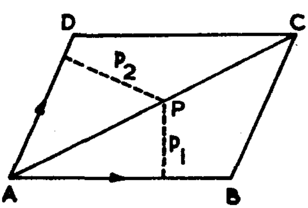

Fig. 154

Let $\mathbf{AB}$ and $\mathbf{AD}$ be vectors representing two angular velocities and let AC (Fig. 154) be the diagonal of the parallelogram formed by AB and AD.

Let P be any point on AC and p_1 and p_2 the lengths of the perpendiculars from P to AB and AD respectively.

The velocity of P due to the rotation AB is $p_1 \times AB$ perpendicular to the plane PAB and out of the page, and this is twice the area of the triangle PAB. The velocity of P due to the rotation AD is $p_2 \times AD$ perpendicular to the plane PAD and into the page, and this is twice the area of the triangle PAD.

Now it is a well-known property of a parallelogram that for any point P on a diagonal the triangles formed by joining P to the extremities of adjacent sides of the parallelogram are equal in area.

That is $$\Delta PAB = \Delta PAD.$$

Therefore the velocities given to P by the two rotations are equal and opposite, that is, P has zero velocity and therefore lies on the axis of the resultant rotation. Therefore the two rotations AB and AD combine to give a rotation about AC.

The magnitude of the resultant rotation may be found by considering the velocity of B.

Due to the rotation AD the velocity of B is AD multiplied by the perpendicular from B to AD, which is equal to the area of the parallelogram $ABCD$. The rotation AB adds no velocity to this.

Let ω be the magnitude of resultant velocity about AC. The velocity of B due to ω is ωp, where p is the perpendicular from B on AC, and since this must be equal to the velocity of B due to AD, ωp must be equal to the area $ABCD$, and hence

$$\omega = AC.$$

Thus the resultant of the angular velocities AB and AD is represented in magnitude and direction by the diagonal AC of the parallelogram.

Thus angular velocity is a vector and, for example, an angular velocity of 15 radians per second about an axis whose direction cosines with reference to axes OX, OY, OZ are $\frac{1}{3}$, $-\frac{2}{3}$, $\frac{2}{3}$, is the vector $5i - 10j + 10k$ rad./sec.

Angular momentum is the product of a moment of inertia and angular velocity, that is a product of a scalar and a vector, and therefore is itself a vector. Thus if a body in turning about an axis whose direction cosines are l, m, n with angular velocity ω and the moment of inertia about the axis of rotation is I, the angular velocity is the vector

$$\omega(li + mj + nk),$$

and the angular momentum the vector

$$I\omega(li + mj + nk).$$

EXERCISES 7 (*a*)

1. If $\mathbf{a}$ and $\mathbf{b}$ are the position vectors of A and B respectively, find the position vector of a point C in AB such that $AC = 2CB$.
2. Calculate the modulus and the unit vector of the sum of the vectors $i - 3j + 4k$, $3i + 2j + 5k$, $-3i + 5j - k$.

3. Forces of 2, 3 and 4 lb. act at the corner O of a cube one along each of the diagonals of the faces which meet at O. Find the magnitude of the resultant and its inclination to each of the edges which meet at O.

4. If the position vectors of points P and Q are $i - 3j + 4k$ and $4i - 5j - 2k$ respectively, find the length of PQ and its direction cosines.

5. The position vectors of points A, B, C are $2i - 3j + 4k$, $4i - 2j + 3k$, $3i - 4j + 2k$ respectively. Find the angle between the vectors **AB** and **BC**.

6. Prove that the modulus of the sum of the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \ldots,$ is

$$\{\sum_r a_r^2 + 2 \sum_{r,s} a_r a_s \cos \alpha_{rs}\}^{1/2}$$

where α_{rs} is the angle between the vectors $\mathbf{a}_r$ and $\mathbf{a}_s$.

7. Find the magnitude of the angular velocity which is the sum of the angular velocities $4i - 3j + 2k$ and $-2i + j + k$, and the direction cosines of the axis of rotation.

8. Vectors $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$, $\mathbf{OC} = \mathbf{c}$ are edges of a tetrahedron. Find in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ the position vector with respect to O of the mid-point of the line joining the mid-points of OA and BC, and hence show that the joins of the mid-points of the opposite edges of a tetrahedron intersect and bisect each other.

9. Vectors $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$, $\mathbf{OC} = \mathbf{c}$ are edges of a tetrahedron. Show that the lines joining the vertices of the tetrahedron to the centroids of opposite faces intersect in a point which divides these lines in the ratio 3 : 1 and find the position vector of this point.

10. The edges OA, OB, OC of a solid tetrahedron are of lengths a, b and c and the angles BOC, COA, AOB are α, β, γ, respectively. Find the distance from O of the centre of gravity G of the tetrahedron and the angle AOG.

7.6 Differentiation of a Vector

A vector may be a function of a scalar quantity such as the time t, that is either its modulus or its direction or both may vary with t. The differential coefficient of the vector with respect to the time takes account of its change of magnitude and direction.

Let $\mathbf{r} = \mathbf{OP}$ be a vector drawn from a fixed point at time t (Fig. 155); at time $t + \delta t$ let the vector be $\mathbf{OP'}$ and let it be denoted by $\mathbf{r} + \delta\mathbf{r}$, so that by vector subtraction $\mathbf{PP'}$ is the vector $\delta\mathbf{r}$.

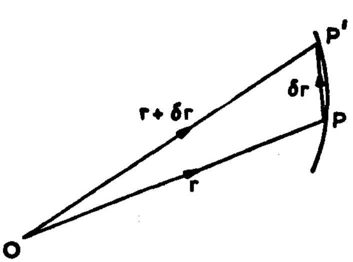

Fig. 155

The vector $\delta\mathbf{r}$ is the increment of the vector $\mathbf{r}$ in the time δt and has both magnitude and direction. Then

$\dfrac{\delta\mathbf{r}}{\delta t}$ is a vector and the derivative of $\mathbf{r}$ is defined as

$$\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt} = \lim_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta t}.$$

Since in the limit the direction of $\delta\mathbf{r}$ is that of a tangent at P to the locus of P the direction of $\dot{\mathbf{r}}$ is along this tangent.

The derivative of the sum of two vectors is easily obtained.

Let

$$\begin{aligned} \mathbf{r} &= \mathbf{r}_1 + \mathbf{r}_2 \\ \delta\mathbf{r} &= \delta\mathbf{r}_1 + \delta\mathbf{r}_2 \\ \dot{\mathbf{r}} &= \lim_{\delta t \to 0} \frac{\delta\mathbf{r}}{\delta t} \\ &= \lim_{\delta t \to 0} \left(\frac{\delta\mathbf{r}_1}{\delta t} + \frac{\delta\mathbf{r}_2}{\delta t} \right) \\ &= \dot{\mathbf{r}}_1 + \dot{\mathbf{r}}_2. \end{aligned}$$

The derivative of the product of a vector and a scalar, both of which vary with the time, is obtained by considering the increments of each.

Let

$$\begin{aligned} \mathbf{r} &= a\mathbf{s} \\ \mathbf{r} + \delta\mathbf{r} &= (a + \delta a)(\mathbf{s} + \delta\mathbf{s}) \\ \delta\mathbf{r} &= a\delta\mathbf{s} + \mathbf{s}\delta a + \delta a \delta\mathbf{s}, \end{aligned}$$

$$\lim_{\delta t \to 0} \frac{\delta\mathbf{r}}{\delta t} = a \lim_{\delta t \to 0} \frac{\delta\mathbf{s}}{\delta t} + \mathbf{s} \lim_{\delta t \to 0} \frac{\delta a}{\delta t} + \lim_{\delta t \to 0} \delta a \frac{\delta\mathbf{s}}{\delta t}.$$

That is

$$\frac{d\mathbf{r}}{dt} = a\frac{d\mathbf{s}}{dt} + \mathbf{s}\frac{da}{dt}.$$

7.7 Derivative in Terms of Components

Let

$$\mathbf{r} = xi + yj + zk.$$

Then since the unit vectors are constant in magnitude and direction we have

$$\dot{\mathbf{r}} = \dot{x}i + \dot{y}j + \dot{z}k.$$

If the vector $\mathbf{r}$ is written as the product of its modulus and the unit vector, its direction cosines being l, m, n, we have

$$\begin{aligned} \mathbf{r} &= r(li + mj + nk) \\ \dot{\mathbf{r}} &= \dot{r}(li + mj + nk) + r(\dot{l}i + \dot{m}j + \dot{n}k). \end{aligned}$$

Thus the derivative of $\mathbf{r}$ has two parts, a vector of magnitude $\dot{r}$ in the same direction as $\mathbf{r}$ and in addition the vector $r(\dot{l}i + \dot{m}j + \dot{n}k)$.

Since

$$l^2 + m^2 + n^2 = 1,$$

we have

$$l\dot{l} + m\dot{m} + n\dot{n} = 0.$$

Hence the vector $r(\dot{l}i + \dot{m}j + \dot{n}k)$ is perpendicular to the vector $\mathbf{r}$.

If $\mathbf{r}$ and $\mathbf{r} + \delta\mathbf{r}$ be the vector at times t and $t + \delta t$, the direction *ratios* of $\mathbf{r} + \delta\mathbf{r}$ are

$$l + \delta l : m + \delta m : n + \delta n.$$

Hence, if $\delta\theta$ be the angle between the vectors $\mathbf{r}$ and $\mathbf{r} + \delta\mathbf{r}$ we have

$$\cos\delta\theta = \frac{l(l + \delta l) + m(m + \delta m) + n(n + \delta n)}{\{(l + \delta l)^2 + (m + \delta m)^2 + (n + \delta n)^2\}^{1/2}}$$

$$= \frac{1}{\{1 + (\delta l)^2 + (\delta m)^2 + (\delta n)^2\}^{1/2}},$$

$$\sin^2\delta\theta = \frac{(\delta l)^2 + (\delta m)^2 + (\delta n)^2}{1 + (\delta l)^2 + (\delta m)^2 + (\delta n)^2},$$

that is $\quad (\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$, approximately,

and hence in the limit

$$\dot{\theta}^2 = (\dot{l}^2 + \dot{m}^2 + \dot{n}^2).$$

Hence the derivative of the vector $\mathbf{r}$ is

$$\dot{\mathbf{r}} = \dot{r}(li + mj + nk) + r\dot{\theta}(\dot{l}i + \dot{m}j + \dot{n}k)/(\dot{l}^2 + \dot{m}^2 + \dot{n}^2)^{1/2}.$$

Thus the component of $\dot{\mathbf{r}}$ due to changing direction of the vector is, as was seen in § 1.3, of magnitude $r\dot{\theta}$ and perpendicular to the vector $\mathbf{r}$.

7.8 Velocity and Acceleration

If $\mathbf{r}$ be the position vector of a point P with respect to a fixed point O, the velocity of P is the rate of change of its position, and the velocity vector $\mathbf{v}$ is

$$\mathbf{v} = \frac{d}{dt}\mathbf{r} = \dot{\mathbf{r}}.$$

If the coordinates of P with reference to fixed axes are (x, y, z) we have

$$\mathbf{r} = xi + yj + zk,$$
$$\mathbf{v} = \dot{\mathbf{r}} = \dot{x}i + \dot{y}j + \dot{z}k.$$

The acceleration of P is the rate at which its velocity is changing and the acceleration vector $\mathbf{a}$ is

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2}{dt^2}\mathbf{r},$$
$$= \dot{\mathbf{v}} = \ddot{\mathbf{r}},$$
$$= \ddot{x}i + \ddot{y}j + \ddot{z}k.$$

Velocity vectors are in fact vectors located at different points at time t and time $t + \delta t$. If velocity vectors OQ are all considered as drawn from the same point for different values of t, the locus of Q is called the *hodograph* of the motion and the acceleration vector at any time being the derivative of the velocity vector is in the direction of the tangent to the hodograph.

7.9 Integration of a Vector

Integration of a vector is defined as the reverse process to differentiation.

Thus if
$$\frac{d\mathbf{a}}{dt} = \mathbf{b},$$
we define the integral of $\mathbf{b}$ with respect to t as
$$\int \mathbf{b}\,dt = \mathbf{a} + \mathbf{c},$$
where $\mathbf{c}$ is an arbitrary constant vector.

Also the definite integral
$$\int_{t_1}^{t_2} \mathbf{b}\,dt = \mathbf{a}_2 - \mathbf{a}_1,$$
where $\mathbf{a}_1$ and $\mathbf{a}_2$ are the values of $\mathbf{a}$ at times t_1 and t_2.

As an example of vector integration consider the parabolic motion of a particle which moves freely under gravity.

Let the motion take place in the xy plane with the x-axis horizontal and the y-axis vertical. Then the acceleration is $-gj$ and if $\mathbf{r}$ is the position vector of the particle at time t we have
$$\ddot{\mathbf{r}} = -gj.$$
Therefore
$$\dot{\mathbf{r}} = -gtj + \mathbf{v}_0,$$
where $\mathbf{v}_0$ is the initial velocity.

Hence
$$\mathbf{r} = -\frac{1}{2}gt^2j + \mathbf{v}_0t + \mathbf{c}.$$

The added vector $\mathbf{c}$ is zero if $\mathbf{r} = 0$ when $t = 0$, and if
$$\mathbf{v}_0 = V\cos\alpha . i + V\sin\alpha . j$$
we have
$$\mathbf{r} = (V\cos\alpha . t)i + \left(V\sin\alpha . t - \frac{1}{2}gt^2\right)j.$$

7.10 Relative Velocity

If $\mathbf{r}_1$ be the position vector of a point P (Fig. 156) and $\mathbf{r}_2$ the position vector of a point Q with respect to the same point O, then PQ is the vector $\mathbf{r}_2 - \mathbf{r}_1$ and gives the position of Q relative to P.

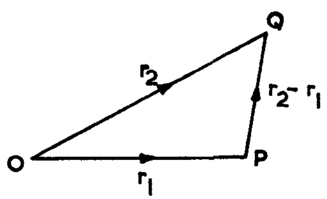

Fig. 156

Then $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$ are the velocities of P and Q with respect to O and $\dot{\mathbf{r}}_2 - \dot{\mathbf{r}}_1$ is the relative velocity of Q with respect to P. This is the vector sum of Q's velocity and P's velocity reversed.

Similarly the relative acceleration of Q with respect to P is $\ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1$.

It is evident that the velocity of Q with

respect to O is the vector sum of its velocity with respect to P and the velocity of P with respect to O.

7.11 Scalar Product of Two Vectors

The scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as the scalar quantity $ab \cos \theta$, where θ is the angle between the vectors.

This scalar product is written $\mathbf{a}.\mathbf{b}$.

Thus, for example, the work done by a force P in a displacement s in a direction making an angle θ with the direction of P is $Ps \cos \theta$, and if the force and displacement are written as vectors we have

$$\mathbf{P}.\mathbf{s} = \text{work done.}$$

Since $\cos 0 = 1$ and $\cos \frac{1}{2}\pi = 0$, it follows that the scalar product of parallel vectors is the product of their moduli and the scalar product of perpendicular vectors is zero.

The square of a vector $\mathbf{a}$ is defined as the scalar product $\mathbf{a}.\mathbf{a}$ and can be written as $\mathbf{a}^2$, and we have

$$\mathbf{a}^2 = a^2.$$

Since the resolved part of $\mathbf{b} + \mathbf{c}$ along the direction of a vector $\mathbf{a}$ is the sum of the resolved parts of $\mathbf{b}$ and $\mathbf{c}$, we have

$$\mathbf{a}.(\mathbf{b} + \mathbf{c}) = \mathbf{a}.\mathbf{b} + \mathbf{a}.\mathbf{c},$$

and thus the distributive law of ordinary multiplication holds for scalar products. Let vectors $\mathbf{a}$ and $\mathbf{b}$ be written as products of their moduli and unit vectors, so that

$$\mathbf{a} = a(l_1 i + m_1 j + n_1 k)$$
$$\mathbf{b} = b(l_2 i + m_2 j + n_2 k).$$

Since $\quad i.i = j.j = k.k = 1,$

and $\quad j.k = k.i = i.j = 0,$

we have $\quad \mathbf{a}.\mathbf{b} = ab(l_1 i + m_1 j + n_1 k).(l_2 i + m_2 j + n_2 k)$

$$= ab(l_1 l_2 + m_1 m_2 + n_1 n_2).$$

The work done by a force P in a small displacement $\delta\mathbf{r}$ is $\mathbf{P}.\delta\mathbf{r}$. Hence, the work done in a finite displacement

$$= \int \mathbf{P}.d\mathbf{r}$$

$$= \int \mathbf{P}.\frac{d\mathbf{r}}{dt}dt$$

$$= \int \mathbf{P}.\mathbf{v}dt.$$

Example 4. *Three non-coplanar vectors* **a**, **b**, **c** *each has unit magnitude and the angles between* **b** *and* **c**, **c** *and* **a**, **a** *and* **b** *are respectively* α, β, γ. *If the vector* **u** *is defined by* $\mathbf{u} = \mathbf{b} - (\mathbf{a}.\mathbf{b})\mathbf{a}$ *prove that* **u** *is perpendicular to* **a** *and has modulus* $\sin\gamma$. *If* $\mathbf{v} = \mathbf{c} - (\mathbf{a}.\mathbf{c})\mathbf{a}$ *and the angle between* **u** *and* **v** *is* A *prove that* $\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos A$. (L.U., Pt. II)

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{b} &= \cos\gamma \\
\mathbf{u} &= \mathbf{b} - \mathbf{a}\cos\gamma \\
\mathbf{a}\cdot\mathbf{u} &= \mathbf{a}\cdot\mathbf{b} - \mathbf{a}^2\cos\gamma \\
&= \cos\gamma - \cos\gamma = 0. \\
\mathbf{u}\cdot\mathbf{u} &= (\mathbf{b} - \mathbf{a}\cos\gamma).(\mathbf{b} - \mathbf{a}\cos\gamma) \\
&= \mathbf{b}^2 - 2\mathbf{b}.\mathbf{a}\cos\gamma + \mathbf{a}^2\cos^2\gamma \\
&= 1 - 2\cos^2\gamma + \cos^2\gamma \\
&= \sin^2\gamma.
\end{aligned}$$

Hence, **u** has modulus $\sin\gamma$ and is perpendicular to **a**.

$$\begin{aligned}
\mathbf{a}\cdot\mathbf{c} &= \cos\beta \\
\mathbf{v} &= \mathbf{c} - \mathbf{a}\cos\beta \\
\mathbf{v}\cdot\mathbf{v} &= (\mathbf{c} - \mathbf{a}\cos\beta).(\mathbf{c} - \mathbf{a}\cos\beta) \\
&= \mathbf{c}^2 - 2\mathbf{c}.\mathbf{a}\cos\beta + \mathbf{a}^2\cos^2\beta \\
&= 1 - 2\cos^2\beta + \cos^2\beta \\
&= \sin^2\beta.
\end{aligned}$$

Therefore, **v** has modulus $\sin\beta$ and

$$\mathbf{u}\cdot\mathbf{v} = \sin\beta\sin\gamma\cos A.$$

Also
$$\begin{aligned}
\mathbf{u}\cdot\mathbf{v} &= (\mathbf{b} - \mathbf{a}\cos\gamma).(\mathbf{c} - \mathbf{a}\cos\beta) \\
&= \mathbf{b}.\mathbf{c} - \mathbf{a}.\mathbf{c}\cos\gamma - \mathbf{b}.\mathbf{a}\cos\beta + \mathbf{a}^2\cos\beta\cos\gamma \\
&= \cos\alpha - \cos\beta\cos\gamma - \cos\beta\cos\gamma + \cos\beta\cos\gamma \\
&= \cos\alpha - \cos\beta\cos\gamma.
\end{aligned}$$

Hence $\cos\alpha = \cos\beta\cos\gamma + \sin\beta\sin\gamma\cos A$.

7.12 Vector Product of Two Vectors

The vector product of two vectors **a** and **b** whose directions include an angle θ is defined as:

(i) a vector,
(ii) of modulus $ab\sin\theta$,
(iii) whose direction is perpendicular to **a** and **b**,
(iv) in the sense that **a**, **b** and the product in this order form a right-handed triad.

The vector product of **a** and **b** is written as $\mathbf{a} \times \mathbf{b}$. Thus if $\hat{\mathbf{n}}$ be a unit vector perpendicular to **a** and **b** we have

$$\mathbf{a} \times \mathbf{b} = ab\sin\theta\,\hat{\mathbf{n}}.$$

The sense of $\hat{\mathbf{n}}$ described in (iv) is such that if **a** and **b** are in the plane of the paper (Fig. 157) $\hat{\mathbf{n}}$ is outwards from the plane of the paper. This is the direction in which a rotation of **a** towards **b** would drive a right-handed screw.

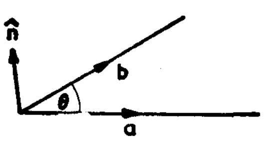

Fig. 157

It follows that $\mathbf{b} \times \mathbf{a}$ is a vector in the opposite direction, that is

$$\mathbf{b} \times \mathbf{a} = -ab\sin\theta\,\hat{\mathbf{n}}.$$

Thus this type of vector multiplication is

not commutative and a change in the order of the vectors in the product changes the sign.

A vector product is also the vector product of one vector and the projection of the other on a direction at right-angles to it.

Thus since the projection of $\mathbf{b} + \mathbf{c}$ on any direction is the sum of the projections of $\mathbf{b}$ and $\mathbf{c}$ on that direction we have

$$\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}.$$

Thus the distributive law of multiplication applies to vector products.

If the vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel $\theta = 0$ and $\mathbf{a} \times \mathbf{b} = 0$.

Thus
$$i \times i = j \times j = k \times k = 0.$$

If the vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular, $\sin\theta = 1$ and $\mathbf{a} \times \mathbf{b} = ab\hat{\mathbf{n}}$.

Thus
$$\begin{aligned} i \times j &= k = -j \times i \\ j \times k &= i = -k \times j \\ k \times i &= j = -i \times k. \end{aligned}$$

If $\mathbf{a}$ and $\mathbf{b}$ are expressed in terms of components so that

$$\begin{aligned} \mathbf{a} &= a(l_1 i + m_1 j + n_1 k) \\ \mathbf{b} &= b(l_2 i + m_2 j + n_2 k) \\ \mathbf{a} \times \mathbf{b} &= ab\{(m_1 n_2 - n_1 m_2) j \times k + (n_1 l_2 - l_1 n_2) k \times i \\ &\qquad + (l_1 m_2 - m_1 l_2) i \times j\} \\ &= ab\{(m_1 n_2 - n_1 m_2) i + (n_1 l_2 - l_1 n_2) j + (l_1 m_2 - m_1 l_2) k\} \\ &= ab \begin{vmatrix} i & j & k \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}. \end{aligned}$$

If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors $a = b = 1$. Then the modulus of $\mathbf{a} \times \mathbf{b}$ is $\sin\theta$.

Therefore

$$\sin^2\theta = (m_1 n_2 - n_1 m_2)^2 + (n_1 l_2 - l_1 n_2)^2 + (l_1 m_2 - m_1 l_2)^2.$$

Example 5. *Find the modulus and direction cosines of the vector* $\mathbf{a} \times \mathbf{b}$ *where* $\mathbf{a} = 2i - 3j + 4k$, $\mathbf{b} = 4i + 3j + 2k$.

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} i & j & k \\ 2 & -3 & 4 \\ 4 & 3 & 2 \end{vmatrix} \\ &= -18i + 12j + 18k \\ &= 6(-3i + 2j + 3k). \end{aligned}$$

Hence the modulus is $6\sqrt{(22)}$ and the direction cosines are

$$-3/\sqrt{(22)}, \quad 2/\sqrt{(22)}, \quad 3/\sqrt{(22)}.$$

Example 6. *If* $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$, $\mathbf{OC} = \mathbf{c}$ *are three vectors drawn from a point, find the area of the triangle ABC and the volume of the tetrahedron OABC.*

The area of the triangle OAB (Fig 158) is $\frac{1}{2}ab \sin AOB$ and this is the

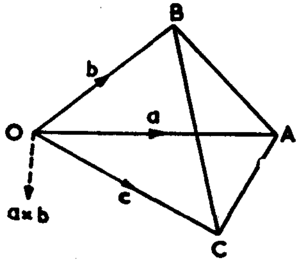

Fig. 158

modulus of the vector product $\frac{1}{2}\mathbf{a} \times \mathbf{b}$.

The direction of the vector $\frac{1}{2}\mathbf{a} \times \mathbf{b}$ is perpendicular to the triangle OAB and hence $\mathbf{c}.\left(\frac{1}{2}\mathbf{a} \times \mathbf{b}\right)$ is a scalar quantity which is the product of $\frac{1}{2}ab \sin AOB$ and the projection of $\mathbf{c}$ on the perpendicular to the triangle OAB.

This is three times the volume of the tetrahedron and hence the volume is

$$\frac{1}{6}\mathbf{c}.(\mathbf{a} \times \mathbf{b}).$$

The area of the triangle ABC is the modulus of the vector $\frac{1}{2}\mathbf{AB} \times \mathbf{AC}$.

$$= \frac{1}{2}(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a})$$

$$= \frac{1}{2}(\mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b}).$$

7.13 Moment of a Force about a Point

Let $\mathbf{r}$ be the position vector with respect to a point O of any point P on the line of action of a force $\mathbf{F}$; then the moment of the force $\mathbf{F}$ about O is the vector product $\mathbf{r} \times \mathbf{F}$.

Let θ be the angle between the vectors $\mathbf{r}$ and $\mathbf{F}$ and let ON (Fig. 159) be the perpendicular from O to the line of action of $\mathbf{F}$. The moment of the force about O is

$$ON.F$$
$$= OP \sin \theta . F$$
$$= rF \sin \theta.$$

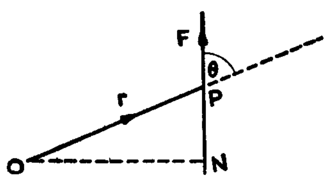

Fig. 159

Thus the moment of the force about O is the modulus of $\mathbf{r} \times \mathbf{F}$ and the axis about which the force will tend to cause rotation is perpendicular to the plane OPN and thus the moment is completely represented by the vector product $\mathbf{r} \times \mathbf{F}$.

The sum of the moments of a number of forces about O is the vector sum of the vector products for the individual forces.

7.14 Moment of a Couple

Let $\mathbf{r}_1$ and $\mathbf{r}_2$ be position vectors with respect to a point O of points on the lines of action of forces $\mathbf{F}$ and $-\mathbf{F}$ respectively (Fig. 160) which form a couple.

The moments of the forces about O are respectively

$$\mathbf{r}_1 \times \mathbf{F} \text{ and } -\mathbf{r}_2 \times \mathbf{F}.$$

The sum of their moments is

$$(\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F},$$

and this is the vector product of a vector joining any two points on the lines of action of the forces and one of the forces.

Thus the moment of the couple about any point is independent of the position of the point.

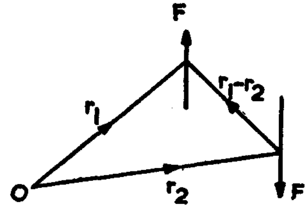

Fig. 160

Example 7. *A force of 10 lb. acts in a direction equally inclined to the axes OX, OY and OZ through the point (3, −2, 5), the unit of length being 1 ft. Find the magnitude and direction cosines of the moment of this force about the origin.*

The force is $$\mathbf{F} = 10\left(\frac{1}{\sqrt{3}}i + \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{3}}k\right).$$

The position vector of the point (3, −2, 5) is

$$\mathbf{r} = 3i - 2j + 5k.$$

The moment about the origin is

$$\mathbf{r} \times \mathbf{F} = \frac{10}{\sqrt{3}}\begin{vmatrix} i & j & k \\ 3 & -2 & 5 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \frac{10}{\sqrt{3}}(-7i + 2j + 5k)$$

$$= 10\sqrt{(26)}\frac{(-7i + 2j + 5k)}{\sqrt{(78)}}.$$

Hence the moment is $10\sqrt{(26)}$ ft.lb. in the direction $(-7i + 2j + 5k)/\sqrt{(78)}$.

7.15 Velocity due to Rotation

Suppose that a body is turning with angular velocity ω about an axis passing through a point O. We can express the velocity of any point P of the body due to the rotation as a vector product.

Let $\boldsymbol{\omega}$ be the angular velocity vector and $\mathbf{r}$ the position vector of P with respect to O. Let PN be the perpendicular from P (Fig. 161) to the axis of rotation and let the angle $PON = \theta$.

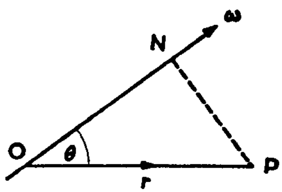

Fig. 161

The velocity of P due to the rotation

$$= \omega . PN$$
$$= \omega r \sin\theta,$$

and this is the modulus of $\boldsymbol{\omega} \times \mathbf{r}$.

With the convention of sign for the angular velocity, P as shown in the figure would tend to move into the plane of the paper perpendicular to the plane NOP, and this is also the direction of the product vector $\boldsymbol{\omega} \times \mathbf{r}$.

Hence the velocity of P is

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}.$$

Example 8. *A body is turning about an axis through the point* (2, 1, 1) *whose direction cosines are* 2/7, 6/7, 3/7 *with angular velocity* 5 *radians per second. Find the velocity of a particle of the body at a point* (4, −5, 2), *the unit of length being* 1 *ft.*

Here

$$\boldsymbol{\omega} = \frac{5}{7}(2i + 6j + 3k)$$

$$\mathbf{r} = 2i - 6j + k$$

$$\boldsymbol{\omega} \times \mathbf{r} = \frac{5}{7}\begin{vmatrix} i & j & k \\ 2 & 6 & 3 \\ 2 & -6 & 1 \end{vmatrix}$$

$$= \frac{5}{7}(24i + 4j - 24k)$$

$$= \frac{20\sqrt{(73)}}{7}\{(6i + j - 6k)/\sqrt{(73)}\}.$$

7.16 Differentiation of Products

The derivative of a scalar product $\mathbf{r}.\mathbf{s}$ is defined by

$$\frac{d}{dt}\mathbf{r}.\mathbf{s} = \lim_{\delta t \to 0} \frac{(\mathbf{r} + \delta\mathbf{r}).(\mathbf{s} + \delta\mathbf{s}) - (\mathbf{r}.\mathbf{s})}{\delta t}$$

$$= \lim_{\delta t \to 0}\left(\mathbf{r}.\frac{\delta\mathbf{s}}{\delta t} + \mathbf{s}.\frac{\delta\mathbf{r}}{\delta t} + \delta\mathbf{r}.\frac{\delta\mathbf{s}}{\delta t}\right)$$

$$= \mathbf{r}.\frac{d\mathbf{s}}{dt} + \mathbf{s}.\frac{d\mathbf{r}}{dt}.$$

If a vector $\mathbf{a}$ has a constant length, then

$$\mathbf{a}.\mathbf{a} = a^2 = \text{constant},$$

and differentiating we have

$$2\mathbf{a}.\frac{d\mathbf{a}}{dt} = 0.$$

Therefore the vectors $\mathbf{a}$ and $\dot{\mathbf{a}}$ are at right-angles.

The kinetic energy of a particle whose velocity is $\mathbf{v}$ is the scalar product

$$T = \frac{1}{2}m\mathbf{v}.\mathbf{v} = \frac{1}{2}mv^2$$

$$\frac{dT}{dt} = m\mathbf{v}.\dot{\mathbf{v}}$$

$$= m\mathbf{v}.\mathbf{a},$$

where $\mathbf{a}$ is the acceleration. If this acceleration is caused by a force $\mathbf{P}$ we have

$$\mathbf{P} = m\mathbf{a}$$

$$\frac{dT}{dt} = \mathbf{P}.\mathbf{v}$$

$$T = \int \mathbf{P}.\mathbf{v}dt + c.$$

This is the integral obtained in § 7.11 for the work done by the force and therefore the change in kinetic energy is equal to the work done.

The derivative of a vector product $\mathbf{r} \times \mathbf{s}$ is defined by

$$\begin{aligned}\frac{d}{dt}\mathbf{r} \times \mathbf{s} &= \lim_{\delta t \to 0} \frac{(\mathbf{r} + \delta\mathbf{r}) \times (\mathbf{s} + \delta\mathbf{s}) - (\mathbf{r} \times \mathbf{s})}{\delta t} \\ &= \lim_{\delta t \to 0} \left(\mathbf{r} \times \frac{\delta\mathbf{s}}{\delta t} + \frac{\delta\mathbf{r}}{\delta t} \times \mathbf{s} + \delta\mathbf{r} \times \frac{\delta\mathbf{s}}{\delta t}\right) \\ &= \mathbf{r} \times \frac{d\mathbf{s}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{s}.\end{aligned}$$

Thus in differentiating a vector product the order of the vectors must be maintained.

Example 9. *If* $\mathbf{r} = ti + 3j + t^2k$, $\mathbf{s} = t^2i - 2tj + tk$, *verify that*

$$\frac{d}{dt}\mathbf{r} \times \mathbf{s} = \mathbf{r} \times \frac{d\mathbf{s}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{s}.$$

$$\dot{\mathbf{r}} = i + 2tk$$
$$\dot{\mathbf{s}} = 2ti - 2j + k$$

$$\mathbf{r} \times \dot{\mathbf{s}} = \begin{vmatrix} i & j & k \\ t & 3 & t^2 \\ 2t & -2 & 1 \end{vmatrix} = (2t^2 + 3)i + (2t^3 - t)j - 8tk.$$

$$\dot{\mathbf{r}} \times \mathbf{s} = \begin{vmatrix} i & j & k \\ 1 & 0 & 2t \\ t^2 & -2t & t \end{vmatrix} = 4t^2i + (2t^3 - t)j - 2tk.$$

$$\mathbf{r} \times \mathbf{s} = \begin{vmatrix} i & j & k \\ t & 3 & t^2 \\ t^2 & -2t & t \end{vmatrix} = (2t^3 + 3t)i + (t^4 - t^2)j - 5t^2k.$$

$$\begin{aligned}\frac{d}{dt}(\mathbf{r} \times \mathbf{s}) &= (6t^2 + 3)i + (4t^3 - 2t)j - 10tk \\ &= \mathbf{r} \times \dot{\mathbf{s}} + \dot{\mathbf{r}} \times \mathbf{s}.\end{aligned}$$

EXERCISES 7 (*b*)

1. The position vector of a particle at time t with respect to fixed axes is $(t^2 + 1)i - 2tj + (t^2 - 1)k$. Find the magnitude of the velocity and of the acceleration at time t and the angle between their directions.

2. The position vector of a particle at time t is $\mathbf{r}$, $\mathbf{v}$ is its velocity and $\mathbf{a}$ its acceleration which is constant. Show that if $\mathbf{r}_0$ and $\mathbf{v}_0$ are the initial values of $\mathbf{r}$ and $\mathbf{v}$

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{a}t,$$

$$\mathbf{r} - \mathbf{r}_0 = \mathbf{v}_0 t + \frac{1}{2}\mathbf{a}t^2,$$

$$\mathbf{v}^2 = \mathbf{v}_0{}^2 + 2\mathbf{a}.(\mathbf{r} - \mathbf{r}_0).$$

3. A wheel is rotating with constant angular velocity ω and an insect starting from the centre at time $t = 0$ is crawling along a spoke with constant velocity u. Show that the position vector of the insect at time t with respect to fixed axes in the plane of the wheel is $iut \cos \omega t + jut \sin \omega t$. Hence show that the acceleration of the insect has components $ut\omega^2$ towards the centre of the wheel and $2u\omega$ in a perpendicular direction.

4. If $\mathbf{a} = a_1 i + a_2 j + a_3 k$, $\mathbf{b} = b_1 i + b_2 j + b_3 k$, $\mathbf{c} = c_1 i + c_2 j + c_3 k$, prove that $$\mathbf{a}.(\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

5 The opposite pairs of edges of a tetrahedron are $\mathbf{a}, \mathbf{a}'$; $\mathbf{b}, \mathbf{b}'$ and $\mathbf{c}, \mathbf{c}'$ respectively and $\mathbf{a}.\mathbf{a}' = \mathbf{b}.\mathbf{b}' = 0$. Prove that $\mathbf{c}.\mathbf{c}' = 0$ and that $\mathbf{a}^2 + \mathbf{a}'^2 = \mathbf{b}^2 + \mathbf{b}'^2 = \mathbf{c}^2 + \mathbf{c}'^2$.

6. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the position vectors of points A, B and C respectively, prove that the area of the triangle ABC is the modulus of

$$\frac{1}{2}(\mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b}).$$

7. Find the unit vector which is perpendicular to the vectors

$$2i - 3j + 6k \text{ and } -6i + 2j + 3k.$$

8. Find the magnitude and direction cosines of the moment about the point $(1, -2, 3)$ of a force $2i + 3j + 6k$ whose line of action passes through the origin.

9. A force $3i - 2j + 4k$ lb. and a force $7i + 4j - 6k$ lb. act on a particle during a displacement $5i - 6j - k$ ft. Find the work done.

10. The position vectors at time t of two aeroplanes A and B are $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. Prove that the relative velocity of A is directed towards B when $\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2 = c(\mathbf{r}_1 - \mathbf{r}_2)$. Prove that if $\dot{\mathbf{r}}_1 - \dot{\mathbf{r}}_2 = \mathbf{a}$, which is constant, and $\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{b}$ initially, the shortest distance between A and B is the modulus of $(\mathbf{a} \times \mathbf{b})/a$. and that the aeroplanes are nearest to each other after time $(\mathbf{a}.\mathbf{b})/a^2$.

11. State the Cartesian components of the vector product $\mathbf{B} \times \mathbf{C}$. By expressing $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ in terms of its Cartesian components, or otherwise, show that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}.\mathbf{C})\mathbf{B} - (\mathbf{A}.\mathbf{B})\mathbf{C}.$$ (L.U., Pt. II)

12. $\mathbf{X}$ is an unknown vector which satisfies the equations
(i) $\mathbf{A} \times \mathbf{X} = \mathbf{B}$, (ii) $\mathbf{A}.\mathbf{X} = a$, where $\mathbf{A}$ and $\mathbf{B}$ are known vectors and a is a known scalar. By multiplying (i) vectorially by $\mathbf{A}$ prove that $\mathbf{X} = (\mathbf{B} \times \mathbf{A} + a\mathbf{A})/A^2$. (L.U., Pt. II)

13. The components of three vectors **A**, **B** and **C** referred to three mutually perpendicular axes are as follows: **A**(1, 2, 3); **B**(2, 0, 1); **C**(−1, 3, 2). Find the components of the vectors $(\mathbf{A}.\mathbf{C})\mathbf{B}$, $(\mathbf{A}.\mathbf{B})\mathbf{C}$ and $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ and hence verify the relation $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}.\mathbf{C})\mathbf{B} - (\mathbf{A}.\mathbf{B})\mathbf{C}$. (L.U., Pt. II)

14. Using the notation $\mathbf{A}.\mathbf{B}$ for the scalar product of two vectors **A** and **B**, and $\mathbf{A} \times \mathbf{B}$ for their vector product, show that a parallelepiped with sides given by the vectors **A**, **B** and **C** has volume $(\mathbf{A} \times \mathbf{B}).\mathbf{C}$. (L.U., Pt. II)

15. A rigid body is rotating at the rate of 5 revs. per sec. about an axis through the origin whose direction ratios are 2, −1, 3. Find the velocity (in ft. per sec.) of the point of the body whose coordinates are (2, 4, 3), the unit of length being 1 ft. (L.U., Pt. II)

16. If **A**, **B** are two vectors and $\mathbf{A} \times \mathbf{B}$ is the vector product, express $\dfrac{d}{d\theta}(\mathbf{A} \times \mathbf{B})$ in terms of the sum of two vector products.

Verify the result when $\mathbf{A} = i\cos\theta + j\sin\theta + k\theta$, $\mathbf{B} = i\sin\theta - j\cos\theta - k\theta$; i, j, k being mutually perpendicular unit vectors. (L.U., Pt. II)

17. Define the vector product $\mathbf{a} \times \mathbf{b}$ of the two vectors **a** and **b**.

A tetrahedron has its vertices at the points $O(0, 0, 0)$, $A(1, 1, 2)$, $B(-1, 2, -1)$, and $C(0, -1, 3)$. By consideration of the vector product $\mathbf{AC} \times \mathbf{AB}$, or otherwise, determine, (i) the area of the face ABC and (ii) the unit vector normal to the face ABC.

Hence, or otherwise, determine the volume of the tetrahedron. (L.U., Pt. II)

18. Show that
$$\int(2\mathbf{r}.\dot{\mathbf{r}} + 2\dot{\mathbf{r}}\ddot{\mathbf{r}})dt = \mathbf{r}^2 + \dot{\mathbf{r}}^2 + c,$$
$$\int\left(\frac{1}{r}\frac{d\mathbf{r}}{dt} - \frac{\mathbf{r}}{r^2}\frac{dr}{dt}\right)dt = \hat{\mathbf{r}} + \mathbf{c}.$$

19. (i) If $\mathbf{A} = 2j + 3i - k$, $\mathbf{B} = i - 2j + 2k$, $\mathbf{C} = 3i + j + k$, show that $\mathbf{A}.(\mathbf{B} \times \mathbf{C}) = 0$ and $(\mathbf{A} - \mathbf{B}) \times \mathbf{C} = 2(\mathbf{A} \times \mathbf{B})$.

(ii) If **r** is the position vector of a variable point, relative to the origin, find the magnitude and direction of the vector $\int \mathbf{r} \times d\mathbf{r}$, where the integral is taken (*a*) round the circle $x = a\cos\theta$, $y = a\sin\theta$, $z = 0$, (*b*) round any closed curve in the plane $z = 0$. (L.U., Pt. II)

20. C is any curve joining two points H, K in a plane, O is a fixed point in the plane, P is a point on C, $d\mathbf{s}$ is the vector element of arc at P and $OP = \mathbf{r}$. If $\hat{\mathbf{r}}$ is the unit vector along **r** and $f(r)$ is a differentiable single valued function of r, show that the value of the line integral $\int_C f'(r)(\hat{\mathbf{r}}.d\mathbf{s})$ depends only on the distances of H and K from O, i.e. is independent of the shape of C. (L.U., Pt. II)

21. Find the vector $\mathbf{x}$ and the scalar λ which satisfy the equations
$$\mathbf{a} \times \mathbf{x} = \mathbf{b} + \lambda\mathbf{a}, \qquad \mathbf{a}\cdot\mathbf{x} = 1,$$
where $\mathbf{a}$ and $\mathbf{b}$ have components (1, 1, 1) and (1, −1, 2) respectively.
(L.U., Pt. II)

7.17 Motion of a Particle

Let (x, y, z) be the coordinates of a particle of mass m with respect to fixed rectangular axes and let the force acting on the particle have components X, Y, Z parallel to the axes.

The position vector of the particle is
$$\mathbf{r} = xi + yj + zk.$$
The velocity at time t is
$$\mathbf{v} = \dot{\mathbf{r}} = \dot{x}i + \dot{y}j + \dot{z}k.$$
The acceleration at time t is
$$\mathbf{f} = \dot{\mathbf{v}} = \ddot{\mathbf{r}} = \ddot{x}i + \ddot{y}j + \ddot{z}k.$$
The force acting on the particle is
$$\mathbf{F} = Xi + Yj + Zk.$$
The momentum is the vector $m\mathbf{v}$, and Newton's second law equating force to rate of change of momentum is
$$\mathbf{F} = \frac{d}{dt}m\mathbf{v},$$
that is, $\mathbf{F} = m\mathbf{f}$, if m is constant.

This equation is equivalent to the three equations
$$X = m\ddot{x}$$
$$Y = m\ddot{y}$$
$$Z = m\ddot{z}.$$
The vector form of the equation is however more concise and is applicable to the motion referred to any system of coordinates.

Since
$$\mathbf{F} = \frac{d}{dt}m\mathbf{v}$$
we have
$$\int_{t_0}^{t_1}\mathbf{F}dt = \Big[m\mathbf{v}\Big]_{t_0}^{t_1}.$$
The vector $\mathbf{I} = \int_{t_0}^{t_1}\mathbf{F}dt$ is called the impulse of the force over the period t_0 to t_1 and the vector equation
$$\mathbf{I} = (m\mathbf{v})_{t_1} - (m\mathbf{v})_{t_0}$$
equating impulse to change of momentum is equivalent to three equations giving change of momentum parallel to each of the axes.

The moment of the force $\mathbf{F}$ about the origin is
$$\mathbf{M} = \mathbf{r} \times \mathbf{F}.$$

The moment of the momentum of the particle about the origin, that is the angular momentum, is

$$\mathbf{H} = \mathbf{r} \times m\mathbf{v}.$$

Now
$$\frac{d\mathbf{H}}{dt} = \dot{\mathbf{r}} \times m\mathbf{v} + \mathbf{r} \times m\dot{\mathbf{v}}.$$

Since $\dot{\mathbf{r}} = \mathbf{v}$ the vector product $\dot{\mathbf{r}} \times m\mathbf{v} = 0$ and we have

$$\begin{aligned}\frac{d\mathbf{H}}{dt} &= \mathbf{r} \times m\dot{\mathbf{v}}\\ &= \mathbf{r} \times \mathbf{F}.\end{aligned}$$

Thus the rate of change of angular momentum about any fixed point is equal to the moment of force about the point. In particular if the force has no moment about the point (as in the case of central forces) the angular momentum about the point is constant.

If $\mathbf{r}$, $\mathbf{v}$ and $\mathbf{F}$ are expressed in terms of components we have

$$\begin{aligned}&\frac{d}{dt}m(xi + yj + zk) \times (\dot{x}i + \dot{y}j + \dot{z}k)\\ &\qquad\qquad = (xi + yj + zk) \times (Xi + Yj + Zk),\end{aligned}$$

that is
$$\frac{d}{dt}m\begin{vmatrix} i & j & k\\ x & y & z\\ \dot{x} & \dot{y} & \dot{z}\end{vmatrix} = \begin{vmatrix} i & j & k\\ x & y & z\\ X & Y & Z\end{vmatrix}.$$

The three component equations are

$$\frac{d}{dt}m(y\dot{z} - z\dot{y}) = yZ - zY,$$

$$\frac{d}{dt}m(z\dot{x} - x\dot{z}) = zX - xZ,$$

$$\frac{d}{dt}m(x\dot{y} - y\dot{x}) = xY - yX.$$

7.18 Motion of a Rigid Body

Let $\mathbf{r}_s = x_s i + y_s j + z_s k$ be the position vector of a particle of mass m_s of a rigid body. Then if M be the total mass and $\mathbf{r} = xi + yj + zk$ the position vector of the centre of gravity we have

$$M\mathbf{r} = \Sigma m_s\mathbf{r}_s.$$

Let $\mathbf{F}_s$ be the force acting on the particle m_s, then

$$m_s\ddot{\mathbf{r}}_s = \mathbf{F}_s.$$

Summing for all the particles of the body we have

$$\Sigma m_s\ddot{\mathbf{r}}_s = \Sigma\mathbf{F}_s,$$

that is,
$$M\ddot{\mathbf{r}} = \mathbf{F},$$

where $\mathbf{F}$ is the sum of the *external* forces acting on the body, the internal forces vanishing in the summation. Thus the centre of gravity moves as if the mass of the body were concentrated there and the external forces acted there.

We have also for the moment of momentum about the origin

$$\frac{d}{dt}\mathbf{r}_s \times m_s\dot{\mathbf{r}}_s = \mathbf{r}_s \times \mathbf{F}_s$$

and

$$\frac{d}{dt}\Sigma\mathbf{r}_s \times m_s\dot{\mathbf{r}}_s = \Sigma\mathbf{r}_s \times \mathbf{F}_s.$$

In the summation of moments of forces the internal forces vanish and therefore the rate of change of the total angular momentum about the origin is equal to the total moment of the external forces about the origin.

Let

$$\mathbf{r}_s = \mathbf{r} + \boldsymbol{\rho}_s$$

so that $\boldsymbol{\rho}_s$ is the position vector of the particle m_s with respect to the centre of gravity, and $\Sigma m_s\boldsymbol{\rho}_s = 0$.

Then

$$\begin{aligned}
&\frac{d}{dt}\Sigma\mathbf{r}_s \times m_s\dot{\mathbf{r}}_s \\
&= \frac{d}{dt}\Sigma m_s(\mathbf{r} + \boldsymbol{\rho}_s) \times (\dot{\mathbf{r}} + \dot{\boldsymbol{\rho}}_s) \\
&= \frac{d}{dt}\{M\mathbf{r} \times \dot{\mathbf{r}} + \mathbf{r} \times \Sigma m_s\dot{\boldsymbol{\rho}}_s + \Sigma m_s\boldsymbol{\rho}_s \times \dot{\mathbf{r}} + \Sigma m_s\boldsymbol{\rho}_s \times \dot{\boldsymbol{\rho}}_s\} \\
&= M\mathbf{r} \times \ddot{\mathbf{r}} + \frac{d}{dt}\Sigma m_s\boldsymbol{\rho}_s \times \dot{\boldsymbol{\rho}}_s.
\end{aligned}$$

$$\begin{aligned}
\Sigma\mathbf{r}_s \times \mathbf{F}_s &= \Sigma(\mathbf{r} \times \mathbf{F}_s + \boldsymbol{\rho}_s \times \mathbf{F}_s) \\
&= \mathbf{r} \times \Sigma\mathbf{F}_s + \Sigma\boldsymbol{\rho}_s \times \mathbf{F}_s \\
&= \mathbf{r} \times M\ddot{\mathbf{r}} + \Sigma\boldsymbol{\rho}_s \times \mathbf{F}_s.
\end{aligned}$$

Therefore

$$\frac{d}{dt}\Sigma m_s\boldsymbol{\rho}_s \times \dot{\boldsymbol{\rho}}_s = \Sigma\boldsymbol{\rho}_s \times \mathbf{F}_s.$$

The quantity $\Sigma\boldsymbol{\rho}_s \times m_s\dot{\boldsymbol{\rho}}_s$ is the moment about the centre of gravity of the momentum relative to the centre of gravity and $\Sigma\boldsymbol{\rho}_s \times \mathbf{F}_s$ is the total moment of the external forces about the centre of gravity.

We have, therefore, that the rate of change of angular momentum about the centre of gravity is equal to the moment of the external forces about the centre of gravity.

The motion relative to the centre of gravity may be represented by a rotation vector $\boldsymbol{\omega} = i\omega_1 + j\omega_2 + k\omega_3$ through the centre of gravity.

Then

$$\dot{\boldsymbol{\rho}}_s = \boldsymbol{\omega} \times \boldsymbol{\rho}_s.$$

The angular momentum is

$$\mathbf{H} = \Sigma m_s\boldsymbol{\rho}_s \times \dot{\boldsymbol{\rho}}_s = \Sigma m_s\boldsymbol{\rho}_s \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_s).$$

The vector product of the three vectors is (see Exercise 7 (*b*) 11)

$$\boldsymbol{\rho}_s \times (\boldsymbol{\omega} \times \boldsymbol{\rho}_s) = \boldsymbol{\rho}_s^2\boldsymbol{\omega} - (\boldsymbol{\rho}_s \cdot \boldsymbol{\omega})\boldsymbol{\rho}_s$$

and

$$\mathbf{H} = \boldsymbol{\omega}\Sigma m_s\boldsymbol{\rho}_s^2 - \Sigma m_s(\boldsymbol{\rho}_s \cdot \boldsymbol{\omega})\boldsymbol{\rho}_s.$$

If $\boldsymbol{\rho}_s = \xi_s i + \eta_s j + \zeta_s k$, the moments and products of inertia about the axes are

$$\begin{aligned} A &= \Sigma m_s(\eta_s^2 + \zeta_s^2), & F &= \Sigma m_s\eta_s\zeta_s, \\ B &= \Sigma m_s(\zeta_s^2 + \xi_s^2), & G &= \Sigma m_s\zeta_s\xi_s, \\ C &= \Sigma m_s(\xi_s^2 + \eta_s^2), & H &= \Sigma m_s\xi_s\eta_s. \end{aligned}$$

Then

$$\begin{aligned} \mathbf{H} = \boldsymbol{\omega}\Sigma m_s(\xi_s^2 + \eta_s^2 + \zeta_s^2) \\ - \Sigma m_s(\xi_s\omega_1 + \eta_s\omega_2 + \zeta_s\omega_3)(i\xi_s + j\eta_s + k\zeta_s) \\ = iH_1 + jH_2 + kH_3, \end{aligned}$$

where

$$\begin{aligned} H_1 &= A\omega_1 - H\omega_2 - G\omega_3, \\ H_2 &= -H\omega_1 + B\omega_2 - F\omega_3, \\ H_3 &= -G\omega_1 - F\omega_2 + C\omega_3. \end{aligned}$$

The motion about the centre of gravity is given by the equation

$$\frac{d\mathbf{H}}{dt} = \Sigma\boldsymbol{\rho}_s \times \mathbf{F}_s.$$

7.19 Kinetic Energy

The kinetic energy of the body is

$$\begin{aligned} T &= \frac{1}{2}\Sigma m_s\dot{\mathbf{r}}_s \cdot \dot{\mathbf{r}}_s \\ &= \frac{1}{2}\Sigma m_s(\dot{\mathbf{r}} + \dot{\boldsymbol{\rho}}_s) \cdot (\dot{\mathbf{r}} + \dot{\boldsymbol{\rho}}_s) \\ &= \frac{1}{2}\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}\Sigma m_s + \dot{\mathbf{r}} \cdot \Sigma m_s\dot{\boldsymbol{\rho}}_s + \frac{1}{2}\Sigma m_s\dot{\boldsymbol{\rho}}_s \cdot \dot{\boldsymbol{\rho}}_s \\ &= \frac{1}{2}M(\dot{\mathbf{r}})^2 + \frac{1}{2}\Sigma m_s(\dot{\boldsymbol{\rho}}_s)^2. \end{aligned}$$

Thus the kinetic energy is the sum of the kinetic energy of the total mass moving with the velocity of the centre of gravity and of the kinetic energy of the relative motion with respect to the centre of gravity.

Writing

$$\begin{aligned} \dot{\boldsymbol{\rho}}_s &= \boldsymbol{\omega} \times \boldsymbol{\rho}_s \\ &= \begin{vmatrix} i & j & k \\ \omega_1 & \omega_2 & \omega_3 \\ \xi_s & \eta_s & \zeta_s \end{vmatrix} \end{aligned}$$

we have

$$
\begin{aligned}
(\dot{\boldsymbol{\rho}}_s)^2 &= (\omega_2\zeta_s - \omega_3\eta_s)^2 + (\omega_3\xi_s - \omega_1\zeta_s)^2 + (\omega_1\eta_s - \omega_2\xi_s)^2 \\
&= \omega_1^2(\eta_s^2 + \zeta_s^2) + \omega_2^2(\zeta_s^2 + \xi_s^2) + \omega_3^2(\xi_s^2 + \eta_s^2) \\
&\qquad - 2\omega_2\omega_3\eta_s\zeta_s - 2\omega_3\omega_1\zeta_s\xi_s - 2\omega_1\omega_2\xi_s\eta_s.
\end{aligned}
$$

$$\frac{1}{2}\Sigma m_s(\dot{\boldsymbol{\rho}}_s)^2 = \frac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2 - 2F\omega_2\omega_3 - 2G\omega_3\omega_1 - 2H\omega_1\omega_2.$$

Now

$$
\begin{aligned}
\frac{dT}{dt} &= \Sigma m_s\dot{\mathbf{r}}_s.\ddot{\mathbf{r}}_s \\
&= \Sigma\dot{\mathbf{r}}_s.\mathbf{F}_s \\
&= \Sigma(\dot{\mathbf{r}} + \dot{\boldsymbol{\rho}}_s).\mathbf{F}_s \\
&= \dot{\mathbf{r}}.\mathbf{F} + \Sigma\dot{\boldsymbol{\rho}}_s.\mathbf{F}_s
\end{aligned}
$$

and

$$
\begin{aligned}
T &= \int\mathbf{F}.\dot{\mathbf{r}}dt + \Sigma\int\mathbf{F}_s.\dot{\boldsymbol{\rho}}_s dt \\
&= \int\mathbf{F}.\dot{\mathbf{r}}dt + \Sigma\int\mathbf{F}_s.(\boldsymbol{\omega} \times \boldsymbol{\rho}_s)dt \\
&= \int\mathbf{F}.\dot{\mathbf{r}}dt + \Sigma\int\boldsymbol{\omega}.(\boldsymbol{\rho}_s \times \mathbf{F}_s)dt.
\end{aligned}
$$

$\Sigma(\boldsymbol{\rho}_s \times \mathbf{F}_s)$ is the total moment of the external forces about the centre of gravity and $\hat{\boldsymbol{\omega}}.\Sigma(\boldsymbol{\rho}_s \times \mathbf{F}_s)$ is the projection of this moment on the axis of rotation, that is the total moment of the external forces about the axis of rotation. Denoting this moment by $\mathbf{M}$ we have

$$T = \int\mathbf{F}.d\mathbf{r} + \int\boldsymbol{\omega}.\mathbf{M}dt.$$

The first integral is the work done by the external forces in the displacement of the centre of gravity and the second integral is the work done by their moments about the centre of gravity in the motion relative to the centre of gravity.

The preceding analysis gives an expression for the kinetic energy of a body which is turning about a fixed point. Let its angular velocity have components ω_1, ω_2, ω_3 at any instant about principal axes fixed in the body about which the moments of inertia are A, B, C respectively. Then the products of inertia being zero we have

$$T = \frac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2).$$

7.20 Coriolis Acceleration

We commonly relate motion to axes fixed relative to the earth which are in fact rotating with the earth's angular velocity, and having measured velocity and acceleration relative to such moving axes and knowing the motion of the moving axes relative to the fixed axes we can find the absolute velocity and acceleration.

Suppose that a body moves so that its displacement, velocity and

acceleration relative to axes OX, OY, OZ are known and that these axes themselves move relative to fixed axes OX_1, OY_1, OZ_1 with the same origin.

The relative motion of the axes $OXYZ$ with respect to the axes $OX_1Y_1Z_1$ may be described by a rotation vector $\boldsymbol{\omega}$ through O.

Let $\mathbf{r}$ be the position vector of a point P relative to $OXYZ$; relative to these axes the velocity of P is $\dot{\mathbf{r}}$ and its acceleration is $\ddot{\mathbf{r}}$.

Let $\mathbf{v}_1$ be the velocity and $\mathbf{f}_1$ the acceleration of P relative to the fixed axes $OX_1Y_1Z_1$.

If $\boldsymbol{\omega}$ were zero $\mathbf{v}_1$ would be equal to $\dot{\mathbf{r}}$, but due to the rotation there is the additional velocity $\boldsymbol{\omega} \times \mathbf{r}$ and we have

$$\mathbf{v}_1 = \frac{d\mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r}.$$

The acceleration $\mathbf{f}_1$ is the velocity of the end point of the velocity vector $\mathbf{v}_1$ in the hodograph, that is when all vectors $\mathbf{v}_1$ are considered as radiating from O. The acceleration $\mathbf{f}_1$ is given, therefore, by replacing $\mathbf{r}$ by $\mathbf{v}_1$ in the formula for $\mathbf{v}_1$ and we have

$$\begin{aligned}\mathbf{f}_1 &= \frac{d\mathbf{v}_1}{dt} + \boldsymbol{\omega} \times \mathbf{v}_1 \\ &= \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r}\right) + \boldsymbol{\omega} \times \left(\frac{d\mathbf{r}}{dt} + \boldsymbol{\omega} \times \mathbf{r}\right) \\ &= \frac{d^2\mathbf{r}}{dt^2} + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r} + 2\boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).\end{aligned}$$

In this expression the first term $\ddot{\mathbf{r}}$ is the acceleration relative to the moving axes. The term $\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ is the familiar centripetal acceleration and the term $\dot{\boldsymbol{\omega}} \times \mathbf{r}$ occurs in addition when $\boldsymbol{\omega}$ varies.

The term $2\boldsymbol{\omega} \times \dot{\mathbf{r}}$ is known as the *Coriolis acceleration* and this acceleration has to be taken into account in the equation of motion $\mathbf{F} = m\mathbf{f}_1$. The acceleration given by this vector product is in a direction perpendicular to the axis of rotation and to the relative velocity $\dot{\mathbf{r}}$.

For example, in latitude λ (North) a parallel to the earth's axis is inclined at an angle λ to the horizontal. If a shell is fired in a direction due North the earth's rotation vector $\boldsymbol{\Omega}$ is in the plane of the trajectory (Fig. 162). The observed velocity $\mathbf{v}$ at any instant is in the direction of a tangent to the trajectory at an angle θ (say) to the horizontal.

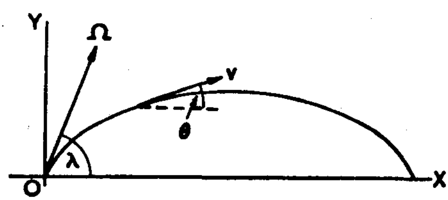

Fig. 162

Then the Coriolis acceleration is $2\boldsymbol{\Omega} \times \mathbf{v} = 2\Omega v \sin(\lambda - \theta)$ in a direction to the left of the trajectory, that is due west.

The reversed effective force $2m\Omega v \sin(\lambda - \theta)$ is to the East and will cause the shell to drift in that direction.

In two-dimensional motion let axes OXY make an angle θ with fixed axes OX_1Y_1, so that they are rotating with angular velocity $\dot{\theta}$ about a perpendicular to the plane OXY through O (Fig. 163). Then a point P distant r from O with coordinates (x, y) relative to OXY has components of velocity $\dot{x}$ and $\dot{y}$ and has also velocity $r\dot{\theta}$ perpendicular to OP.

The acceleration of P has component $\ddot{x}$ and $\ddot{y}$ relative to OXY (Fig. 164). It has also components $r\dot{\theta}^2$ and $r\ddot{\theta}$ along and perpendicular to OP.

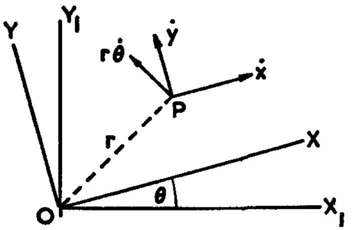

Fig. 163

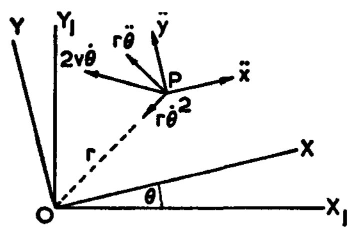

Fig. 164

It has also the Coriolis acceleration $2v\dot{\theta}$ in a direction perpendicular to the direction of its relative velocity where $v = (\dot{x}^2 + \dot{y}^2)^{1/2}$.

This acceleration is of importance in finding the accelerations of mechanisms with rotating slotted links.

7.21 Euler's Equations

Suppose that a rigid body is turning about a fixed point O with angular velocity $\boldsymbol{\omega}$. Let axes OX, OY, OZ be principal axes at O and let the moments of inertia about these axes be A, B, C respectively. Then if $\boldsymbol{\omega} = i\omega_1 + j\omega_2 + k\omega_3$ the angular momentum about O is, from § 7.18,

$$\mathbf{H} = iA\omega_1 + jB\omega_2 + kC\omega_3.$$

Now the axes OX, OY, OZ are themselves rotating with angular velocity $\boldsymbol{\omega}$ about fixed axes through O, so that the rate of change of angular momentum is the velocity of the extremity of the angular momentum vector, that is

$$\frac{d\mathbf{H}}{dt} + \boldsymbol{\omega} \times \mathbf{H}.$$

Therefore if $\mathbf{L}$ be the moment of the external forces about O we have

$$\frac{d\mathbf{H}}{dt} + \boldsymbol{\omega} \times \mathbf{H} = \mathbf{L}.$$

If $\mathbf{L} = iL_1 + jL_2 + kL_3$, we have the equations

$$\begin{aligned} A\dot{\omega}_1 - (B - C)\omega_2\omega_3 &= L_1, \\ B\dot{\omega}_2 - (C - A)\omega_3\omega_1 &= L_2, \\ C\dot{\omega}_3 - (A - B)\omega_1\omega_2 &= L_3. \end{aligned}$$

These equations, which are known as Euler's dynamical equations, determine the rotational motion of the body about the fixed point.

EXERCISES 7 (*c*)

1. Show that the rate of change of angular momentum of a particle about any fixed point is equal to the moment of the applied forces about the point. If the position vector of a particle with respect to fixed axes is
$$\mathbf{r} = (ir \sin \theta \cos \phi + jr \sin \theta \sin \phi + kr \cos \theta),$$
show that its angular momentum about the origin is the vector
$$mr\dot{\theta}(-i \sin \phi + j \cos \phi) + mr\dot{\phi} \sin \theta(-i \cos \theta \cos \phi - j \cos \theta \sin \phi + k \sin \theta).$$

2. Let $\mathbf{r}$ be the position vector of the centre of gravity of a rigid body with respect to fixed axes and $\boldsymbol{\omega}$ its angular velocity. Show that the position vector of the instantaneous centre relative to the centre of gravity is $\boldsymbol{\rho}$, where $\dot{\mathbf{r}} + \boldsymbol{\omega} \times \boldsymbol{\rho} = 0$. If $\mathbf{r} = ix + jy$ and $\boldsymbol{\omega} = k\omega$, show that $\omega\boldsymbol{\rho} = -i\dot{x} + j\dot{y}$.

3. Taking the x-axis due east, the y-axis due north and the z-axis vertical, show that for a car moving with uniform velocity v along a straight road in a direction θ east of north the Coriolis acceleration is the vector $2\Omega v(-i \sin \lambda \sin \theta + j \sin \lambda \cos \theta - k \cos \lambda \cos \theta)$, where λ is the latitude and Ω the earth's rotational velocity. Hence show that for a car of mass m there is a pressure of the tyres on the road towards the driver's right of magnitude $2mv\Omega \sin \lambda$.
   If $m = 1$ ton, $v = 85$ mp.h., $\lambda = 52°$, $\Omega = 10^{-5} \times 7{\cdot}29$ rad./sec., show that this force is about 1 lb.wt.

CHAPTER 8

LAGRANGE'S EQUATIONS

8.1 Introduction

When a dynamical system has n degrees of freedom it is possible to write down n equations involving the n coordinates of the system and their first and second derivatives.

These equations, which are known as Lagrange's equations, provide a simple and uniform approach to the solution of dynamical problems. They involve partial differentiation of the energy of the system with respect to each of the coordinates, and when the kinetic and potential energy has been expressed in terms of the coordinates there is little difficulty in writing down the equations. The unknown constraints which limit the degrees of freedom of the system are automatically eliminated by the equations so that the n coordinates are determined by the n equations.

8.2 Generalized Coordinates

When a particle is moving in a plane its position is determined by two coordinates, which may be Cartesian coordinates (x, y) or polar coordinates (r, θ) or any pair of numbers which determine its position. When a rigid body moves in a plane the position of every particle of the body is determined by three coordinates, for example, the Cartesian coordinates (x, y) of its centre of gravity and the angle θ which a straight line in the body makes with a fixed direction, and the body is said to have three degrees of freedom. Any three numbers which determine the position of the body, whether they be lengths or angles, are coordinates in the generalized sense. For a rigid body moving freely in three-dimensional space we need six coordinates to determine its position, for example, three coordinates to fix the position of its centre of gravity and for motion relative to the centre of gravity two angles giving the direction of some line through the centre of gravity and the angle turned by the body about this line.

The number of degrees of freedom of a system is the least number of coordinates required to determine its position. In what follows we shall suppose all these coordinates to be capable of varying independently within the constraints which limit the degrees of freedom. A system for which this is true is said to be *holonomic*.

The derivatives of generalized coordinates are called generalized velocities; thus in polar coordinates the generalized velocities are $\dot{r}$ and $\dot{\theta}$.

8.3 Spherical Polar Coordinates

The position of a particle P in three dimensions may be given in terms of the spherical polar coordinates r, θ, ϕ, with respect to a frame of reference $OXYZ$ (Fig. 165).

Here $r = OP$, is the distance of the particle from O,

θ is the inclination of OP to OZ,

ϕ is the angle which ON, the projection of OP on the plane OXY, makes with OX, ϕ being positive in the sense of rotation of OX towards OY. Hence, since $ON = r \sin\theta$, the Cartesian coordinates of P are

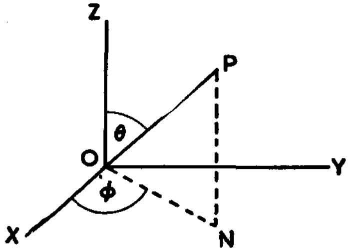

Fig. 165

$$x = r\sin\theta\cos\phi,$$
$$y = r\sin\theta\sin\phi,$$
$$z = r\cos\theta.$$

We have on differentiating with respect to the time

$$\dot{x} = \dot{r}\sin\theta\cos\phi + r\dot{\theta}\cos\theta\cos\phi - r\dot{\phi}\sin\theta\sin\phi,$$
$$\dot{y} = \dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\phi,$$
$$\dot{z} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta.$$

Hence,

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2\sin^2\theta + r^2\dot{\theta}^2\cos^2\theta + r^2\dot{\phi}^2\sin^2\theta + 2r\dot{r}\dot{\theta}\sin\theta\cos\theta,$$
$$\dot{z}^2 = \dot{r}^2\cos^2\theta + r^2\dot{\theta}^2\sin^2\theta - 2r\dot{r}\dot{\theta}\sin\theta\cos\theta,$$
$$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta.$$

Thus the kinetic energy of a particle of mass m at P is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$
$$= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2\sin^2\theta).$$

The angles θ and ϕ are called the Euler angles specifying the orientation of the line OP.

8.4 Kinetic Energy in Terms of Generalized Coordinates

When suitable coordinates have been chosen the first step in the formation of Lagrange's equations is to express the kinetic energy of the system in terms of the generalized coordinates and velocities.

Example 1. *A particle of mass m is moving in a plane. Find its kinetic energy when its polar coordinates are (r, θ).*

The components of velocity are $\dot{r}$ and $r\dot{\theta}$ and the kinetic energy is

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2).$$

Example 2. *A particle of mass m moves on the smooth inner surface of a sphere of radius a. Find its kinetic energy in suitable coordinates.*

If axes $OXYZ$ be taken at the centre of the sphere with OZ vertically downwards we may write the spherical polar coordinates of the particle as (a, θ, ϕ) (Fig. 166). Then from § 8.3, the kinetic energy is

$$T = \frac{1}{2}m(a^2\dot{\theta}^2 + a^2\dot{\phi}^2 \sin^2 \theta).$$

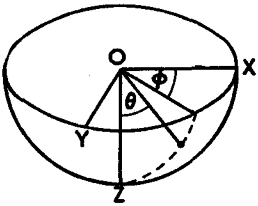

Fig. 166

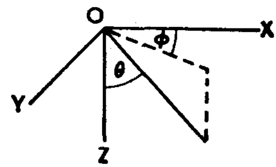

Fig. 167

Example 3. *A uniform rod of mass m and length 2a is freely hinged at one end to a fixed point O. Find its kinetic energy.*

With axes $OXYZ$, OZ being vertically downwards, the position of the rod is described by the spherical polar angles θ and ϕ (Fig. 167). The kinetic energy of a particle of mass δm distant r from O is, since r is constant and $\dot{r} = 0$,

$$\frac{1}{2}\delta m(r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2 \theta).$$

Hence the kinetic energy of the rod is

$$\begin{aligned} T &= \frac{1}{2}\Sigma\delta m(r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2 \theta) \\ &= \frac{1}{2}(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta)\Sigma\delta m r^2 \\ &= \frac{1}{2}m\frac{4}{3}a^2(\dot{\theta}^2 + \dot{\phi}^2 \sin^2 \theta). \end{aligned}$$

Example 4. *A uniform rod AB of mass m and length 2a is suspended by a small light ring at A from a horizontal rail and is initially at rest. A force R is applied at A along the rail (Fig.* 168). *If x is the distance moved by A and θ the inclination of the rod to the vertical, express the kinetic energy of the rod in terms of x, θ, ẋ and θ̇.*

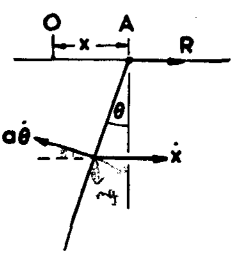

Fig. 168

The velocity of A is $\dot{x}$ and the velocity of the centre of gravity C with respect to A is $a\dot{\theta}$. Hence the kinetic energy of translation of the rod is

$$\frac{1}{2}m(\dot{x}^2 + a^2\dot{\theta}^2 - 2\dot{x}a\dot{\theta} \cos \theta).$$

There is also the kinetic energy of rotation $\frac{1}{2}m\frac{a^2}{3}\dot{\theta}^2$, therefore

$$T = \frac{1}{2}m(\dot{x}^2 + \frac{4}{3}a^2\dot{\theta}^2 - 2\dot{x}a\dot{\theta} \cos \theta).$$

Example 5. *A uniform rod AB of mass m_1 and length $2a$ is pivoted at A and a uniform rod BC of mass m_2 and length $2b$ is freely joined to it at B and the rods move in a vertical plane. Find the kinetic energy of the system.*

Let θ and ϕ be the angles made by AB and BC respectively with the vertical (Fig. 169). The kinetic energy of AB is

$$\frac{1}{2}\cdot\frac{4}{3}m_1a^2\dot{\theta}^2.$$

The point B has velocity $2a\dot{\theta}$ perpendicular to AB, the velocity of the centre of gravity G of BC with respect to B is $b\dot{\phi}$ perpendicular to BC, and the kinetic energy of translation of BC is

$$\frac{1}{2}m_2\left\{4a^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 4ab\dot{\theta}\dot{\phi}\cos(\phi-\theta)\right\}.$$

The rod BC has also the rotational kinetic energy $\frac{1}{2}m_2\frac{b^2}{3}\dot{\phi}^2$, and hence

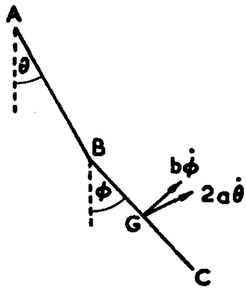

Fig. 169

$$T = \frac{1}{2}\cdot\frac{4}{3}m_1a^2\dot{\theta}^2 + \frac{1}{2}m_2\left\{4a^2\dot{\theta}^2 + \frac{4}{3}b^2\dot{\phi}^2 + 4ab\dot{\theta}\dot{\phi}\cos(\phi-\theta)\right\}.$$

It will be noticed that in each of the above examples the kinetic energy is a quadratic form in terms of the generalized velocities and also involves the generalized coordinates themselves.

8.5 The Work Function

The second step in the formation of Lagrange's equations is to find the work done by the external forces when each of the coordinates is given a small arbitrary increment.

In § 8.4, Example 1, let the force acting on the particle have components R along the radius vector and F perpendicular to it in the direction of θ increasing.

If r is increased by δr the work done by the force is $R\delta r$, and if θ is increased by $\delta\theta$ the work done by the force is $Fr\delta\theta$.

If we write
$$\delta W = R\delta r + Fr\delta\theta,$$

W is a function of r and θ such that the work done when r increases by δr is $\frac{\partial W}{\partial r}\delta r$, and the work done when θ increases by $\delta\theta$ is $\frac{\partial W}{\partial \theta}\delta\theta$.

W, if such a function exists, is called the work function and its derivative with respect to each of the coordinates is called the generalized force corresponding to that coordinate.

These generalized forces have the dimensions of a force if the corresponding coordinate is a length, and have the dimensions of the moment of a force if the coordinate is an angle.

In § 8.4, Examples 2 and 3, gravity is the only external force doing

work and work is done only as θ varies, so that in each case

$$\delta W = -mg \sin\theta \, a\delta\theta.$$

Then

$$\begin{aligned} W &= mga \cos\theta + \text{constant} \\ &= -V, \end{aligned}$$

where V is the potential energy.

In § 8.4, Example 4, work is done by the force R as x increases by δx and work is done by gravity as θ increases by $\delta\theta$, so that

$$\delta W = R\delta x - mg \sin\theta \, a\delta\theta.$$

We can if we wish write,

$$W = Rx + mga \cos\theta + \text{constant},$$

but this is not necessary since only the generalized forces $\dfrac{\partial W}{\partial x}$ and $\dfrac{\partial W}{\partial \theta}$ are required.

In § 8.4, Example 5, it is simpler to write down the potential energy

$$-W = V = -m_1 ga \cos\theta - m_2 g(2a \cos\theta + b \cos\phi),$$

and we have the generalized forces

$$\frac{\partial W}{\partial \theta} = -(m_1 + 2m_2)ag \sin\theta,$$

$$\frac{\partial W}{\partial \phi} = -m_2 gb \sin\phi.$$

8.6 The Form of Lagrange's Equations

Let r, θ, ϕ, etc., be the generalized coordinates of a system, T its kinetic energy and W the work function.

Then Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) - \frac{\partial T}{\partial r} = \frac{\partial W}{\partial r},$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta},$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = \frac{\partial W}{\partial \phi},$$

$$\cdots\cdots\cdots\cdots$$

there being as many equations as there are coordinates. It should be noted that the kinetic energy T is differentiated partially with respect to each generalized velocity as well as with respect to each generalized coordinate, the velocities and coordinates being considered independent in these differentiations.

Before proving Lagrange's equations let us write down the solutions to the examples of § 8.4.

In Example 1,

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2),$$

$$\frac{\partial T}{\partial \dot{\theta}} = mr^2\dot{\theta},$$

$$\frac{\partial T}{\partial \theta} = 0,$$

$$\frac{\partial W}{\partial \theta} = Fr,$$

therefore Lagrange's first equation is

$$\frac{d}{dt}(mr^2\dot{\theta}) = Fr. \tag{1}$$

$$\frac{\partial T}{\partial \dot{r}} = m\dot{r},$$

$$\frac{\partial T}{\partial r} = mr\dot{\theta}^2,$$

$$\frac{\partial W}{\partial r} = R,$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{r}}\right) = m\ddot{r},$$

therefore Lagrange's second equation is

$$m(\ddot{r} - r\dot{\theta}^2) = R. \tag{2}$$

These equations are, of course, the usual equations obtained by equating force to mass $\times$ acceleration in polar coordinates.

In Example 2,

$$T = \frac{1}{2}m(a^2\dot{\theta}^2 + a^2\dot{\phi}^2 \sin^2 \theta),$$

$$\frac{\partial T}{\partial \dot{\theta}} = ma^2\dot{\theta},$$

$$\frac{\partial T}{\partial \theta} = ma^2\dot{\phi}^2 \sin \theta \cos \theta,$$

$$\frac{\partial W}{\partial \theta} = -mga \sin \theta,$$

and we have

$$ma^2\ddot{\theta} - ma^2\dot{\phi}^2 \sin\theta\cos\theta = -mga\sin\theta. \tag{1}$$

$$\frac{\partial T}{\partial \dot{\phi}} = ma^2\dot{\phi}\sin^2\theta,$$

$$\frac{\partial T}{\partial \phi} = 0,$$

$$\frac{\partial W}{\partial \phi} = 0,$$

and we have

$$\frac{d}{dt}(ma^2\dot{\phi}\sin^2\theta) = 0. \tag{2}$$

If $\dot{\phi}$ is initially equal to ω and θ to α,

we have
$$\dot{\phi}\sin^2\theta = \text{constant} = \omega\sin^2\alpha.$$

Substituting for $\dot{\phi}$ in (1) we have an equation in θ only

$$a\ddot{\theta} = \frac{a\omega^2\sin^4\alpha\cos\theta}{\sin^3\theta} - g\sin\theta.$$

Integrating with respect to θ we have

$$\frac{1}{2}a\dot{\theta}^2 = -\frac{1}{2}a\omega^2\frac{\sin^4\alpha}{\sin^2\theta} + g\cos\theta + \text{constant}.$$

In Example 3,

$$T = \frac{1}{2}m\frac{4}{3}a^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta),$$

$$\frac{\partial T}{\partial \dot{\theta}} = \frac{4}{3}ma^2\dot{\theta},\ \frac{\partial T}{\partial \theta} = \frac{4}{3}ma^2\dot{\phi}^2\sin\theta\cos\theta,$$

$$\frac{\partial W}{\partial \theta} = -mga\sin\theta,$$

therefore

$$\frac{4}{3}ma^2\ddot{\theta} - \frac{4}{3}ma^2\dot{\phi}^2\sin\theta\cos\theta = -mga\sin\theta. \tag{1}$$

$$\frac{\partial T}{\partial \dot{\phi}} = \frac{4}{3}ma^2\dot{\phi}\sin^2\theta,\ \frac{\partial T}{\partial \phi} = \frac{\partial W}{\partial \phi} = 0,$$

therefore

$$\frac{4}{3}ma^2\frac{d}{dt}(\dot{\phi}\sin^2\theta) = 0. \tag{2}$$

If initially $\dot{\phi} = \omega,\ \theta = \alpha,\ \dot{\theta} = 0,$

$$\dot{\phi}\sin^2\theta = \omega\sin^2\alpha,$$

$$a\ddot{\theta} = \frac{a\omega^2\sin^4\alpha\cos\theta}{\sin^3\theta} - \frac{3}{4}g\sin\theta,$$

$$a\dot{\theta}^2 = a\omega^2\sin^2\alpha\left(1 - \frac{\sin^2\alpha}{\sin^2\theta}\right) + \frac{3}{2}g(\cos\theta - \cos\alpha).$$

In Example 4,

$$T = \frac{1}{2}m\left(\dot{x}^2 + \frac{4}{3}a^2\dot{\theta}^2 - 2\dot{x}a\dot{\theta}\cos\theta\right),$$

$$\frac{\partial T}{\partial \dot{x}} = m(\dot{x} - a\dot{\theta}\cos\theta), \quad \frac{\partial T}{\partial x} = 0,$$

$$\frac{\partial W}{\partial x} = R,$$

therefore
$$m(\ddot{x} - a\ddot{\theta}\cos\theta + a\dot{\theta}^2\sin\theta) = R. \tag{1}$$

$$\frac{\partial T}{\partial \dot{\theta}} = \frac{4}{3}ma^2\dot{\theta} - ma\dot{x}\cos\theta,$$

$$\frac{\partial T}{\partial \theta} = m\dot{x}a\dot{\theta}\sin\theta,$$

$$\frac{\partial W}{\partial \theta} = -mga\sin\theta,$$

therefore
$$\frac{4}{3}ma^2\ddot{\theta} - ma\ddot{x}\cos\theta = -mga\sin\theta. \tag{2}$$

Elimination of $\ddot{x}$ leads to an equation in θ,

$$m\left(\frac{4}{3}a\ddot{\theta} - a\ddot{\theta}\cos^2\theta + a\dot{\theta}^2\sin\theta\cos\theta\right) = R\cos\theta - mg\sin\theta.$$

In Example 5,

$$T = \frac{1}{2}\cdot\frac{4}{3}m_1a^2\dot{\theta}^2 + \frac{1}{2}m_2\left\{4a^2\dot{\theta}^2 + \frac{4}{3}b^2\dot{\phi}^2 + 4ab\dot{\theta}\dot{\phi}\cos(\phi - \theta)\right\},$$

$$\frac{\partial W}{\partial \theta} = -(m_1 + 2m_2)ga\sin\theta,$$

$$\frac{\partial W}{\partial \phi} = -m_2gb\sin\phi,$$

and Lagrange's equations are easily seen to be

$$\frac{4}{3}(m_1 + 3m_2)a\ddot{\theta} + 2m_2b\{\ddot{\phi}\cos(\phi - \theta) - \dot{\phi}^2\sin(\phi - \theta)\}$$
$$= -(m_1 + 2m_2)g\sin\theta. \tag{1}$$

$$\frac{4}{3}m_2b\ddot{\phi} + 2m_2a\{\ddot{\theta}\cos(\phi - \theta) + \dot{\theta}^2\sin(\phi - \theta)\} = -m_2g\sin\phi. \tag{2}$$

8.7 Proof of Lagrange's Equations

Consider a particle of a system of mass m whose Cartesian coordinates are (x, y, z) and let the force acting on the particle have components (X, Y, Z). We shall use Σ to denote the sum of any of these quantities taken over all the particles of the system. The force system (X, Y, Z) will include both internal and external forces.

We shall assume that the coordinates of any particle can be expressed in terms of generalized coordinates p, q, r, etc. For simplicity we shall assume that the system has two degrees of freedom only with generalized coordinates p and q, but the proof is easily extended to a system with n degrees of freedom.

In the formation of Lagrange's equations we have seen that in the differentiations the generalized coordinates and the generalized velocities are considered as independent quantities, and this fact should be borne in mind when following the proof.

We need two preliminary lemmas.

Lemma 1.
$$\frac{\partial \dot{x}}{\partial \dot{p}} = \frac{\partial x}{\partial p}.$$

Since x is a function of p and q, we have

$$\dot{x} = \frac{\partial x}{\partial p}\frac{dp}{dt} + \frac{\partial x}{\partial q}\frac{dq}{dt},$$
$$= \frac{\partial x}{\partial p}\dot{p} + \frac{\partial x}{\partial q}\dot{q}.$$

Remembering that p, $\dot{p}$, q, $\dot{q}$ are considered as independent quantities, we have at once

$$\frac{\partial \dot{x}}{\partial \dot{p}} = \frac{\partial x}{\partial p},$$

and similarly
$$\frac{\partial \dot{y}}{\partial \dot{p}} = \frac{\partial y}{\partial p}, \quad \frac{\partial \dot{z}}{\partial \dot{p}} = \frac{\partial z}{\partial p}.$$

Lemma 2.
$$\frac{d}{dt}\left(\frac{\partial x}{\partial p}\right) = \frac{\partial \dot{x}}{\partial p}.$$

We have
$$\dot{x} = \frac{\partial x}{\partial p}\dot{p} + \frac{\partial x}{\partial q}\dot{q},$$

therefore
$$\frac{\partial \dot{x}}{\partial p} = \frac{\partial^2 x}{\partial p^2}\dot{p} + \frac{\partial^2 x}{\partial p \partial q}\dot{q}.$$

Also
$$\frac{d}{dt}\left(\frac{\partial x}{\partial p}\right) = \frac{\partial^2 x}{\partial p^2}\dot{p} + \frac{\partial^2 x}{\partial p \partial q}\dot{q}.$$

Hence the lemma is proved, and a similar result is true for y and z.

Proof.

The kinetic energy of the system is

$$T = \Sigma \frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2).$$

We have

$$\frac{\partial T}{\partial p} = \Sigma m\left(\dot{x}\frac{\partial \dot{x}}{\partial p} + \dot{y}\frac{\partial \dot{y}}{\partial p} + \dot{z}\frac{\partial \dot{z}}{\partial p}\right)$$

$$= \Sigma m\left(\dot{x}\frac{\partial x}{\partial p} + \dot{y}\frac{\partial y}{\partial p} + \dot{z}\frac{\partial z}{\partial p}\right).$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}}\right) = \Sigma m\left(\ddot{x}\frac{\partial x}{\partial p} + \ddot{y}\frac{\partial y}{\partial p} + \ddot{z}\frac{\partial z}{\partial p}\right) + \Sigma m\left(\dot{x}\frac{d}{dt}\frac{\partial x}{\partial p} + \dot{y}\frac{d}{dt}\frac{\partial y}{\partial p} + \dot{z}\frac{d}{dt}\frac{\partial z}{\partial p}\right).$$

Remembering that $m\ddot{x} = X$, $m\ddot{y} = Y$, $m\ddot{z} = Z$, and using lemma 2, we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}}\right) = \Sigma\left(X\frac{\partial x}{\partial p} + Y\frac{\partial y}{\partial p} + Z\frac{\partial z}{\partial p}\right) + \Sigma m\left(\dot{x}\frac{\partial \dot{x}}{\partial p} + \dot{y}\frac{\partial \dot{y}}{\partial p} + \dot{z}\frac{\partial \dot{z}}{\partial p}\right).$$

The last term is clearly $\dfrac{\partial T}{\partial p}$ and we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}}\right) - \frac{\partial T}{\partial p} = \Sigma\left(X\frac{\partial x}{\partial p} + Y\frac{\partial y}{\partial p} + Z\frac{\partial z}{\partial p}\right).$$

As p increases by δp, the increase in x is $\dfrac{\partial x}{\partial p}\delta p$ and the work done by the force X is $X\dfrac{\partial x}{\partial p}\delta p$.

Hence the work done by the external forces acting on the system as p increases by δp is

$$\Sigma\left(X\frac{\partial x}{\partial p} + Y\frac{\partial y}{\partial p} + Z\frac{\partial z}{\partial p}\right)\delta p.$$

Hence if W is the work function

$$\frac{\partial W}{\partial p}\delta p = \Sigma\left(X\frac{\partial x}{\partial p} + Y\frac{\partial y}{\partial p} + Z\frac{\partial z}{\partial p}\right)\delta p,$$

and we have Lagrange's equation for p,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{p}}\right) - \frac{\partial T}{\partial p} = \frac{\partial W}{\partial p}.$$

If we write $L = T + W$, since W does not contain the velocities

$$\frac{\partial W}{\partial \dot{p}} = 0 \text{ and we have } \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}}\right) - \frac{\partial L}{\partial p} = 0,$$

and a similar equation for each coordinate. L is called the Lagrangian function.

8.8 The Energy Equation

The energy equation can be deduced from Lagrange's equations as follows.

We have on multiplying each equation by the corresponding generalized velocity,

$$\left(\dot{p}\frac{d}{dt}\frac{\partial T}{\partial \dot{p}} + \dot{q}\frac{d}{dt}\frac{\partial T}{\partial \dot{q}}\right) - \left(\dot{p}\frac{\partial T}{\partial p} + \dot{q}\frac{\partial T}{\partial q}\right)$$
$$= \dot{p}\frac{\partial W}{\partial p} + \dot{q}\frac{\partial W}{\partial q} = \frac{dW}{dt},$$

that is

$$\frac{d}{dt}\left(\dot{p}\frac{\partial T}{\partial \dot{p}} + \dot{q}\frac{\partial T}{\partial \dot{q}}\right) - \left(\ddot{p}\frac{\partial T}{\partial \dot{p}} + \ddot{q}\frac{\partial T}{\partial \dot{q}} + \dot{p}\frac{\partial T}{\partial p} + \dot{q}\frac{\partial T}{\partial q}\right) = \frac{dW}{dt}.$$

Now $$\frac{dT}{dt} = \frac{\partial T}{\partial \dot{p}}\ddot{p} + \frac{\partial T}{\partial \dot{q}}\ddot{q} + \frac{\partial T}{\partial p}\dot{p} + \frac{\partial T}{\partial q}\dot{q}.$$

Also since T is of the form $a\dot{p}^2 + b\dot{q}^2 + c\dot{p}\dot{q}$, where a, b, c are functions of p and q,

$$\dot{p}\frac{\partial T}{\partial \dot{p}} + \dot{q}\frac{\partial T}{\partial \dot{q}} = \dot{p}(2a\dot{p} + c\dot{q}) + \dot{q}(2b\dot{q} + c\dot{p}) = 2T.$$

Therefore $$\frac{d}{dt}(2T) - \frac{dT}{dt} = \frac{dW}{dt},$$

and we have $$T = W + \text{constant},$$
or $$T + V = \text{constant}.$$

The energy equation involves only first derivatives of the coordinates and is often used in conjunction with Lagrange's equations or to replace one of them.

8.9 Oscillations about a Steady State

When Lagrange's equations have been formed it will often be perceived that a steady state of motion is possible when all the coordinates except one are constant and the velocity corresponding to this one coordinate is constant.

The conditions for a steady state are readily obtained from the equations of motion.

Consider the motion of a particle on the inner surface of a smooth spherical bowl. Lagrange's equations obtained in § 8.6, Example 2, are

$$a\ddot{\theta} - a\dot{\phi}^2 \sin\theta\cos\theta = -g\sin\theta, \tag{1}$$
$$(\dot{\phi}\sin^2\theta) = \text{constant}. \tag{2}$$

If a steady state with $\theta = \alpha$ and $\dot{\phi} = \omega$ (a constant) is possible, in this position $\dot{\theta}$ would be zero and $\ddot{\theta} = \ddot{\phi} = 0$.

If $\theta = \alpha$, $\ddot{\phi} = 0$ from (2).

If $\ddot{\theta} = 0$, for $\theta = \alpha$ and $\dot{\phi} = \omega$ we have from (1)

$$a\omega^2 \sin\alpha\cos\alpha = g\sin\alpha,$$

that is $$\cos\alpha = \frac{g}{a\omega^2}.$$

Thus for any value of a, there is a value of ω such that steady motion is possible if $\dot{\theta}$ is zero in this position.

Also for any value of ω there is a value of a for which steady motion is possible, provided that $a\omega^2 > g$.

If the particle is in a state of steady motion and disturbed we may find the period of its oscillation about the steady state.

We have in this example from (2)

$$\dot{\phi} \sin^2 \theta = \omega \sin^2 \alpha,$$

and substituting in (1)

$$a\ddot{\theta} - \frac{a\omega^2 \sin^4 \alpha \cos \theta}{\sin^3 \theta} + g \sin \theta = 0.$$

Let $\theta = \alpha + \delta$ where δ is a small quantity and $\ddot{\theta} = \ddot{\delta}$.

To the first order of small quantities

$$\begin{aligned}
\cos \theta &= \cos \alpha - \delta \sin \alpha, \\
\sin \theta &= \sin \alpha + \delta \cos \alpha, \\
\frac{1}{\sin^3 \theta} &= \frac{1}{\sin^3 \alpha}(1 + \delta \cot \alpha)^{-3} \\
&= \frac{1 - 3\delta \cot \alpha}{\sin^3 \alpha}, \\
\frac{a\omega^2 \sin^4 \alpha \cos \theta}{\sin^3 \theta} &= \frac{g}{\cos \alpha}.\sin \alpha \cos \alpha(1 - \delta \tan \alpha)(1 - 3\delta \cot \alpha) \\
&= g \sin \alpha\{1 - \delta(\tan \alpha + 3 \cot \alpha)\}, \\
g \sin \theta - \frac{a\omega^2 \sin^4 \alpha \cos \theta}{\sin^3 \theta} &= g \sin \alpha . \delta(4 \cot \alpha + \tan \alpha) \\
&= g\delta . \frac{(1 + 3 \cos^2 \alpha)}{\cos \alpha}.
\end{aligned}$$

Hence we have

$$\frac{a \cos \alpha}{1 + 3 \cos^2 \alpha}\ddot{\delta} + g\delta = 0,$$

and the period is that of a simple pendulum of length

$$a \cos \alpha/(1 + 3 \cos^2 \alpha).$$

Example 6. *A bead of mass m can slide on a smooth fixed straight wire which is inclined at an angle α to the vertical. A uniform rod of mass 3m and length 2a is hinged to the bead and can move freely in a vertical plane through the wire. Find the angular velocity of the rod when its inclination to the vertical is θ if the system starts from rest with $\theta = 0$.*

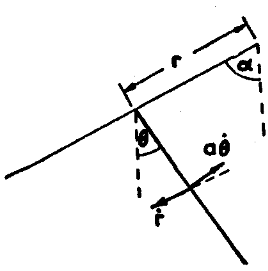

Fig. 170

Let r be the distance moved by the bead down the wire and θ the inclination of the rod to the vertical (Fig. 170). The velocity of the bead is $\dot{r}$; the centre of gravity of the rod has this

velocity and also a relative velocity $a\dot{\theta}$ perpendicular to the rod, and its velocity is

$$\{\dot{r}^2 + a^2\dot{\theta}^2 - 2\dot{r}a\dot{\theta}\sin(\alpha + \theta)\}^{1/2}.$$

The rotational kinetic energy of the rod is $\frac{1}{2}.3m\frac{a^2}{3}\dot{\theta}^2$ and hence we have

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}3m\left\{\dot{r}^2 + \frac{4}{3}a^2\dot{\theta}^2 - 2\dot{r}a\dot{\theta}\sin(\alpha + \theta)\right\},$$

$$W = mgr\cos\alpha + 3mg(r\cos\alpha + a\cos\theta).$$

$$\frac{\partial T}{\partial \dot{r}} = 4m\dot{r} - 3ma\dot{\theta}\sin(\alpha + \theta),$$

$$\frac{\partial T}{\partial r} = 0$$

$$\frac{\partial W}{\partial r} = 4mg\cos\alpha.$$

$$\frac{\partial T}{\partial \dot{\theta}} = 4ma^2\dot{\theta} - 3ma\dot{r}\sin(\alpha + \theta).$$

$$\frac{\partial T}{\partial \theta} = -3m\dot{r}a\dot{\theta}\cos(\alpha + \theta),$$

$$\frac{\partial W}{\partial \theta} = -3mga\sin\theta.$$

Lagrange's equations are

$$4m\ddot{r} - 3ma\ddot{\theta}\sin(\alpha + \theta) - 3ma\dot{\theta}^2\cos(\alpha + \theta) = 4mg\cos\alpha,$$
$$4ma^2\ddot{\theta} - 3ma\ddot{r}\sin(\alpha + \theta) = -3mga\sin\theta.$$

Eliminating $\ddot{r}$ we have

$$16ma^2\ddot{\theta} - 9ma^2\ddot{\theta}\sin^2(\alpha + \theta) - 9ma^2\dot{\theta}^2\sin(\alpha + \theta)\cos(\alpha + \theta)$$
$$= 12mga\cos\alpha\sin(\alpha + \theta) - 12\,mga\sin\theta.$$

Integrating with respect to θ,

$$8ma^2\dot{\theta}^2 - \frac{9}{2}ma^2\dot{\theta}^2\sin^2(\alpha + \theta) = 12mga\{\cos\theta - \cos\alpha\cos(\alpha + \theta) - \sin^2\alpha\}.$$

Hence, $ma\dot{\theta}^2\{16 - 9\sin^2(\alpha + \theta)\} = 24mg\sin\alpha\{\sin(\alpha + \theta) - \sin\alpha\}.$

Example 7. *A small bead of mass m can move on a smooth circular wire of radius a, which can rotate freely about a vertical diameter, about which its moment of inertia is λma^2. If the system is set in motion with angular momentum ma^2h about the vertical diameter and left to itself, prove that the angular distance θ of the bead from the lowest point of the wire is given by*

$$a\ddot{\theta} + \sin\theta\{g - ah^2\cos\theta/(\lambda + \sin^2\theta)^2\} = 0.$$

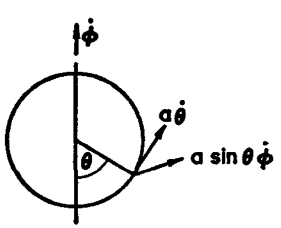

Fig. 171

Show that a position of relative equilibrium $\theta = \alpha$ is possible with the wire rotating at a steady rate ω, if $a\omega^2\cos\alpha = g$ and $\omega^2 > g/a$.
Show also that the length of the equivalent simple pendulum for small oscillations about this steady state is $\dfrac{a\cos\alpha(\lambda + \sin^2\alpha)}{\sin^2\alpha(\lambda + 1 + 3\cos^2\alpha)}$. (L.U.)

Let $\dot{\phi}$ be the angular velocity of the wire about the vertical and θ the inclination to the vertical of the radius to the

wire (Fig. 171). The bead has velocity $a \sin\theta.\dot\phi$ due to the rotation $\dot\phi$ and has also the velocity $a\dot\theta$ in the plane of the wire.

The angular momentum is $\lambda ma^2\dot\phi + ma^2\sin^2\theta.\dot\phi$ and the initial value of this quantity is ma^2h.

We have
$$T = \frac{1}{2}\lambda ma^2\dot\phi^2 + \frac{1}{2}m(a^2\sin^2\theta.\dot\phi^2 + a^2\dot\theta^2),$$
$$W = mga\cos\theta.$$

Lagrange's equations are
$$ma^2\ddot\theta - ma^2\sin\theta\cos\theta.\dot\phi^2 = -mga\sin\theta, \tag{1}$$
$$\frac{d}{dt}(\lambda ma^2 + ma^2\sin^2\theta)\dot\phi = 0. \tag{2}$$

Hence, using the initial condition,
$$(\lambda + \sin^2\theta)\dot\phi = h.$$

Substituting for $\dot\phi$ in (1)
$$a\ddot\theta - \frac{h^2a\sin\theta\cos\theta}{(\lambda+\sin^2\theta)^2} = -g\sin\theta.$$

From (1) if $\theta = \alpha$ and $\dot\phi = \omega$, $\ddot\theta = 0$ if
$$a\sin\alpha\cos\alpha.\omega^2 = g\sin\alpha,$$
that is
$$a\omega^2\cos\alpha = g.$$

Therefore a steady state is possible if $a\omega^2 > g$ and $a\omega^2\cos\alpha = g$.

We have $h = (\lambda + \sin^2\alpha)\omega$, and hence
$$a\ddot\theta + \sin\theta\left\{g - \frac{a(\lambda+\sin^2\alpha)^2\omega^2\cos\theta}{(\lambda+\sin^2\theta)^2}\right\} = 0,$$
that is
$$a\ddot\theta + g\sin\theta\left\{1 - \frac{(\lambda+\sin^2\alpha)^2\cos\theta}{(\lambda+\sin^2\theta)^2\cos\alpha}\right\} = 0.$$

Let $\theta = \alpha + \delta$, where δ is small. Approximately we have
$$\frac{\cos\theta}{\cos\alpha} = 1 - \delta\tan\alpha,$$
$$\lambda + \sin^2\theta = \lambda + \sin^2\alpha + 2\delta\sin\alpha\cos\alpha,$$
$$\left\{\frac{\lambda+\sin^2\alpha}{\lambda+\sin^2\theta}\right\}^2 = 1 - \frac{4\delta\sin\alpha\cos\alpha}{\lambda+\sin^2\alpha},$$
$$1 - \frac{(\lambda+\sin^2\alpha)^2\cos\theta}{(\lambda+\sin^2\theta)^2\cos\alpha} = \delta\left(\tan\alpha + \frac{4\sin\alpha\cos\alpha}{\lambda+\sin^2\alpha}\right),$$
$$= \frac{\delta\sin\alpha(\lambda+\sin^2\alpha+4\cos^2\alpha)}{\cos\alpha(\lambda+\sin^2\alpha)}.$$

Hence
$$a\cos\alpha(\lambda+\sin^2\alpha)\ddot\delta + g\sin^2\alpha(\lambda+1+3\cos^2\alpha)\delta = 0,$$
and the length of the equivalent simple pendulum follows.

Example 8. *A uniform circular disc of radius a is suspended by two vertical inextensible strings, each of length l, attached at the same level to points of its circumference at a distance 2d apart. If one of the strings is cut, determine the initial angular accelerations of the remaining string and the disc, and show that the tension in this string is immediately altered in the ratio* $2a^2 : (a^2 + 2d^2)$.

(L.U.)

Let θ be the inclination to the vertical of the remaining string and ϕ the inclination to the horizontal of the radius to the point of attachment at time t. Initially $\theta = 0$, $\phi = \alpha$, where $\cos\alpha = d/a$ (Fig. 172).

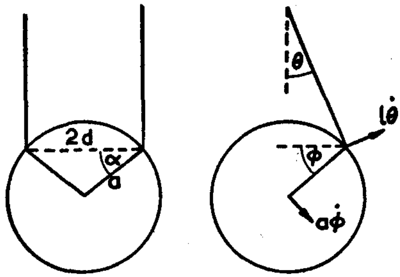

Fig. 172

The centre of the disc has components of velocity $l\dot\theta$ and $a\dot\phi$ including an angle $\dfrac{\pi}{2} - \phi + \theta$. The rotational kinetic energy is $\dfrac{1}{2}m\dfrac{a^2}{2}\dot\phi^2$ and we have

$$T = \frac{1}{2}m\left\{l^2\dot\theta^2 + \frac{3}{2}a^2\dot\phi^2 + 2al\dot\theta\dot\phi\sin(\phi - \theta)\right\},$$

$$W = mg(l\cos\theta + a\sin\phi) + \text{constant}.$$

Therefore Lagrange's equations are

$$\frac{d}{dt}m\{l\dot\theta + a\dot\phi\sin(\phi - \theta)\} + ma\dot\theta\dot\phi\cos(\phi - \theta) = -mg\sin\theta,$$

$$\frac{d}{dt}m\left\{\frac{3}{2}a\dot\phi + l\dot\theta\sin(\phi - \theta)\right\} - ml\dot\theta\dot\phi\cos(\phi - \theta) = mg\cos\phi.$$

Therefore $$l\ddot\theta + a\ddot\phi\sin(\phi - \theta) + a\dot\phi^2\cos(\phi - \theta) = -g\sin\theta,$$

$$\frac{3}{2}a\ddot\phi + l\ddot\theta\sin(\phi - \theta) - l\dot\theta^2\cos(\phi - \theta) = g\cos\phi.$$

In the initial position $\dot\theta = \dot\phi = 0$, $\theta = 0$, $\phi = \alpha$, and we have

$$l\ddot\theta + a\ddot\phi\sin\alpha = 0,$$

$$l\ddot\theta\sin\alpha + \frac{3}{2}a\ddot\phi = g\cos\alpha,$$

$$a\ddot\phi = \frac{g\cos\alpha}{\frac{1}{2} + \cos^2\alpha},$$

$$= \frac{2gad}{a^2 + 2d^2}.$$

$$l\ddot\theta = -\frac{g\sin\alpha\cos\alpha}{\frac{1}{2} + \cos^2\alpha},$$

$$= -\frac{2gd(a^2 - d^2)^{1/2}}{a^2 + 2d^2}.$$

If R be the tension in the string

$$mg - R\cos\theta = m\frac{d^2}{dt^2}(l\cos\theta + a\sin\phi),$$

$$= m(-l\sin\theta.\ddot\theta - \cos\theta.\dot\theta^2 + a\cos\phi.\ddot\phi - a\sin\phi.\dot\phi^2).$$

Therefore initially $mg - R = ma\cos\alpha.\ddot{\phi}$,

$$= mg\frac{\cos^2\alpha}{\frac{1}{2} + \cos^2\alpha},$$

$$R = \frac{mg}{1 + 2\cos^2\alpha} = mg\frac{a^2}{a^2 + 2d^2}.$$

Before the string was cut R was $\frac{1}{2}mg$ and hence the tension is altered in the ratio $2a^2 : a^2 + 2d^2$.

EXERCISES 8 (*a*)

1. Write down Lagrange's equations for a uniform sphere rolling down the line of greatest slope of a plane inclined at an angle α to the horizontal, and show that the acceleration is $5g\sin\alpha/7$.

2. A light inextensible string passes over a pulley. Attached to one end is a mass m and attached to the other end is another pulley. Over this pulley passes a light inextensible string with masses m and $2m$ attached to its ends. Each of the pulleys is a uniform circular disc of mass m and radius a in frictionless bearings. Use Lagrange's equations to obtain the angular accelerations of each of the pulleys.

3. Two masses, $2m$ and m, are attached to the ends of a light inelastic string passing over a pulley of mass m with its axis fixed horizontally. From m is suspended another mass m by means of an elastic string of unstretched length a and modulus mg. If the system is released from rest with the elastic string unstretched and the pulley is considered as a solid circular disc, prove that each mass will move with simple harmonic motion, and find the period and the distance over which the mass $2m$ oscillates. (L.U., Pt. II)

4. Two particles A and B, each of mass m, are attached to the ends of a light stiff spiral spring AB and the system is placed on a smooth horizontal table. A blow of impulse I is applied to A in the direction AB. Prove that the greatest compression is $I/\sqrt{(2mS)}$ where S is the stiffness of the spring. Prove also, that when the spring regains its natural length for the first time it has moved forward a distance $\pi I/2\sqrt{(2mS)}$. (L.U., Pt. II)

5. Two particles, A and B, of equal mass m are attached to the ends of a light spring which exerts a tension of amount s per unit extension. Initially the particles are at rest on a smooth horizontal table with the spring just taut, and a constant force of magnitude sa is then applied to particle B in the direction AB. Obtain the differential equations for the displacements, x and y, of the particles A and B respectively at time t, and show that $y = \frac{1}{4}a\omega^2t^2, y - x = \frac{1}{2}a(1 - \cos\omega t)$, where $m\omega^2 = 2s$. (L.U., Pt. II)

6. A uniform rod of mass m and length $2a$ is free to turn in a vertical plane about one end which is fixed. A light inextensible string of length $2a$ is attached to the other end of the rod and carries a particle of mass m at its free end. The system moves in a vertical plane and the

rod and the string are inclined at angles θ and ϕ respectively to the vertical. Write down Lagrange's equations of motion in terms of θ and ϕ.

7. A uniform rod of length $2a$ is held in a horizontal position by two light inextensible vertical strings each of length l attached to its ends. One of the strings is suddenly cut. Write down Lagrange's equations for the inclination θ of the other string to the vertical and the inclination ϕ of the rod to the horizontal. Deduce the initial acceleration of the rod.

8. A particle is free to move on the smooth inner surface of a sphere of centre O and radius a and OZ is the vertical through O. The particle is initially in the horizontal plane through O and is given an angular velocity ω about O. Show that if r is the distance of the particle from OZ at any instant,
$$\dot{r}^2 = (1 - r^2/a^2)^{3/2}\{2ga - \omega^2a^4(1 - r^2/a^2)^{1/2}/r^2\}.$$

9. A uniform rod AB of length $2a$ can turn freely about the end A which is fixed. Initially B is below A, the rod is inclined at an angle α to the vertical and is made to rotate about the vertical through A with angular velocity ω. Show that if $4a\omega^2 \cos \alpha = 3g$ the rod remains inclined at α to the vertical, and that in any case its inclination to the vertical lies between α and β where $\cos \beta = -\varrho + (1 - 2\varrho \cos \alpha + \varrho^2)^{1/2}$ and $\varrho = a\omega \sin^2 \alpha/(3g)$.

10. A uniform rod AB of length $2a$ can turn freely about the end A which is fixed. Show that steady motion is possible with the rod inclined at an angle α to the vertical and turning about the vertical through O with angular velocity ω if $4a\omega^2 \cos \alpha = 3g$. If the rod is slightly disturbed from this position show that the period of a small oscillation is the same as that of a simple pendulum of length $4a \cos \alpha/\{3(1 + 3\cos^2 \alpha)\}$.

11. Three flywheels A, B, C have moments of inertia I_1, I_2, I_3 respectively and are connected by light gears so that their angular velocities are in the ratios $n_1 : n_2 : n_3$. A torque T is applied to the flywheel A. Write down the kinetic energy of the system and the work done by the torque in any position and use Lagrange's equations to find the angular acceleration of the flywheel C.

12. A uniform sphere of mass m and radius a rolls down a line of greatest slope of a plane surface of a wedge which is inclined at an angle α to the horizontal. The wedge is of mass M and rests on a smooth horizontal plane. Find the acceleration of the wedge.

13. A truck of total mass M has four wheels each of which is a uniform circular disc of mass m. The truck rolls without slipping down a plane of inclination α. A uniform circular cylinder of mass m, lies on the floor of the truck, which is parallel to the plane, with its axis perpendicular to the line of greatest slope of the plane and can roll without slipping on the floor. Find the acceleration of the truck and the cylinder down the plane.

14. A uniform rod of mass $3m$ and length $2a$ has its centre fixed and a mass m attached to one end. The rod is set rotating about a vertical

axis through its centre with angular velocity ω when its inclination to the vertical is α. Show that steady motion is possible in this position if $2a\omega^2 \cos \alpha = g$ and find the length of the equivalent simple pendulum for oscillations about this position.

15. A uniform rod AB of length c and mass M, is attached to a fixed point A of a smooth horizontal table, about which it is free to turn. It is smoothly jointed at B to a uniform rod BD of length l and mass m, which slides through a small swivel ring fixed to the table at C, where $AC = b$. If $\theta = \angle CAB$ and $\phi = \angle ACB$, prove that

$$\dot{\phi} = c\dot{\theta}(b \cos \theta - c)/a^2 \text{ where } a^2 = b^2 + c^2 - 2bc \cos \theta,$$

and that the kinetic energy T of the system is given by

$$6T = c^2\dot{\theta}^2\{M + 3m + ml(b \cos \theta - c)^2(l - 3a)/a^4\}.$$

If $\dot{\theta}$ is constrained to be constant by a couple G applied to AB, show that $G = \partial T/\partial\theta$ and that it vanishes whenever the rods are collinear or at right-angles. (L.U.)

16. Two uniform rods AB, BC, of mass $3M$ and M and lengths a and $4a$ respectively are smoothly hinged at B and pivoted to a fixed pin at A. If the system is set moving in a vertical plane, under gravity, prove that the kinetic energy of the system is

$$T = Ma^2\dot{\theta}^2 + \frac{8}{3}Ma^2\dot{\phi}^2 + 2Ma^2\dot{\theta}\dot{\phi} \cos (\theta - \phi),$$

where θ and ϕ are the angles which AB and BC make with the downward vertical at any instant, and find the potential energy.

Find the initial angular accelerations of the two rods if the system is set in motion with the rods in a line inclined at an angle α to the downward vertical. (L.U.)

17. One end A of a heavy uniform rod AB can slide on a smooth vertical wire OZ below O, the other end being attached by a light inextensible string to O and $OB = AB$. If the system is free to move so that the plane AOB has an angular velocity $\dot{\phi}$ when the angle AOB is θ, show that the kinetic energy is

$$T = \frac{1}{2}ma^2\left(\frac{4}{8}\dot{\theta}^2 + 8 \sin^2 \theta . \dot{\theta}^2 + \sin^2 \theta . \dot{\phi}^2\right).$$

Find the Lagrange equations of motion and show that a state of steady motion is possible in which A is at a depth d below O and $\dot{\phi} = 3(g/d)^{1/2}$. (L.U.)

18. A uniform rod AB of mass m and length $2a$ is at rest at time $t = 0$, suspended from a fixed smooth horizontal rail by means of a small ring at A of mass m. A force F is applied at A along the rail. If x is the distance moved by A from its initial position O and the angle OAB is $\frac{1}{2}\pi - \theta$, obtain the equations of motion of the system.

If A is constrained to move with constant acceleration g, show that the rod oscillates between the vertical $\theta = 0$ and the horizontal position $\theta = \pi/2$, and that

$$F = \frac{mg}{8} \{19 - 12 \sin \theta + 9(\sin 2\theta - \cos 2\theta)\}. \qquad \text{(L.U.)}$$

19. A uniform rod AB of mass m and length $2a$ is at rest suspended from a fixed smooth horizontal rail by means of a small light ring at A. A force F applied at A along the rod makes A move with constant acceleration f. Show that the rod oscillates between its original direction and one in which it is inclined at an angle 2α with the downward vertical, given by $\tan \alpha = f/g$. Show that when its inclination is θ,

$$F=\frac{1}{8}m\{11f - 12g \sin \theta - 9(f \cos 2\theta - g \sin 2\theta)\}. \qquad \text{(L.U.)}$$

20. A heavy bead of mass m is free to slide on a smooth uniform rod of mass $3\lambda m/4$ and length $2a$, which is turning about one end. If θ and ϕ are the Euler angles specifying the orientation of the rod at time t, and x is the distance of the bead from the fixed end, and initially $\dot{\phi} = \omega$, $\theta = \alpha$, $\dot{\theta} = \dot{x} = x = 0$, find the initial values of the generalized accelerations.

8.10 The Top

A symmetrical top is a rigid body which is a solid of revolution. We shall consider the motion of a top when it is given a rotation about its axis of symmetry and one point of the axis is fixed. It is convenient to take as generalized coordinates to describe the motion the Eulerian angles θ, ϕ and ψ.

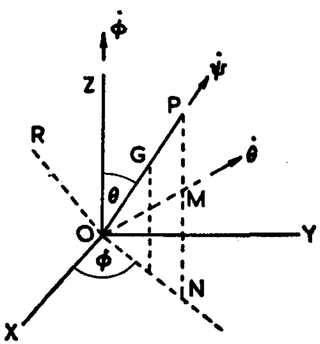

Fig. 173

Let OP (Fig. 173) be the axis of the top, O the fixed point, OZ the vertical through O and OX and OY fixed perpendicular axes. θ is the angle which OP makes with OZ, ϕ is the angle which the projection ON of OP on the plane OXY makes with OX. ψ is the angle through which the top turns about its axis, being positive if the rotation would drive a right-handed screw in the direction OP.

The corresponding generalized velocities are:

$\dot{\theta}$, which is the angular velocity of the axis about a line OM through O perpendicular to the plane ZOP,

$\dot{\phi}$, which is the angular velocity of the axis about OZ,

$\dot{\psi}$, which is the angular velocity of the top about its axis.

Motion due to change of θ is called *nutation*, motion due to change of ϕ is called *precession*. The general motion of the top is one of nutation and precession.

8.11 Kinetic and Potential Energy

The top has three angular velocities $\dot{\theta}$, $\dot{\phi}$, and $\dot{\psi}$, about OM, OZ, and OP respectively. Let OR be perpendicular to OP in the plane ZOP. Then the angular velocity $\dot{\phi}$ about OZ may be resolved into

components $\dot\phi \cos\theta$ about OP and $\dot\phi \sin\theta$ about OR. We thus have angular velocities in three perpendicular directions,

$$\dot\psi + \dot\phi\cos\theta \text{ about } OP,$$
$$\dot\phi\sin\theta \text{ about } OR,$$
$$\dot\theta \text{ about } OM.$$

Let A be the amount of inertia of the top about its axis and B the moment of inertia about any perpendicular axis through O. The products of inertia about OP, OR and OM are zero and the kinetic energy is, by § 7.19,

$$T = \frac{1}{2}\{A(\dot\psi + \dot\phi\cos\theta)^2 + B(\dot\phi\sin\theta)^2 + B\dot\theta^2\}.$$

Let G be the centre of gravity of the top and h its distance from O. Then the potential energy is $V = mgh\cos\theta$, and the work function is $W = -V = -mgh\cos\theta$.

8.12 Equations of Motion of the Top

Lagrange's equations for the top are:

$$\frac{d}{dt}(B\dot\theta) + A(\dot\psi + \dot\phi\cos\theta)\dot\phi\sin\theta - B\dot\phi^2\sin\theta\cos\theta = mgh\sin\theta, \quad (1)$$

$$\frac{d}{dt}\{A(\dot\psi + \dot\phi\cos\theta)\cos\theta + B\dot\phi\sin^2\theta\} = 0, \quad (2)$$

$$\frac{d}{dt}\{A(\dot\psi + \dot\phi\cos\theta)\} = 0. \quad (3)$$

From (3) we have that $\dot\psi + \dot\phi\cos\theta$ is constant, and denoting this constant by N (the spin about the axis of the top), we have

$$\dot\psi + \dot\phi\cos\theta = N. \quad (4)$$

Substituting this value in (2) we have

$$AN\cos\theta + B\dot\phi\sin^2\theta = K \text{ (a constant)}. \quad (5)$$

Instead of (1) we may use the energy equation and we have, remembering that N is constant,

$$\frac{1}{2}B(\dot\theta^2 + \dot\phi^2\sin^2\theta) + mgh\cos\theta = E \text{ (a constant)}. \quad (6)$$

8.13 Description of the Motion

Let the initial conditions which determine the constants of integration be $\theta = \alpha$, $\dot\theta = \dot\phi = 0$ and $\dot\psi = N$. Then the three top equations become

$$\dot\psi + \dot\phi\cos\theta = N, \quad (1)$$
$$AN\cos\theta + B\dot\phi\sin^2\theta = AN\cos\alpha, \quad (2)$$

$$\frac{1}{2}B\dot\phi^2\sin^2\theta + \frac{1}{2}B\dot\theta^2 + mgh\cos\theta = mgh\cos\alpha. \quad (3)$$

Eliminating $\dot{\phi}$ between (2) and (3) we have

$$\dot{\theta}^2 \sin^2\theta = \frac{2mgh}{B}\sin^2\theta(\cos\alpha - \cos\theta) - \frac{A^2N^2}{B^2}(\cos\alpha - \cos\theta)^2.$$

Hence if $\dot{\theta} = 0$ we have

either $\qquad \cos\alpha - \cos\theta = 0,$

or $$\sin^2\theta = \frac{A^2N^2}{2mghB}(\cos\alpha - \cos\theta).$$

Writing $s = \dfrac{A^2N^2}{4mghB}$, the last equation becomes

$$\cos^2\theta - 2s\cos\theta - 1 + 2s\cos\alpha = 0,$$
$$\cos\theta = s \pm \sqrt{(s^2 - 2s\cos\alpha + 1)}.$$

The surd with the positive sign gives a value of $\cos\theta$ greater than 1 and hence $\dot{\theta} = 0$ if $\theta = \alpha$ or $\theta = \beta$, where

$$\cos\beta = s - \sqrt{(s^2 - 2s\cos\alpha + 1)},$$

and it is easily seen that $\beta > \alpha$.

Thus the axis of the top oscillates between $\theta = \alpha$ and $\theta = \beta$. When $\theta = \alpha$, $\dot{\phi} = 0$, and when $\theta = \beta$, $\dot{\phi} = \dfrac{AN}{2Bs} = \dfrac{2mgh}{AN}$.

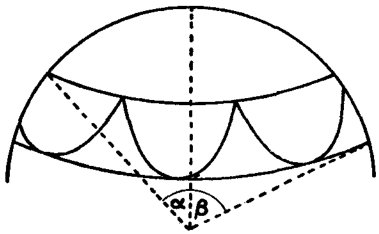

Fig. 174

The motion of the axis of the top thus lies between two cones of semi-vertical angles α and β and its trace on a unit sphere with centre at O is as shown in Fig. 174.

8.14 Stability in the Vertical Position

If initially $\theta = 0$, Lagrange's equation for ϕ gives

$$\dot{\phi}\sin^2\theta = \frac{AN}{B}(1 - \cos\theta).$$

Substituting for $\dot{\phi}$ in Lagrange's equation for θ we have

$$B\ddot{\theta} + \frac{A^2N^2(1 - \cos\theta)}{B\sin\theta} - \frac{A^2N^2}{B}\frac{(1 - \cos\theta)^2\cos\theta}{\sin^3\theta} - mgh\sin\theta = 0.$$

that is
$$B\ddot{\theta} + \left\{\frac{A^2N^2}{B}.\frac{(1-\cos\theta)^2}{\sin^4\theta} - mgh\right\}\sin\theta = 0,$$
$$B\ddot{\theta} + \left\{\frac{A^2N^2}{B}.\frac{1}{(1+\cos\theta)^2} - mgh\right\}\sin\theta = 0.$$
If θ is small so that squares and higher powers may be neglected we have
$$B\ddot{\theta} + \left(\frac{A^2N^2}{4B} - mgh\right)\theta = 0.$$
Thus the motion is simple harmonic about $\theta = 0$ and stable if
$$A^2N^2 > 4Bmgh,$$
and the periodic time is
$$2\pi\left\{\frac{A^2N^2}{4B^2} - \frac{mgh}{B}\right\}^{-1/2}.$$
The motion of a rotating projectile about its centre of gravity is analogous to the motion of a top. The shell, whose moment of inertia about its axis is A and about a perpendicular axis through its centre of gravity is B, is given a spin N. In place of the moment $mgh \sin\theta$ of the weight of the top we have the moment $\mu \sin\theta$ of the aerodynamic forces on the shell where θ is the yaw of the shell from its trajectory. Then the shell is stable when it emerges from the muzzle of the gun with a slight yaw if
$$A^2N^2 > 4B\mu.$$
The quantity $s = \dfrac{A^2N^2}{4B\mu}$ is called the stability coefficient of the shell and must be greater than unity.

8.15 Steady Precession

Lagrange's equation for θ is (§ 8.12 (1))
$$B\ddot{\theta} + AN\dot{\phi}\sin\theta - B\dot{\phi}^2\sin\theta\cos\theta = mgh\sin\theta.$$
A state of steady precession in which θ is constant is possible if $\ddot{\theta} = 0$, that is
$$B\dot{\phi}^2\cos\theta - AN\dot{\phi} + mgh = 0,$$
that is
$$\dot{\phi} = \frac{AN \pm \sqrt{(A^2N^2 - 4Bmgh\cos\theta)}}{2B\cos\theta}.$$
Thus provided that $A^2N^2 > 4Bmgh\cos\theta$ there are two values of $\dot{\phi}$ for any value of θ at which steady precession is possible. One is a fast rate difficult to obtain outside a laboratory while the slow rate is easily obtained with a child's top. It should be noted that $\ddot{\phi}$ and $\dot{\theta}$ are related by Lagrange's equation for ϕ, and $\dot{\theta} = 0$ implies $\ddot{\phi} = 0$.

An important case of steady precession occurs when the centre of

gravity of the top is at the fixed point, that is $h = 0$. In this case there is only one speed of precession given by

$$B\dot{\phi} \cos \theta - AN = 0.$$

Writing $N = \dot{\psi} + \dot{\phi} \cos \theta$, this gives the relation between $\dot{\phi}$ and $\dot{\psi}$, that is

$$\dot{\phi} = -\frac{A\dot{\psi}}{(A - B) \cos \theta}.$$

Thus if $A > B$, $\dot{\phi}$ and $\dot{\psi}$ have opposite signs and if $A < B$ they have the same sign.

8.16 The Gyroscopic Moment

When the centre of gravity coincides with the fixed point it is possible to have steady precession with $\ddot{\theta} = 0$ for any value of $\dot{\phi}$ if there is a turning moment to make $\ddot{\theta} = 0$.

Let the work function be such that

$$\frac{\partial W}{\partial \theta} = M, \quad \frac{\partial W}{\partial \phi} = \frac{\partial W}{\partial \psi} = 0.$$

Then work is done by the moment M only as θ changes and the moment is about the axis OM (Fig. 173) perpendicular to the axis of the top and to OZ.

Lagrange's equation for θ becomes

$$B\ddot{\theta} + AN\dot{\phi} \sin \theta - B\dot{\phi}^2 \sin \theta \cos \theta = M,$$

and if $\ddot{\theta} = 0$ we have $M = AN\dot{\phi} \sin \theta - B\dot{\phi}^2 \sin \theta \cos \theta$.

In particular if $\theta = \frac{\pi}{2}$,

$$M = AN\dot{\phi} = A\dot{\psi}\dot{\phi}.$$

The moment $-M$ is called the *gyroscopic moment* and is of considerable importance in engineering. For example, if a shaft carrying a rotating wheel is made to turn in a given plane the moment M must be exerted to make it do so, that is, the wheel will exert a moment $-M$ on the bearings. If an aeroplane with a propellor rotating in a clockwise direction as seen by the pilot turns left the shaft will exert the moment $-M$ on the aeroplane in the sense tending to lift the nose.

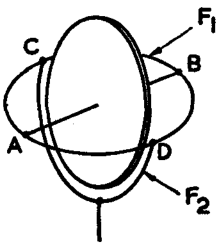

Fig. 175

8.17 The Gyroscope

A gyroscope consists essentially of a wheel or disc free to turn about a horizontal axle AB which is held in a frame F_1 (Fig. 175). The frame F_1 may itself be held in a frame F_2 so that it is free to turn about the horizontal axis CD which is perpendicular to AB, while the frame F_2 can itself

turn about a vertical axis. The centre of gravity is in line with the vertical axis and fixed so that the motion is that of a top with its centre of gravity at the fixed point. If the gyroscope is held in the frame F_1 only and supported at B the moment of the weight about the horizontal through B perpendicular to AB will cause precession about the vertical through B.

If the gyroscope has moment of inertia I about its axis and spin ω and turns about the vertical with angular velocity $\dot{\phi}$ the gyroscopic moment required to cause this precession is $M = I\dot{\phi}\omega$ about CD (Fig. 176). This may be caused by a weight placed at B.

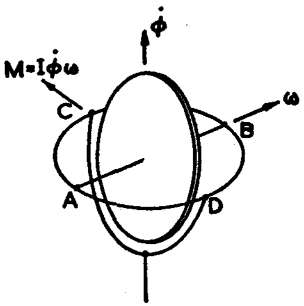

Fig. 176

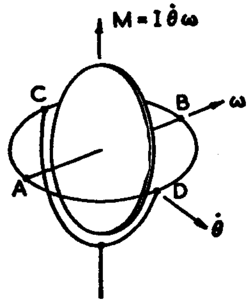

Fig. 177

If it turns about CD with angular velocity $\dot{\theta}$ (Fig. 177) the gyroscopic moment required to cause this precession is $M = I\dot{\theta}\omega$ about the vertical. The direction of the gyroscopic moments may be remembered by writing them as the vector products $I\dot{\boldsymbol{\phi}} \times \boldsymbol{\omega}$ and $I\dot{\boldsymbol{\theta}} \times \boldsymbol{\omega}$.

8.18 The Gyroscopic Compass

The parallel to the earth's axis in latitude North λ lies in a meridian plane inclined at an angle λ to the North line (Fig. 178). The earth's angular velocity Ω has therefore components $\Omega \cos \lambda$ about the North line and $\Omega \sin \lambda$ about the vertical (Fig. 179). Let the axis of a gyroscope in this latitude be inclined at an angle θ to the North line.

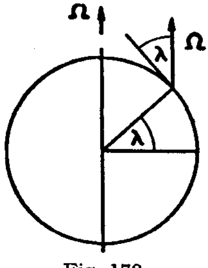

Fig. 178

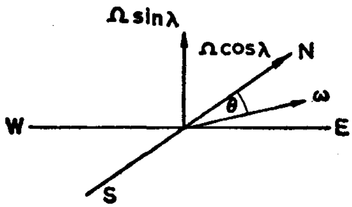

Fig. 179

Let its spin be ω, its moment of inertia about its axis A and about a perpendicular through its centre of gravity B. As θ changes the axis has angular velocity $\dot{\theta}$ about the vertical and the components of angular velocity about the axis of the gyroscope and two perpendicular axes are then

$$\omega + \Omega \cos \lambda \cos \theta,\ \Omega \cos \lambda \sin \theta,\ \Omega \sin \lambda - \dot{\theta}.$$

The kinetic energy is

$$T = \frac{1}{2}\{A(\omega + \Omega \cos \lambda \cos \theta)^2 + B\Omega^2 \cos^2 \lambda \sin^2 \theta + B(\Omega \sin \lambda - \dot{\theta})^2\}.$$

We have

$$\frac{\partial T}{\partial \omega} = A(\omega + \Omega \cos \lambda \cos \theta) = \text{constant} = AN,$$

$$\frac{\partial T}{\partial \dot{\theta}} = B(\dot{\theta} - \Omega \sin \lambda),$$

$$\frac{\partial T}{\partial \theta} = -A(\omega + \Omega \cos \lambda \cos \theta)\Omega \cos \lambda \sin \theta + B\Omega^2 \cos^2 \lambda \sin \theta \cos \theta,$$

$$\frac{\partial W}{\partial \theta} = 0,$$

and Lagrange's equation for θ is

$$B\ddot{\theta} + AN\Omega \cos \lambda \sin \theta - B\Omega^2 \cos^2 \lambda \sin \theta \cos \theta = 0.$$

Neglecting Ω^2 we have

$$B\ddot{\theta} + AN\Omega \cos \lambda \sin \theta = 0,$$

and this equation represents an oscillatory motion about $\theta = 0$. This oscillation is damped out by frictional forces so that the axis of the gyroscope comes to point due North. In a ship's compass the gyroscope floats in mercury and its frame is loaded centrally by an attached mass. It is the variation in the direction of the gravity pull on this mass which causes the axis to precess so that it comes to point North.

EXERCISES 8 (*b*)

1. A top consists of a light pin through the centre of a uniform circular disc of radius 4 in. with 3 in. of the pin below the disc. Show that for steady precession in which the rim does not touch the ground the spin must exceed 27·4 rad./sec.

2. A top of mass 5 lb. spins on a perfectly rough horizontal plane. Its moment of inertia about its axis is 18 lb.in.^{2} and about a perpendicular through its point is 198 lb.in.^{2}. The distance of the centre of gravity from the point is 6 in. Find the least spin required for steady precession with the axis of the top inclined at 30° to the vertical and the rate of precession at this spin.

3. A top of mass 5 lb. spins on a perfectly rough horizontal plane. Its moment of inertia about its axis and a perpendicular through its point are 18 lb.in.^{2} and 198 lb.in.^{2} respectively and its centre of gravity is 6 in. from its point. If the top is spinning with its axis vertical at 300 rad./sec. show that it is stable and find the period of its small oscillations if it is disturbed from this position.

4. A shell has moments of inertia 22·5 lb.in.^{2} and 200 lb.in.^{2} about its axis of symmetry and about a perpendicular through its centre of gravity respectively. When its axis is inclined at a small angle θ to the trajectory the aerodynamic turning moment about the centre of gravity is 186 sin θ ft.lb. at a certain velocity. Find the spin required to make the shell just stable at this velocity. If the spin is in fact 1645 rad./sec., find the period of small oscillations about the trajectory and show that the rate of precession about the trajectory is nearly 93 rad./sec.

5. A light rod AB of length 2 ft. which can turn freely about A carries a uniform disc of mass 5 lb. and radius 1 ft. at the end B in a plane perpendicular to the rod. If the disc is given a rotation of 300 rad./sec. about AB find the possible speeds of steady precession of the rod about the vertical inclined at 30° to the vertical.

If the rod is constrained to precess at twice the lower of these rates, find the couple exerted.

6. A uniform circular disc of mass 4 lb. and radius 1 ft. rotates at 100 rad./sec. about an axle through its centre inclined at 30° to its plane. Find the couple exerted by the axle on its bearings.

7. The flywheel of a car has moment of inertia 15 lb.ft.^{2} and turns about its axis, which is horizontal and in the direction of motion of the car at 3700 r.p.m. when the car is travelling at 60 m.p.h. The distance between the front and rear axles of the car is 8 ft. If the car takes a left-hand bend travelling in a circle of radius 40 ft. at this speed, show that the pressure on the front wheels is increased by about 50 lb.

8. The propeller of an aircraft is rotating at 1500 r.p.m. in a clockwise direction as seen by the pilot and its moment of inertia about the axis of rotation is 500 ft.lb. As the aircraft lands and its main wheels touch the runway a rear wheel on the fuselage 40 ft. behind them begins to descend at 10 ft./sec. Find the magnitude and direction of the couple tending to make the aircraft swerve.

9. A pair of wheels and axle are rolling along a straight railway track at a speed of 75 m.p.h. The effective radius of the wheels is 2 ft., the effective width of the track is 5 ft., and the moment of inertia of the system about the centre line of the axle is 3000 lb.ft.^{2}. One rail has a track imperfection consisting of a rise of 1/10 in. acquired in a distance of 30 in., the new level then remaining constant. If the formation of the rise is given by the equation $y = 0{\cdot}1 \sin^2 \pi x/60$, on the assumption that the wheel remains in contact with the rail, determine the magnitude and direction of the gyroscopic couple brought into existence at the instant the wheel has its maximum vertical velocity.

(C.U.)

10. A turn indicator in an aircraft consists of a wheel of moment of inertia I turning about a horizontal axle with angular velocity ω. The axle is mounted in a frame which can turn about a perpendicular horizontal axis through the centre of the wheel. A mass m rigidly attached to the frame is at a depth h below the centre of the wheel. Show that if the system is given a rotation Ω about a vertical axis the frame will take up a position inclined at an angle θ to the vertical, where $\tan\theta = \dfrac{I\omega\Omega}{mgh}$.

8.19 Small Oscillations

A simplified form of Lagrange's equations is used to give the equations of motion of a system which is oscillating about a position of stable equilibrium with two or more degrees of freedom.

The differential equations of most oscillations of this type are essentially non-linear. For example, the equation of motion of a simple pendulum, $l\ddot{\theta} = -g\sin\theta$, cannot be solved in terms of elementary functions, but if squares and higher powers of θ are neglected the equation is $l\ddot{\theta} = -g\theta$, and can be integrated at once. Similar approximations are made to linearize the equations of motion when there are two or more degrees of freedom and the equations are given directly by a linearized form of Lagrange's equations. The linearized equations may be obtained by direct approximation in Lagrange's equations, as may be seen from the following example.

Example 9. *A uniform rod AB of mass $2m$ and length $2a$ is free to turn about the end A. A second rod BC of mass $3m$ and length $2a$ is freely hinged to the first at B and the system moves in a vertical plane. Find the equations of small oscillations about the position of stable equilibrium.*

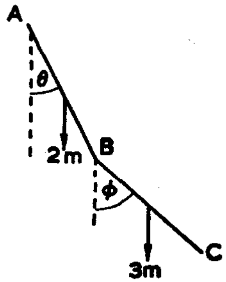

Fig. 180

Let θ and ϕ be the inclinations of the rods to the vertical (Fig. 180). The kinetic energy of AB is

$$\frac{1}{2}\cdot 2m\cdot\frac{4}{3}a^2\dot{\theta}^2.$$

The centre of gravity of BC has velocities $2a\dot{\theta}$ and $a\dot{\phi}$ inclined at angles θ and ϕ respectively to the horizontal. The kinetic energy of BC is

$$\frac{1}{2}\cdot 3m\left\{a^2\dot{\phi}^2 + 4a^2\dot{\theta}^2 + 4a^2\dot{\theta}\dot{\phi}\cos(\phi-\theta) + \frac{a^2}{3}\dot{\phi}^2\right\}.$$

The total kinetic energy of the system is

$$T = \frac{1}{2}ma^2\left\{\frac{44}{3}\dot{\theta}^2 + 12\dot{\theta}\dot{\phi}\cos(\phi-\theta) + 4\dot{\phi}^2\right\}.$$

The potential energy is

$$\begin{aligned} V &= -2mga\cos\theta - 3mg(2a\cos\theta + a\cos\phi) \\ &= -mga(8\cos\theta + 3\cos\phi). \end{aligned}$$

Lagrange's equations are

$$ma^2\left\{\frac{44}{3}\ddot{\theta} + 6\ddot{\phi}\cos(\phi-\theta) - 6\dot{\phi}^2\sin(\phi-\theta)\right\} + 8mga\sin\theta = 0,$$

$$ma^2\{4\ddot{\phi} + 6\ddot{\theta}\cos(\phi-\theta) + 6\dot{\theta}^2\sin(\phi-\theta)\} + 3mga\sin\phi = 0.$$

Neglecting squares and higher powers of θ, ϕ, $\dot\theta$ and $\dot\phi$ we have

$$\frac{44}{3}\ddot\theta + 6\ddot\phi + 8(g/a)\theta = 0,$$

$$6\ddot\theta + 4\ddot\phi + 3(g/a)\phi = 0.$$

These are the linearized equations which describe the motion.

8.20 Linearized Form of Lagrange's Equations

Let us suppose that the motion has two degrees of freedom and can be expressed in terms of generalized coordinates p and q. We shall further suppose that p and q are both zero in the position of equilibrium.

The kinetic energy of the system is

$$T = \frac{1}{2}(a\dot p^2 + 2b\dot p\dot q + c\dot q^2),$$

where a, b and c are functions of p and q. It is evident that the terms $\frac{\partial T}{\partial p}$ and $\frac{\partial T}{\partial q}$ involve squares of the generalized velocities and can be omitted. Also if a, b, c be expanded in powers of p and q and a_0, b_0, c_0 are their values when $p = q = 0$, to the first order of small quantities we have

$$\frac{\partial T}{\partial \dot p} = a_0\dot p + b_0\dot q,$$

$$\frac{\partial T}{\partial \dot q} = b_0\dot p + c_0\dot q.$$

Thus we may write

$$T_0 = \frac{1}{2}(a_0\dot p^2 + 2b_0\dot p\dot q + c_0\dot q^2),$$

which is the value of the kinetic energy as the system passes through the equilibrium position, and take the first terms in Lagrange's equations as $\frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot p}\right)$ and $\frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot q}\right)$ respectively.

It is usually more convenient to use the potential energy rather than the work function in these problems. Since the potential energy V is a function of p and q we can use the Taylor series expansion for V in the neighbourhead of $p = q = 0$.

We have

$$V = (V)_0 + p\left(\frac{\partial V}{\partial p}\right)_0 + q\left(\frac{\partial V}{\partial q}\right)_0$$
$$+ \frac{1}{2!}\left\{p^2\left(\frac{\partial^2 V}{\partial p^2}\right)_0 + 2pq\left(\frac{\partial^2 V}{\partial p\partial q}\right)_0 + q^2\left(\frac{\partial^2 V}{\partial q^2}\right)_0\right\} + \ldots.$$

Now in a position of equilibrium the potential energy must have a stationary value, and the condition for this when $p = q = 0$ is

$$\left(\frac{\partial V}{\partial p}\right)_0 = 0, \quad \left(\frac{\partial V}{\partial q}\right)_0 = 0.$$

For stable equilibrium the turning value must be a minimum, for which the conditions are

$$\left(\frac{\partial^2 V}{\partial p^2}\right)_0 > 0, \quad \left(\frac{\partial^2 V}{\partial p^2}\right)_0 \left(\frac{\partial^2 V}{\partial q^2}\right)_0 - \left(\frac{\partial^2 V}{\partial p \partial q}\right)_0^2 > 0.$$

Thus we may write the generalized forces in Lagrange's equations as

$$-\frac{\partial V_0}{\partial p} \text{ and } -\frac{\partial V_0}{\partial q},$$

where $V_0 = \frac{1}{2}(\alpha p^2 + 2\beta pq + \gamma q^2)$ and $\alpha > 0$, $\alpha\gamma - \beta^2 > 0$.

Then Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{p}}\right) + \frac{\partial V_0}{\partial p} = 0,$$

$$\frac{d}{dt}\left(\frac{\partial T_0}{\partial \dot{q}}\right) + \frac{\partial V_0}{\partial q} = 0.$$

It is not necessary to form the function V_0 as the necessary approximation can usually be made in $\dfrac{\partial V}{\partial p}$ and $\dfrac{\partial V}{\partial q}$.

The method described is easily generalized for a system with any number of degrees of freedom.

Example 10. *A uniform rod of mass M and length 2a can turn freely about a fixed horizontal axis through its mid-point. A particle of mass m is attached to one end of the rod by a light inextensible string of length l. Find the equations of motion for small oscillations about the position of stable equilibrium.*

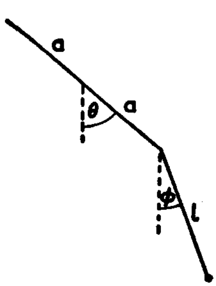

Fig. 181

Let the rod be inclined at an angle θ and the string at an angle ϕ to the vertical (Fig. 181). The particle has the velocity $a\dot{\theta}$ of the upper end of the string and the relative velocity $l\dot{\phi}$, both in a horizontal direction when $\theta = \phi = 0$. Therefore

$$T_0 = \frac{1}{2}M\frac{a^2}{3}\dot{\theta}^2 + \frac{1}{2}m(a\dot{\theta} + l\dot{\phi})^2.$$

$$V = -mg(a\cos\theta + l\cos\phi)$$

$$= -mg\left\{a\left(1 - \frac{\theta^2}{2}\right) + l\left(1 - \frac{\phi^2}{2}\right)\right\} + \dots.$$

$$V_0 = mg\left(\frac{1}{2}a\theta^2 + \frac{1}{2}l\phi^2\right) + \text{constant.}$$

$$\frac{\partial T_0}{\partial \dot\theta} = \left(M\frac{a^2}{3} + ma^2\right)\dot\theta + mal\dot\phi,$$

$$\frac{\partial T_0}{\partial \dot\phi} = ml^2\dot\phi + mal\dot\theta,$$

$$\frac{\partial V_0}{\partial \theta} = mga\theta,$$

$$\frac{\partial V_0}{\partial \phi} = mgl\phi.$$

Lagrange's linearized equations are

$$\left(\frac{1}{3}M + m\right)a\ddot\theta + ml\ddot\phi + mg\theta = 0,$$

$$ma\ddot\theta + ml\ddot\phi + mg\phi = 0.$$

8.21 Solution of the Equations of Motion

When there are two degrees of freedom the linearized equations of motion are of the form

$$a\ddot p + b\ddot q + \alpha p + \beta q = 0,$$
$$b\ddot p + c\ddot q + \beta p + \gamma q = 0.$$

The equations are solved by assuming solutions

$$p = A \sin(\omega t + \varepsilon),$$
$$q = B \sin(\omega t + \varepsilon).$$

Substituting in the differential equations, we have

$$A(a\omega^2 - \alpha) + B(b\omega^2 - \beta) = 0,$$
$$A(b\omega^2 - \beta) + B(c\omega^2 - \gamma) = 0.$$

Eliminating A and B we have

$$\begin{vmatrix} a\omega^2 - \alpha & b\omega^2 - \beta \\ b\omega^2 - \beta & c\omega^2 - \gamma \end{vmatrix} = 0,$$

that is $$(a\omega^2 - \alpha)(c\omega^2 - \gamma) - (b\omega^2 - \beta)^2 = 0.$$

This equation gives two values of ω^2 which are real and positive. This follows from the fact that T_0 and V_0 are positive quadratic forms and the theorem of real positive roots is proved in textbooks of algebra.

We have, therefore, two values ω_1 and ω_2 which satisfy the equation and the complete solution is

$$p = A_1 \sin(\omega_1 t + \varepsilon_1) + A_2 \sin(\omega_2 t + \varepsilon_2),$$
$$q = B_1 \sin(\omega_1 t + \varepsilon_1) + B_2 \sin(\omega_2 t + \varepsilon_2).$$

The constants A_1, B_1 and A_2, B_2 are connected by the equations

$$A_1(a\omega_1^2 - \alpha) + B_1(b\omega_1^2 - \beta) = 0,$$
$$A_2(a\omega_2^2 - \alpha) + B_2(b\omega_2^2 - \beta) = 0,$$

so that there are four arbitrary constants in the complete solution.

It is clear that the motion has two periods, $2\pi/\omega_1$ and $2\pi/\omega_2$.

Example 11. *Find the complete solution of the equations of motion of the double pendulum* (Example 9)

$$44\ddot{\theta} + 18\ddot{\phi} + 24(g/a)\theta = 0,$$
$$6\ddot{\theta} + 4\ddot{\phi} + 3(g/a)\phi = 0.$$

Substituting
$$\theta = A \sin (\omega t + \varepsilon),$$
$$\phi = B \sin (\omega t + \varepsilon)$$

we have
$$A\left(44\frac{a\omega^2}{g} - 24\right) + B\left(18\frac{a\omega^2}{g}\right) = 0,$$
$$A\left(\frac{6a\omega^2}{g}\right) + B\left(4\frac{a\omega^2}{g} - 3\right) = 0.$$

Writing $\lambda = \dfrac{a\omega^2}{g}$, we have

$$\begin{vmatrix} 44\lambda - 24 & 18\lambda \\ 6\lambda & 4\lambda - 3 \end{vmatrix} = 0,$$

that is
$$17\lambda^2 - 57\lambda + 18 = 0,$$
$$(17\lambda - 6)(\lambda - 3) = 0,$$
$$\lambda = 3 \text{ or } \frac{6}{17}.$$

Hence we have two values of ω,

$$\omega_1 = \sqrt{(3g/a)},$$
$$\omega_2 = \sqrt{(6g/17a)},$$

and the corresponding periods are $2\pi/\omega_1$ and $2\pi/\omega_2$.

We have
$$A_1\left(44\frac{a\omega_1^2}{g} - 24\right) + B_1\left(18\frac{a\omega_1^2}{g}\right) = 0,$$

that is
$$108A_1 + 54B_1 = 0,$$
$$B_1 = -2A_1.$$

Also
$$A_2\left(44\frac{a\omega_2^2}{g} - 24\right) + B_2\left(18\frac{a\omega_2^2}{g}\right) = 0,$$
$$A_2\left(\frac{44 \times 6}{17} - 24\right) + B_2\left(\frac{18 \times 6}{17}\right) = 0,$$
$$B_2 = \frac{4}{3}A_2.$$

Hence the complete solution is

$$\theta = A_1 \sin (\omega_1 t + \varepsilon_1) + A_2 \sin (\omega_2 t + \varepsilon_2),$$
$$\phi = -2A_1 \sin (\omega_1 t + \varepsilon_1) + \frac{4}{3}A_2 \sin (\omega_2 t + \varepsilon_2).$$

The four constants A_1, A_2, ε_1, ε_2 may be determined from the initia values of θ, ϕ, $\dot{\theta}$ and $\dot{\phi}$.

8.22 Normal Modes

We have seen that when an oscillation has two degrees of freedom the solution is of the form

$$p = A_1 \sin (\omega_1 t + \varepsilon_1) + A_2 \sin (\omega_2 t + \varepsilon_2),$$
$$q = B_1 \sin (\omega_1 t + \varepsilon_1) + B_2 \sin (\omega_2 t + \varepsilon_2),$$

where B_1 and B_2 can be expressed in terms of A_1 and A_2 respectively. We can separate the oscillations with periods $2\pi/\omega_1$ and $2\pi/\omega_2$ by solving these equations, and we have

$$B_2p - A_2q = (A_1B_2 - A_2B_1)\sin(\omega_1 t + \varepsilon_1),$$
$$B_1p - A_1q = (B_1A_2 - B_2A_1)\sin(\omega_2 t + \varepsilon_2).$$

Hence, with suitable initial conditions an oscillation is possible in which $B_1p - A_1q = 0$ and the system oscillates with period $2\pi/\omega_1$; similarly an oscillation is possible in which $B_2p - A_2q = 0$ and the system oscillates with period $2\pi/\omega_2$. For any initial conditions the complete oscillation is the sum of the two oscillations and the two separate oscillations are called the *normal modes* of oscillation.

In Example 11 we found the solutions

$$\theta = A_1\sin(\omega_1 t + \varepsilon_1) + A_2\sin(\omega_2 t + \varepsilon_2),$$

$$\phi = -2A_1\sin(\omega_1 t + \varepsilon_1) + \frac{4}{3}A_2\sin(\omega_2 t + \varepsilon_2).$$

Hence
$$\frac{4}{3}\theta - \phi = \frac{10}{3}A_1\sin(\omega_1 t + \varepsilon_1),$$

$$2\theta + \phi = \frac{10}{3}A_2\sin(\omega_2 t + \varepsilon_2).$$

and the normal modes are variations of $\frac{4}{3}\theta - \phi$ with period $2\pi/\omega_1$ and variations of $2\theta + \phi$ with period $2\pi/\omega_2$.

8.23 Normal Modes from Equations of Motion

The normal modes may be found directly from the equations of motion by a method of multipliers. Let the equations of motion be

$$a\ddot{p} + b\ddot{q} + \alpha p + \beta q = 0, \quad b\ddot{p} + c\ddot{q} + \beta p + \gamma q = 0.$$

Multiplying the first equation by λ, the second by μ and adding we have,

$$(a\lambda + b\mu)\ddot{p} + (b\lambda + c\mu)\ddot{q} + (\alpha\lambda + \beta\mu)p + (\beta\lambda + \gamma\mu)q = 0. \quad (1)$$

This is the equation of a normal mode if

$$\frac{a\lambda + b\mu}{\alpha\lambda + \beta\mu} = \frac{b\lambda + c\mu}{\beta\lambda + \gamma\mu}.$$

This equation can be solved to give two values of the ratio of λ to μ, and the equation (1) becomes

$$\left(\frac{a\lambda + b\mu}{\alpha\lambda + \beta\mu}\frac{d^2}{dt^2} + 1\right)\{(\alpha\lambda + \beta\mu)p + (\beta\lambda + \gamma\mu)q\} = 0.$$

Thus $(\alpha\lambda + \beta\mu)p + (\beta\lambda + \gamma\mu)q$ is a normal mode.

Example 12. *Find the normal modes of the oscillation*

$$44\ddot{\theta} + 18\ddot{\phi} + 24(g/a)\theta = 0, \quad (1)$$
$$6\ddot{\theta} + 4\ddot{\phi} + 3(g/a)\phi = 0. \quad (2)$$

We have
$$\frac{44\lambda + 6\mu}{24\lambda} = \frac{18\lambda + 4\mu}{3\mu},$$

that is
$$24\lambda^2 - 2\lambda\mu - \mu^2 = 0,$$
$$\frac{\lambda}{\mu} = \frac{1}{4} \text{ or } -\frac{1}{6}.$$

Taking $\lambda = 1$ and $\mu = 4$ we add (2) multiplied by 4 to (1) and we have

$$68\ddot{\theta} + 34\ddot{\phi} + (g/a)(24\theta + 12\phi) = 0,$$
$$17a(2\ddot{\theta} + \ddot{\phi}) + 6g(2\theta + \phi) = 0.$$

Taking $\lambda = 1$ and $\mu = -6$ we have

$$8\ddot{\theta} - 6\ddot{\phi} + (g/a)(24\theta - 18\phi) = 0,$$
$$a(4\ddot{\theta} - 3\ddot{\phi}) + 3g(4\theta - 3\phi) = 0.$$

Thus the normal modes are $2\theta + \phi$ and $4\theta - 3\phi$ and their periods are $2\pi(17a/6g)^{1/2}$ and $2\pi(a/3g)^{1/2}$.

8.24 Oscillations with Three Degrees of Freedom

When there are three degrees of freedom the equations of motion are deduced in the same way as for two degrees of freedom and if p, q and r are the generalized coordinates the equations are of the form

$$a_{11}\ddot{p} + a_{21}\ddot{q} + a_{31}\ddot{r} + c_{11}p + c_{21}q + c_{31}r = 0,$$
$$a_{12}\ddot{p} + a_{22}\ddot{q} + a_{32}\ddot{r} + c_{12}p + c_{22}q + c_{32}r = 0,$$
$$a_{13}\ddot{p} + a_{23}\ddot{q} + a_{33}\ddot{r} + c_{13}p + c_{23}q + c_{33}r = 0,$$

where $a_{rs} = a_{sr}$ and $c_{rs} = c_{sr}$, for $r, s = 1, 2, 3$.

Substituting
$$p = P \sin(\omega t + \varepsilon),$$
$$q = Q \sin(\omega t + \varepsilon),$$
$$r = R \sin(\omega t + \varepsilon),$$

we obtain the equations

$$\left.\begin{aligned} P(a_{11}\omega^2 - c_{11}) + Q(a_{21}\omega^2 - c_{21}) + R(a_{31}\omega^2 - c_{31}) = 0,\\ P(a_{12}\omega^2 - c_{12}) + Q(a_{22}\omega^2 - c_{22}) + R(a_{32}\omega^2 - c_{32}) = 0,\\ P(a_{13}\omega^2 - c_{13}) + Q(a_{23}\omega^2 - c_{23}) + R(a_{33}\omega^2 - c_{33}) = 0. \end{aligned}\right\} \quad (1)$$

Elimination of P, Q and R gives the equation

$$\begin{vmatrix} a_{11}\omega^2 - c_{11} & a_{21}\omega^2 - c_{21} & a_{31}\omega^2 - c_{31} \\ a_{12}\omega^2 - c_{12} & a_{22}\omega^2 - c_{22} & a_{32}\omega^2 - c_{32} \\ a_{13}\omega^2 - c_{13} & a_{23}\omega^2 - c_{23} & a_{33}\omega^2 - c_{33} \end{vmatrix} = 0.$$

This cubic equation gives three real positive values of ω^2 and hence the three periods of the oscillation are found. To obtain the complete solution relations between P, Q and R for each value of ω are obtained from any two of the equations (1).

The complete solution is then

$$p = P_1 \sin(\omega_1 t + \varepsilon_1) + P_2 \sin(\omega_2 t + \varepsilon_2) + P_3 \sin(\omega_3 t + \varepsilon_3),$$
$$q = Q_1 \sin(\omega_1 t + \varepsilon_1) + Q_2 \sin(\omega_2 t + \varepsilon_2) + Q_3 \sin(\omega_3 t + \varepsilon_3),$$
$$r = R_1 \sin(\omega_1 t + \varepsilon_1) + R_2 \sin(\omega_2 t + \varepsilon_2) + R_3 \sin(\omega_3 t + \varepsilon_3).$$

The normal modes can be obtained by solving these equations for

$$\sin(\omega_1 t + \varepsilon_1),\ \sin(\omega_2 t + \varepsilon_2) \text{ and } \sin(\omega_3 t + \varepsilon_3).$$

Similar methods apply to the solution of the equations of oscillations with any number of degrees of freedom.

Example 13. *Two double pendulums ABC, $A'B'C'$, smoothly jointed at B and B' are freely suspended from fixed points A, A' at the same level and smoothly connected by a rod BB'. The lengths AB, BC, BB', $B'C'$, $B'A'$, AA' are each equal to $2a$. The rods AB, BB', $B'A'$ are each uniform and of mass m, and the rods BC, $B'C'$ are each uniform and of mass $4m$. The rods AB, BC, $B'C'$ make angles θ, ϕ, ψ with the downward vertical. Prove that $\phi - \psi$ is a normal coordinate for small oscillations of the system (in a vertical plane through AA') about the equilibrium configuration, the length of the equivalent simple pendulum in the corresponding normal mode of oscillation being $4a/3$.*

Show that the remaining normal coordinates are

$$5\theta - 2\phi - 2\psi$$

and $$4\theta + \phi + \psi$$

and that the corresponding equivalent simple pendulums are of lengths $a/3$ and $44a/15$. (L.U.)

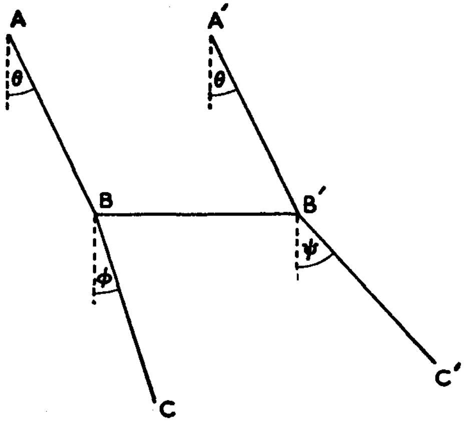

Fig. 182

The angles θ, ϕ, ψ (Fig. 182) being the generalized coordinates, the kinetic energies of the rods are:

$$AB \ldots \frac{1}{2}\cdot\frac{4}{3}ma^2\dot\theta^2,$$

$$A'B' \ldots \frac{1}{2}\cdot\frac{4}{3}ma^2\dot\theta^2,$$

$$BB' \ldots \frac{1}{2}m(2a\dot\theta)^2,$$

$$BC \ldots \frac{1}{2}\cdot 4m\left\{\frac{4}{3}a^2\dot\phi^2 + 4a^2\dot\theta^2 + 4a^2\dot\theta\dot\phi\cos(\phi - \theta)\right\},$$

$$B'C' \ldots \frac{1}{2}\cdot 4m\left\{\frac{4}{3}a^2\dot\psi^2 + 4a^2\dot\theta^2 + 4a^2\dot\theta\dot\psi\cos(\psi - \theta)\right\}.$$

Hence

$$T_0 = \frac{1}{2}ma^2\left\{\frac{116}{3}\dot\theta^2 + \frac{16}{3}\dot\phi^2 + \frac{16}{3}\dot\psi^2 + 16\dot\theta\dot\phi + 16\dot\theta\dot\psi\right\}.$$

$$V = -4mga\cos\theta - 4mg(2a\cos\theta + a\cos\phi) - 4mg(2a\cos\theta + a\cos\psi),$$

$$V_0 = \frac{1}{2}mga(20\theta^2 + 4\phi^2 + 4\psi^2).$$

The equations of motion are

$$\frac{116}{3}\ddot{\theta} + \quad 8\ddot{\phi} + 8\ddot{\psi} + 20(g/a)\theta = 0, \tag{1}$$

$$8\ddot{\theta} + \frac{16}{3}\ddot{\phi} \qquad + \; 4(g/a)\phi = 0, \tag{2}$$

$$8\ddot{\theta} + \qquad \frac{16}{3}\ddot{\psi} + \; 4(g/a)\psi = 0. \qquad (3$$

Subtracting (3) from (2) we have

$$\frac{4a}{3}(\ddot{\phi} - \ddot{\psi}) + g(\phi - \psi) = 0,$$

and hence $\phi - \psi$ is a normal mode with period $2\pi\sqrt{(4a/3g)}$.
Adding (2) and (3) we have

$$12\ddot{\theta} + 4(\ddot{\phi} + \ddot{\psi}) + 3(g/a)\,(\phi + \psi) = 0. \tag{4}$$

From (1) we have
$$29\ddot{\theta} + 6(\ddot{\phi} + \ddot{\psi}) + 15(g/a)\theta = 0. \tag{5}$$

Subtracting (4) multiplied by 2 from (5) we have

$$(5\ddot{\theta} - 2\ddot{\phi} - 2\ddot{\psi}) + 3(g/a)\,(5\theta - 2\phi - 2\psi) = 0.$$

Multiplying (4) by 5 and (5) by 4 and adding we have

$$44(4\ddot{\theta} + \ddot{\phi} + \ddot{\psi}) + 15(g/a)(4\theta + \phi + \psi) = 0.$$

Hence the normal modes are $\phi - \psi$, $5\theta - 2\phi - 2\psi$ and $4\theta + \phi + \psi$ and the lengths of the equivalent simple pendulums follow.

8.25 Oscillations of a Shaft with Flywheels

The torsional oscillation of a shaft carrying flywheels is easily found by using Lagrange's equations. Let three flywheels whose moments of inertia are I_1, I_2, I_3 be fitted on a light shaft which is such that the torque required to give it a twist θ per unit length is $\mu\theta$. Let l_1 and l_2 be the free lengths of shaft between the flywheels (Fig. 183).

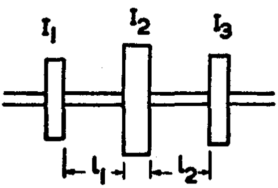

Fig. 183

Let θ_1, θ_2, θ_3 be the small angular displacements of the flywheels from their mean positions.

The kinetic energy is

$$T = \frac{1}{2}(I_1\dot{\theta}_1^2 + I_2\dot{\theta}_2^2 \quad I_3\dot{\theta}_3^2).$$

The energy stored in the shaft is

$$V = \frac{1}{2}\frac{\mu}{l_1}(\theta_1 - \theta_2)^2 + \frac{1}{2}\frac{\mu}{l_2}(\theta_2 - \theta_3)^2.$$

Lagrange's equations are

$$I_1\ddot{\theta}_1 + \frac{\mu}{l_1}(\theta_1 - \theta_2) = 0,$$

$$I_2\ddot{\theta}_2 + \frac{\mu}{l_1}(\theta_2 - \theta_1) + \frac{\mu}{l_2}(\theta_2 - \theta_3) = 0,$$

$$I_3\ddot{\theta}_3 + \frac{\mu}{l_2}(\theta_3 - \theta_2) = 0.$$

Assuming an oscillation $\sin \omega t$, the equation for ω^2 is

$$\begin{vmatrix} I_1\omega^2 - \frac{\mu}{l_1} & \frac{\mu}{l_1} & 0 \\ \frac{\mu}{l_1} & I_2\omega^2 - \frac{\mu}{l_1} - \frac{\mu}{l_2} & \frac{\mu}{l_2} \\ 0 & \frac{\mu}{l_2} & I_3\omega^2 - \frac{\mu}{l_2} \end{vmatrix} = 0,$$

that is

$$\omega^4 l_1 l_2 I_1 I_2 I_3 - \mu\omega^2\{l_1 I_1(I_2 + I_3) + l_2 I_3(I_1 + I_2)\} + \mu^2(I_1 + I_2 + I_3) = 0.$$

Thus there are two frequencies of oscillation, the absence of the third being accounted for by the relation $I_1\ddot{\theta}_1 + I_2\ddot{\theta}_2 + I_3\ddot{\theta}_3 = 0$. A similar method can be applied to find the $n - 1$ frequencies of a shaft with n flywheels.

EXERCISES 8 (c)

1. Find the complete solution of the equations

$$35\ddot{\theta} - 6\ddot{\phi} + 13g\theta = 0,$$
$$-6\ddot{\theta} + 30\ddot{\phi} + 13g\phi = 0,$$

given that initially $\theta = \phi = a$, $\dot{\theta} = \dot{\phi} = 0$.

2. Two particles of equal mass m and distance a apart are attached to a taut string at equal distances ka from the fixed end points. Obtain the simultaneous differential equations for small transverse displacements x, y of the particles when the tension in the string has the constant value $kman^2$. Show that, if the particles start from rest at $t = 0$ with $x = b$, $y = 0$ then x and y can be expressed in the forms $C(\cos nt + \cos \lambda nt)$, $C(\cos nt - \cos \lambda nt)$ respectively, and evaluate the constants λ, C. (L.U., Pt. II)

3. A light string of length $3a$ is stretched horizontally between two fixed points and masses $8m$ and $3m$ are attached to the points of trisection. Gravity is neglected and the tension of the string in equilibrium is T.

If both masses are drawn aside a small distance c and released show that the displacements of the masses at time t are

$$-(c/14)\cos(3T/4ma)^{1/2}t + (15c/14)\cos(T/6ma)^{1/2}t,$$
$$(4c/14)\cos(3T/4ma)^{1/2}t + (10c/14)\cos(T/6ma)^{1/2}t.$$

4. A light string is stretched horizontally between two points $9a$ apart and masses m and $2m$ are attached at points distant a and $5a$ respectively from one end. Gravity is neglected and the tension of the string in equilibrium is $\frac{4}{3}mg$. Find the periods of the normal modes of oscillation of the masses.

5. A uniform circular ring of mass $2m$ and radius a can turn in a vertical plane about a point of the ring which is fixed. A bead of mass m can slide freely on the ring. Show that the normal oscillations about the position of stable equilibrium have periods $2\pi(2a/g)^{1/2}$ and $2\pi(2a/3g)^{1/2}$.

6. A light inelastic string AB of length $7a$ is attached to a fixed point at A. A particle of mass m is attached to the string at B and a similar particle at a point C of the string where $AC = 4a$. The system oscillates in a vertical plane the portions AC and CB of the string making small angles θ and ϕ respectively with the vertical. Show that $\theta + \phi$ and $4\theta - 3\phi$ are normal modes and find their equivalent simple pendulums.

7. A uniform rod AB of mass $2m$ and length $4a$ is free to turn in a vertical plane about A. A particle of mass m is attached by a light inextensible string of length a to B and the system oscillates in a vertical plane about the position of stable equilibrium. If the rod and the string are inclined at angles θ and ϕ respectively to the vertical show that normal modes are $6\theta + \phi$ and $4\theta - 3\phi$ and find their periods.

8. A uniform rod of mass $2m$ and length $4a$ can turn freely about a fixed horizontal axis through its mid-point. A particle of mass m is attached to one end of the rod by a light inextensible string of length a. For oscillations in a vertical plane about the position of stable equilibrium with the rod inclined at θ and the string at ϕ to the vertical show that the normal modes are $3\theta + \phi$ and $2\theta - \phi$ and find their periods.

9. A uniform rod AB, of length $2a$ and mass $8m$, can turn freely about a fixed horizontal axis through its mid-point O. A string of length a has one end fastened to B and the other to a particle of mass m. If the system is slightly disturbed from its position of stable equilibrium so that the rod and string move in the same vertical plane, find the periods of the normal modes of oscillation. (L.U.)

10. A uniform disc of mass $2m$ and radius $2a$ is free to turn about a horizontal axis through its centre perpendicular to the disc. A particle of mass m is attached to a point on the circumference of the disc by a light inextensible string of length $3a$. If the system oscillates in a vertical plane about the position of stable equilibrium, find the periods of the normal modes.

11. A uniform shaft of torsional rigidity μ free to rotate in bearings carries three flywheels whose moments of inertia are I, $2I$ and I. The fly-

wheels are equally spaced at distances l apart, the wheel of largest moment being central. Calculate the periods of oscillation of the wheels about their mean position.

12. Four flywheels, each of moment of inertia I are equally spaced at distance l apart on a uniform shaft of torsional rigidity μ. Show that the squares of the principal periods of oscillation of the wheels about their mean positions are in the ratio $1 : 2 + \sqrt{2} : 2 - \sqrt{2}$.

13. A uniform rod of length $2a$ is hung from a fixed point by a string of length $2b$ fastened to one end of the rod. The system makes small oscillations in a vertical plane. Show that if $b = \frac{1}{3}a$ the periods of the normal modes are $2\pi(5a/4g)^{1/2}$ and $2\pi(a/12g)^{1/2}$.

14. A uniform rod of mass m and length $2a$ is suspended from a fixed point by an inextensible string of length b attached to one end of the rod and the system oscillates in a vertical plane through the fixed point. If b is small prove that the two normal modes have periods approximately equal to those of simple pendulums of length $4a/3$ and $b/4$. (L.U.)

15. AB is a uniform rod of length $2a$ suspended from a fixed point O by an inextensible string OC of length $5a/6$ attached to a point C of the rod such that $AC = 2a/3$. Find the periods and normal modes of oscillation in a vertical plane. (L.U.)

16. A double pendulum consists of two uniform rods AB and BC; AB is of mass $2m$ and length $8a$, BC is of mass m and length $2a$. AB is free to turn about A, which is fixed, and BC is freely hinged to AB at B. If the rods oscillate in a vertical plane about the position of stable equilibrium AB and BC being inclined at angles θ and ϕ respectively to the vertical, show that $4\theta - 3\phi$ and $12\theta + \phi$ are normal modes and find the length of equivalent simple pendulums for these modes.

17. A uniform rod AB of mass $2m$ and length $2a$ is freely hinged at A. A uniform rod BC of mass m and length $4a$ is hinged to AB at B. The rods oscillate in a vertical plane about the position of stable equilibrium, being inclined at angles θ and ϕ respectively to the vertical. Show that $\theta + \phi$ and $2\theta - \phi$ are normal modes and find their periods.

18. To a point on a hoop of mass m and radius a, is freely hinged one end B of a uniform rod AB of mass $3m$ and length $4a$, and the other end A is freely hinged to a fixed point. If the system makes small oscillations under gravity in a vertical plane about the position of equilibrium, prove that the normal coordinates are $2\theta - \phi$ and $5\theta + \phi$, where θ is the inclination of AB to the vertical and ϕ the inclination to the vertical of the diameter of the hoop which passes through B. (L.U.)

CHAPTER 9

STATICS OF A RIGID BODY

9.1 Terminology

Statics deals with the forces which act on a rigid body which is in equilibrium, that is, at rest. The fundamental concept in statics is that of a rigid body, which is in no way deformed by the forces acting on it. In practice every body is more or less deformable but the error in assuming perfect rigidity is usually a second-order effect and in most engineering problems a solution based on the concept of rigidity is adequate. There are, however, certain problems, notably that of finding the stresses in the members of an over-stiff framework, that can only be solved by assuming a deformation.

A rigid body can have contact with another rigid body in a point. The effect of a force on a rigid body depends on the line along which the force acts and not on the particular point of the line at which it acts. This is the principle of transmissibility of force and follows from consideration of the manner in which force is transmitted from particle to particle in the body.

Other concepts which give the required degree of accuracy in the solution of practical problems are those of a smooth surface, such that the force of reaction to a body in contact with it is normal to the surface, a smooth hinge, such that there is no restraining couple to prevent it turning, and a light inextensible string.

A light rod is one whose weight is negligible compared with the other forces associated with it. It is well to remember that such a rod if it does not carry a load at some point of its length must be in direct tension or compression since the forces acting at its ends must balance each other.

9.2 Forces

It is shown in § 1.1 that it follows from Newton's second law of motion that force is a vector, and the vectorial character of forces can also be demonstrated experimentally. Forces can, therefore, be added vectorially or resolved into components. Thus the resultant of forces P and Q of specified magnitudes and directions is a force R, that is, a vector which is the third side of a triangle formed by placing the vectors P and Q end to end (Fig. 184).

A force P which is inclined at an angle θ to a direction OX and $\frac{1}{2}\pi - \theta$ to a perpendicular direction OY (Fig. 185) has components $P \cos \theta$ and

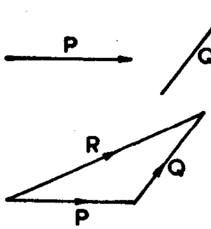

Fig. 184

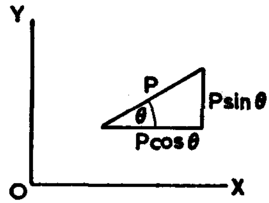

Fig. 185

$P \sin\theta$ parallel to OX and OY and the vector may be written in terms of its components as

$$\mathbf{P} = iP\cos\theta + jP\sin\theta.$$

The sum of a number of forces acting at a point, of magnitudes $P_1, P_2 \ldots$, inclined at angles $\theta_1, \theta_2, \ldots$, respectively to OX is the sum of the components and is the vector

$$i(P_1\cos\theta_1 + P_2\cos\theta_2 + \ldots) + j(P_1\sin\theta_1 + P_2\sin\theta_2 + \ldots).$$

Force is a vector which acts at a point, but by virtue of the principle of *transmissibility of force it may be taken as acting at any point on* its line of action. Thus force is said to be a line-localized vector. The resultant of two forces must be taken as acting along a line through the point of intersection of the lines of action of the two forces.

If a body is in equilibrium the forces acting on it must balance, that is, the vector sum of the forces must be zero. Therefore the components of the vector sum in perpendicular directions must individually be zero.

9.3 Moments and Couples

Two forces of magnitude P acting in parallel directions at points A and B, where AB is not parallel to the forces (Fig. 186), will not have the same effect on the body on which they act. The force P at B has a tendency to turn the body about A which the force at A lacks. This turning effect is measured by the moment Ph, where h is the distance between the lines of action of the forces.

Thus a force P acting at B is equivalent to a force P acting at A

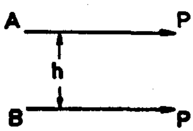

Fig. 186

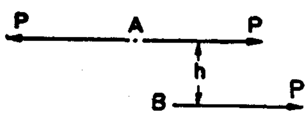

Fig. 187

together with a turning moment Ph. We may think of this turning moment as being equivalent to a couple since the force at B (Fig. 187) taken in conjunction with two equal and opposite forces at A (which

will not affect the equilibrium of the body) form a force P at A and a couple of moment Ph.

It follows that two parallel forces P and Q have the same effect on a body as their vector sum $P + Q$ whose line of action is such that the forces P and Q have equal and opposite moments about a point of it.

A number of forces $P, Q, R, \ldots$, acting on a body have, therefore, the same effect as forces $P, Q, R, \ldots$, acting at some one point of the body together with a total turning moment about the point.

The forces are, therefore, equivalent to a single force X at any point, X being the vector sum of $P, Q, R, \ldots$, together with a turning moment G about the point, and the turning moment G may be taken as due to a couple. It is shown in § 7.14 that the moment of a couple about any point is the same.

The moment of a force about a point is itself a vector whose direction is perpendicular to the plane of the force and the point (see § 7.13). If the forces acting on a body are coplanar their moments about any point in the plane are parallel and may be added arithmetically. When the forces are not coplanar their moments about any point may be added vectorially.

Example 1. *A cantilever is built into a wall over a length of* 21 *in. The masonry pressure varies uniformly from an upward thrust of* 8 *lb. per in. run where the beam enters the wall to a downward thrust of* 6 *lb. per in. run at its end. Reduce the masonry thrusts to an upward force where the beam enters the wall and a couple.*

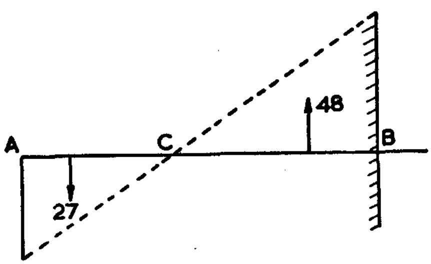

Fig. 188

The thrust is upwards over 12 in. and varies from 0 at C (Fig. 188) to 8 lb. per in. at B.

The thrust at x from C is $\frac{2}{3}x$ and the total upward thrust

$$= \int_0^{12} \frac{2}{3} x dx = 48 \text{ lb.}$$

The moment of this thrust about B

$$= -\int_0^{12} \frac{2}{3} x(12 - x) dx = -192 \text{ lb.in.}$$

Similarly, the downward thrust in AC

$$= \int_0^9 \frac{2}{3} y dy = 27 \text{ lb.},$$

and its moment about B

$$= \int_0^9 \frac{2}{3} y(y + 12) dy = 486 \text{ lb.in.}$$

Hence, the thrusts are equivalent to an upward thrust of 21 lb. at B together with a couple whose moment is 294 lb.in. = 24·5 lb.ft.

9.4 Conditions for Equilibrium

It follows from the previous section that if a body is in equilibrium under the action of a number of forces the vector sum of the forces must be zero and also the sum of the moments of the forces about some point must be zero.

If the forces are coplanar, let X and Y be the components of the resultant acting at some point O and G the sum of the moments about O. Then for equilibrium we must have $X = Y = G = 0$, that is, there are three equations which must be satisfied. It follows that only three unknown quantities such as forces, directions or lengths, may be found by applying the conditions of equilibrium.

If we take the point O as the origin the moment of the system of forces about a point (x_1, y_1) (Fig. 189) is

$$G + y_1 X - x_1 Y,$$

and if $G = X = Y = 0$, this moment is also zero.

If the moment about O and about points (x_1, y_1) and (x_2, y_2) are all zero, we have $G = 0$ and

$$y_1 X - x_1 Y = 0,$$
$$y_2 X - x_2 Y = 0,$$

and hence either $X = Y = 0$ or $y_1/x_1 = y_2/x_2$; in which case the origin and the points (x_1, y_1), (x_2, y_2) are collinear. Hence it is a sufficient condition for equilibrium if the moments of the forces of the system about three non-collinear points are zero. Therefore one, two or three of the conditions for equilibrium may be obtained from moment equations.

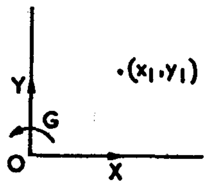

Fig. 189

The triangle of forces for three forces acting on a rigid body in equilibrium follows from these equations since the forces must be concurrent if one is not to have a moment about the intersection of the lines of action of the other two and the vectors which represent the three forces must form a closed triangle. Lami's theorem that each of these three forces is proportional to the sine of the angle between the other

two is merely the application of the sine rule to the triangle of forces.

The polygon of forces is formed by placing all the forces acting on a body end to end and for equilibrium it is evident that the polygon must be a closed figure. This is a necessary condition for equilibrium but not sufficient unless the sum of the moments of the forces about some point is also zero. When the body is not in equilibrium the vector which closes the polygon represents the resultant of the forces in magnitude and direction.

9.5 Solution of Problems

The solution of statical problems involves the application to one or more rigid bodies of the conditions of equilibrium given in the preceding section. It is important that a clear figure be drawn, and all the forces acting on each body marked on the figure. Where one body presses against another an unknown reaction must be assumed to act which, if the bodies be smooth, will be perpendicular to the common tangent plane. If a body is supported by a string a tension must be assumed in the string and this tension is unchanged when the string passes over a smooth peg or pulley. If a body is hinged to another body and there is no friction at the hinge the force acting there on either body may be taken as a single force acting through the centre of the hinge. If the direction of this force is unknown it may be expressed as two forces in directions at right-angles.

When all the forces have been marked the equations which express the conditions for equilibrium of each body should be written down and numbered. This completes the statical part of the problem, although some algebraic manipulation may be needed to complete the solution. Geometrical relations between the angles and lengths in the figure may have to enter into the calculations.

Example 2. *A step ladder of weight $2W$ consists of two equal parts, jointed at the top, and held together by a rope half-way between the top and bottom so that when the rope is tight the angle between the two halves of the ladder is $2\tan^{-1} 6/13$. A man of weight $5W$ mounts the ladder and stops two-thirds of the way up. Neglecting friction between the ladder and the ground, find the tension in the rope and the reaction at the hinge.*

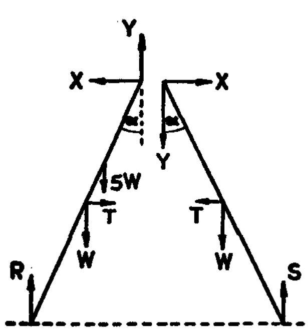

Fig. 190

Let l be the length of the ladder and $\tan \alpha = \dfrac{6}{13}$ (Fig. 190). The forces acting on the two halves of the ladder are shown in the figure.

For the left-hand half we have, equating components to zero,

$$X - T = 0, \tag{1}$$

$$R - 6W + Y = 0. \tag{2}$$

Taking moments about the top we have

$$W \times \frac{l}{2}\sin\alpha + 5W \times \frac{l}{3}\sin\alpha + T \times \frac{l}{2}\cos\alpha - R \times l\sin\alpha = 0,$$

that is
$$26W - 12R + 13T = 0. \tag{3}$$

Similarly, for the right-hand half we have, equating components to zero,

$$X - T = 0, \tag{4}$$
$$S - W - Y = 0. \tag{5}$$

Taking moments about the top we have

$$S \times l\sin\alpha - W \times \frac{l}{2}\sin\alpha - T \times \frac{l}{2}\cos\alpha = 0,$$

that is
$$12S - 6W - 13T = 0. \tag{6}$$

Hence,
$$T = 2W,$$
$$X = 2W,$$
$$Y = \frac{5}{3}W.$$

Example 3. *A straight uniform rod BC, 3 ft. long and weighing 5 lb., is hung by two strings AB and CD each 3 ft. long. The points A and D are in a horizontal plane and are 7 ft. apart. A weight of 10 lb. is hung from B. What is the least force that must be applied at C so that AB may make an angle of 45° with the vertical, the point C being below B? What are then the tensions in the strings?* (Q.E.)

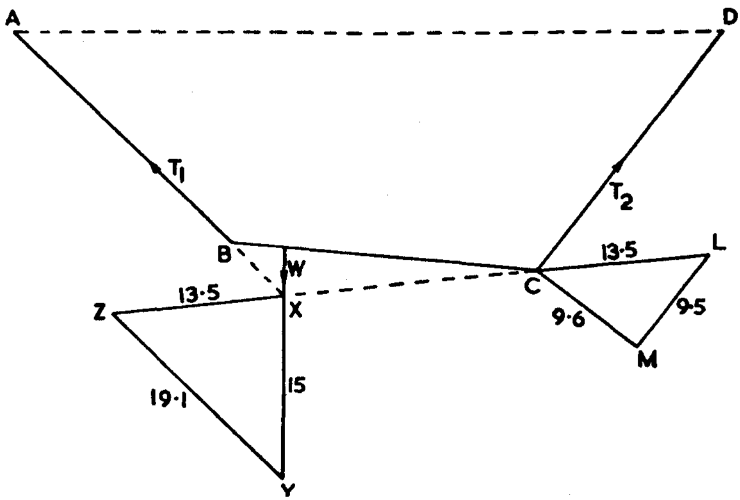

Fig. 191

This problem is simply solved by graphical methods. There are four forces acting on the rod, the tensions T_1 and T_2 in the strings, the weight of 15 lb. acting in the rod at 0·5 ft. from B and the unknown reaction R at C (Fig. 191).

The position of B is given and the position of C is found by drawing.

Let the lines of action of T_1 and $W(= 15$ lb.) meet at X. Since the forces

T_2 and R pass through C, the resultant of T_1 and W must be along the line CX. Thus the triangle XYZ, with $XY = 15$, is a triangle of forces for W, T_1 and the resultant of T_2 and R.

Therefore $T_1 = 19{\cdot}1$ lb. and the resultant of T_2 and R is $13{\cdot}5$ lb.

Produce XC to L where $CL = 13{\cdot}5$. Then whatever be the direction of R, T_2 and R are given by the lengths CM and ML, ML being parallel to CD. CM is least when CM is perpendicular to ML and in this case

$$R = 9{\cdot}6 \text{ lb.}$$
$$T_2 = 9{\cdot}5 \text{ lb.}$$

9.6 Forces in Three Dimensions

When the forces which act on a body in equilibrium are not coplanar we have three equations obtained by equating to zero the vector sum of the forces, since the vector has components in three mutually perpendicular directions. In addition, the sum of the moments of the forces about some point must be zero. The moment of a force is itself a vector with three components and equating these components to zero gives three further equations. There are, therefore, in all six conditions for equilibrium.

It is shown in § 7.13 that the vector which is the moment of a force about a point has a component along any axis through the point which is the moment of the force about the axis. A force acting at a point may be resolved into two components, one parallel to the axis and one in a plane perpendicular to the axis. The moment of the force about the axis is the moment of the second component about the point in which the plane cuts the axis.

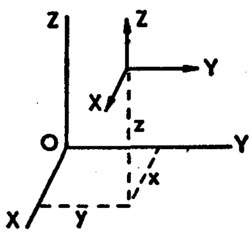

Fig. 192

Thus a force whose components are (X, Y, Z) acting at a point (x, y, z) has moments about OX, OY, OZ which are respectively (Fig. 192),

$$yZ - zY,\ zX - xZ,\ xY - yX.$$

The moment of the force about the point O is the vector sum of these components and its magnitude is

$$\{(yZ - zY)^2 + (zX - xZ)^2 + (xY - yX)^2\}^{1/2}.$$

Example 4. *A uniform platform of weight W is in the form of a rectangle of sides a and b. The platform is supported in a horizontal plane by means of a smooth joint at one corner and ropes are attached to two diagonally opposite corners and to a point at a height h vertically above the joint. Find the tensions in the ropes and the reaction at the joint.* (L.U., Pt. II)

Let OX, OY be two sides of the rectangle and OZ the vertical through the joint (Fig. 193) and let T_1 and T_2 be the tensions in the ropes.

The vertical components of the tensions are $T_1h/(a^2 + h^2)^{1/2}$ and $T_2h/(b^2 + h^2)^{1/2}$ respectively.

Equating to zero the moments about OX and OY, we have

$$W\frac{b}{2} - \frac{T_2 h}{\sqrt{(b^2 + h^2)}} \times b = 0,$$

$$W\frac{a}{2} - \frac{T_1 h}{\sqrt{(a^2 + h^2)}} \times a = 0.$$

Hence $$T_1 = \frac{1}{2}W(1 + a^2/h^2)^{1/2},$$

$$T_2 = \frac{1}{2}W(1 + b^2/h^2)^{1/2}.$$

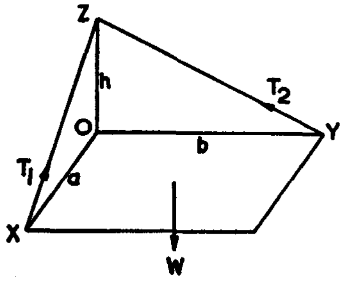

Fig. 193

The resolved parts of T_1 and T_2 vertically are then each equal to $\frac{1}{2}W$ and there is therefore no vertical component of reaction at O.

The horizontal components parallel to OX and OY respectively are $T_1 a/(a^2 + h^2)^{1/2}$ and $T_2 b/(b^2 + h^2)^{1/2}$, that is $\frac{1}{2}Wa/h$ and $\frac{1}{2}Wb/h$.

The reaction at the joint is therefore $\frac{1}{2}W(a^2 + b^2)^{1/2}/h$.

EXERCISES 9 (a)

1. Two uniform planks, AB and AC, each of length 10 ft., are freely hinged at A, and stand in a vertical plane in equilibrium with B and C on a smooth horizontal plane. The mid-points of the planks are joined by a light inextensible rope of length 5 ft., and the masses of AB and AC are 50 lb., and 30 lb. respectively. Find the tension in the rope and the horizontal and vertical components of the reaction at the hinge. (L.U.)

2. ABC is an equilateral triangle of side $8a$, fixed in a vertical plane with BC horizontal and A above BC. A particle of weight W is attached to A, B, C by three light elastic strings of the same material and thickness. When the particle is at the mid-point of BC the strings have their natural lengths. If the particle can rest in equilibrium at a depth $3a$ below BC, show that the modulus of the strings is $20W/(6 + 5\sqrt{3})$. (L.U.)

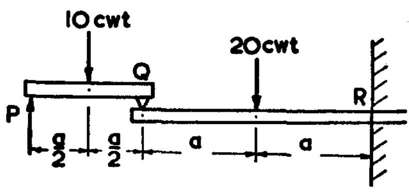

Fig. 194

3. A beam QR is rigidly built into a wall at R and at Q supports another beam PQ, which is also supported at P. The beams are loaded as shown in Fig. 194. Find the reactions at P, Q and R and the couple at R due to the applied loads.

4. Two beams, AB and CD, are rigidly fixed at A and D, and support another beam BF. The forces on the beams are shown in Fig. 195. Find the distance marked x on the figure in order that there shall be no vertical reaction at D. Find also the reaction at A and the fixing couples at A and D. Neglect the weights of the beams. (Q.E.)

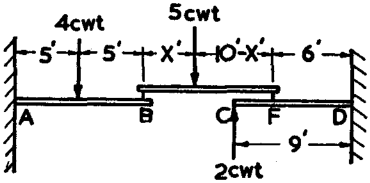

Fig. 195

5. AB and BC are two uniform bars of the same material weighing w oz. per inch length, and are smoothly jointed together at B. AB is 25 in. long and is smoothly hinged to a fixed pivot at A. BC is 52 in. long and the end C is constrained to slide in a smooth horizontal groove which is in line with A, the bars and the groove being in the same vertical plane with B 20 in. above AC. What horizontal force, acting along the groove, is necessary to preserve equilibrium, and what is then the direction of the reaction at the hinge A? (L.U.)

6. Two equal uniform smooth solid cylinders, each of radius r and of length nr, rest symmetrically inside a fixed horizontal cylinder of radius $5r$, the axes of all three being parallel. A sphere of radius $2r$ and of the same material as the two cylinders is placed upon them, so that the centres of gravity of the three are in the same vertical plane at right-angles to the axes of the cylinders. Show that the cylinders will separate if $3n < 4\sqrt{30} - 16$. (L.U., Pt. II)

7. Two uniform smooth spheres, each of weight W and radius b, rest inside a hollow cylinder of radius a ($< 2b$), fixed with its base horizontal. Show that the reaction between the curved surface of the cylinder and each sphere is $(a - b)W/\sqrt{(2ab - a^2)}$, and find the reaction between the two spheres.

8. In Fig. 196 AB is a light spar being used as a temporary derrick to raise the large uniform spar CD of weight 1000 lb., into position. The lower ends B and D of the spars rest in recesses in the ground and the upper ends A and C are joined by a rope AC. Another rope AE is attached to A and the spar CD is raised by pulling on this rope at E. Find the pull in AE required to support the spar CD in the position shown.

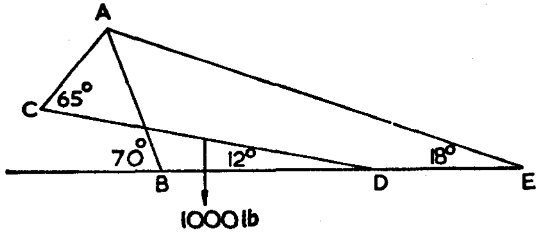

Fig. 196

9. Two equal uniform rods AB, AC, smoothly jointed at A, each of length $2a$ and weight W, rest in equilibrium in a vertical plane astride a smooth fixed cylinder of radius r whose axis is horizontal. The ends B, C of the rods are jointed by an inextensible string of length $4l$ (where $l < a$) which passes underneath and remains clear of the cylinder. Show that the string is taut if $r > \dfrac{l^3}{a\sqrt{(a^2 - l^2)}}$, and find an expression for the reaction between a rod and the cylinder in terms of W, a and l.

10. Three equal light bars AB, BC, CD form a linkage in a vertical plane being pin-jointed to each other at B and C and to fixed points on ground level at A and D. The angles ABC and BCD are each 120°. A weight of 12 lb. is applied to the bar BC at E, where $BC = 3BE$, and the linkage is held in equilibrium by a force P applied at F to the bar CD, where $CD = 3CF$, the direction of P being perpendicular to CD. Find the magnitude of the force P and the magnitude and inclination to the vertical of the reaction at D.

11. A pier for a bridge consists of two trestles supported on mudsills as shown in Fig. 197. The load on the trestles are 12 and 17 tons respectively, and the pier itself weighs 4 tons, the centre of gravity being midway between the trestles. If the pressure of the mudsills on the earth is to be uniform and not more than $\frac{3}{4}$ ton per sq. foot, find the number of mudsills required, and the distance marked x on the figure. The mudsills are timber planks each 12 ft. long and 9 inches broad, and the distribution of pressure perpendicular to the plane of the sketch may be assumed uniform.

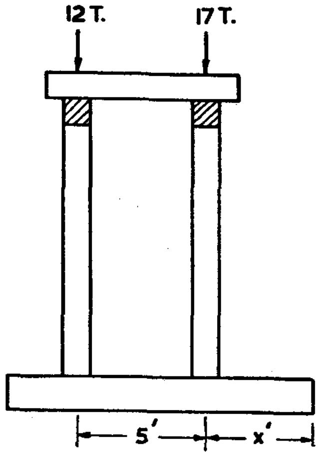

Fig. 197

12. Fig. 198 represents a uniform rectangular girder weighing 20 tons being launched across a gap. The girder moves on frictionless rollers at A; T is the pull in the hauling tackle and P the pull in the preventer tackle. Find the values of T and P for this position of the girder. T and P are inclined at angles 55° and $7\frac{1}{2}$° respectively and the girder at 18° to the horizontal, $AD = 48$ ft., $DC = 7$ ft., $AB = 14$ ft.

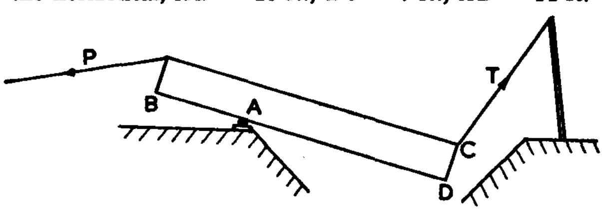

Fig. 198

13. The four fixed points A, B, C, D are the corners of a square of side $2a$ in a horizontal plane. A uniform equilateral triangular plate DCO of side $2a$ is smoothly hinged at D and C, and the point O (which is higher than D and C), is connected to A and B by taut light strings each of length $3a$.

    Find the tensions in the strings and the vertical components of the reactions at D and C. (L.U., Pt. II)

14. Three smooth spheres, each of radius r, rest in a spherical cup of radius R, and a fourth sphere also of radius r is placed symmetrically on them. All the spheres have the same mass. Show that, if equilibrium is possible with the three lower spheres still touching one another, R must not be greater than $r(1 + 2\sqrt{11})$. (L.U., Pt. I)

15. A smooth sphere of radius R and weight W rests in a horizontal frame consisting of three thin rods each of length $2R$ pin-jointed together at their ends, the frame being supported by vertical reactions at the joints. Show that the horizontal reaction between the rods at each joint is $W/(3\sqrt{6})$. and find the greatest bending moment in each rod. (L.U., Pt. I)

9.7 Friction

(1) When two bodies are in contact the direction of the force of friction on either of them at the point of contact is in the opposite direction to that in which either tends to move relative to the other.

(2) If the bodies are in equilibrium the force of friction is just sufficient to prevent motion and may be found from the conditions of equilibrium of the body.

(3) Limiting friction is the maximum tangential force, exerted when equilibrium is on the point of being broken and this limiting friction is proportional to the normal reaction between surfaces. The limiting friction is denoted by μR, where R is the normal reaction, and μ is called the coefficient of friction.

(4) The amount of the limiting friction is independent of the area of contact between the surfaces provided the normal reaction is the same.

(5) When motion takes place the direction of friction is opposite to the direction of relative motion and independent of the velocity.

These are the laws of what is called dry friction between surfaces. They are really rules for the mathematical treatment of friction based on experimental results. Thus within certain limits the ratio of the friction to the normal pressure is found to be fairly constant for given materials polished to the same extent, although for very high pressures the friction increases more rapidly and seizure may take place. The so-called wet friction between lubricated surfaces is not governed by these laws.

Modern theory suggests that the force of friction is due to the non-rigidity of bodies. When one body presses on another there is always an *area* of contact, much smaller than the apparent area of contact,

which depends on the normal thrust between the bodies. Over this small area pressure and consequently temperature may be very high, so that there may be fusion of the materials of which the bodies are composed over the area of contact. Thus friction would be proportional to the area of contact and therefore to the normal thrust as assumed in the above laws.

9.8 Angle of Friction

Let R be the normal reaction at a point O (Fig. 199) and F the force of friction acting in a direction perpendicular to R. The total action at O is a force of magnitude $(R^2 + F^2)^{1/2}$ acting in a direction making an angle $\tan^{-1}(F/R)$ with the normal reaction.

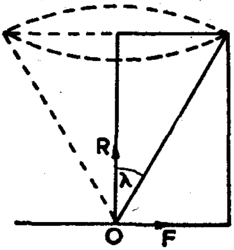

Fig. 199

If friction is limiting $F = \mu R$ and the action at O makes an angle $\tan^{-1} \mu$ with the normal reaction. This angle is called the angle of friction and is denoted by λ.

Thus $$\mu = \tan \lambda.$$

If the direction in which the body tends to move varies, the line of action of the force at O will always lie on a cone with vertex at O and semivertical angle λ. If the force at O lies within this cone the body will not move along the surface. It follows that if P be the resultant of the other forces acting on the body its line of action must pass through O and must lie inside or on the cone of friction for equilibrium.

9.9 Circle of Friction

Consider a shaft of radius r rotating in bearings with a certain clearance and let the load on the shaft be W. There will be a line of contact between the shaft and the bearings. Suppose in the first instance that this line is below the centre of the shaft at A (Fig. 200), with the shaft turning in the direction shown. Then the normal reaction R will

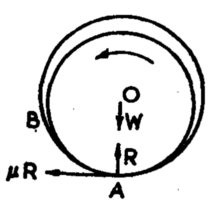

Fig. 200

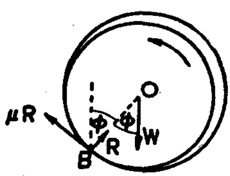

Fig. 201

balance the weight and there will be an unbalanced frictional force μR tending to move the centre of gravity to the left and therefore tending to move the point of contact from A towards B. A position of relative equilibrium is at B, where OB is inclined at ϕ to the vertical (Fig. 201)

if $\qquad R \sin \phi = \mu R \cos \phi,$
that is $\qquad \tan \phi = \mu,$
or $\qquad \phi = \lambda$, the angle of friction.
In this position $\qquad \mu R = W \sin \lambda,$
and hence there is a retarding couple $Wr \sin \lambda$.

The force at B is tangential to a circle of radius $r \sin \lambda$ which is called the circle of friction. Thus if a link in a machine is pin-connected at both ends the line of thrust in the link will be tangential to the friction circles at both ends.

Example 5. *Two equal uniform cylinders are placed in contact with their axes parallel and horizontal on a horizontal plane. A third equal cylinder is rested symmetrically on the first two with its axis parallel to theirs. Determine, graphically or otherwise, the limiting values of the coefficients of friction between cylinders and between cylinder and plane for equilibrium to be possible.*

The plane containing the axes of the upper and one of the lower cylinders is inclined at 60° to the horizontal. Let the reactions and the limiting frictional forces be as shown in Fig. 202.

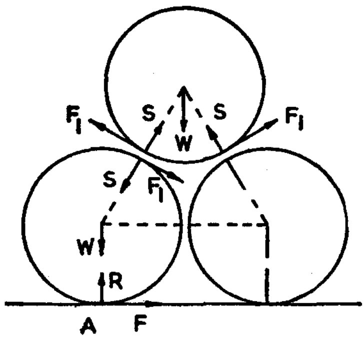

Fig. 202

The lower cylinders will rotate unless $F = F_1$.
For the upper cylinder we have

$$S \cos 30° + F_1 \sin 30° = \frac{1}{2}W. \tag{1}$$

For one of the lower cylinders we have

$$S \cos 30° + F_1 \sin 30° + W = R, \tag{2}$$
$$S \sin 30° - F_1 \cos 30° - F = 0. \tag{3}$$

Hence $\qquad \dfrac{F}{S} = \dfrac{\sin 30°}{1 + \cos 30°} = \tan 15°.$

Also $\qquad R = \dfrac{3}{2}W$, and from (2) and (3)

$$F = \frac{1}{2}W\frac{\sin 30°}{1 + \cos 30°} = \frac{1}{2}W \tan 15°;$$

$$\frac{F}{R} = \frac{1}{3} \tan 15°.$$

Hence, the coefficient of friction between the cylinders and between the cylinders and plane must not be less than tan 15° and $\frac{1}{3}$ tan 15° respectively.

Graphically, it is evident that since the forces R and F pass through the point A, the resultant of S and F_1 must also pass through A and must be inclined at 15° to the vertical and therefore to the radius.

Similarly since $R - W = \frac{1}{2}W = \frac{1}{3}R$, the resultant of $\frac{1}{3}R$ and F is inclined at 15° to the vertical and $F/R = \frac{1}{3}$ tan 15°.

Example 6. *A uniform beam AB, shown in plan in Fig.* 203, 10 *ft. long and weighing* 400 *lb., rests on a rough horizontal plane, making contact with it at the ends only. The end A is anchored by means of a rope to a point C at the same level, BAC being a right-angle, and a horizontal force of* 100 *lb. is applied to the middle*

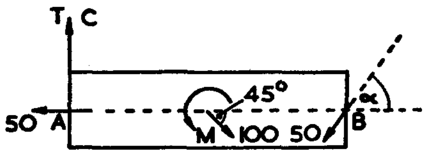

Fig. 203

point of the beam in the direction shown. The coefficient of friction between the beam and the plane is 0·25. *Find the magnitude of the couple applied to the beam in an anti-clockwise direction which will just cause it to move.* (C.U.)

The point A must move in a direction perpendicular to AC, hence the frictional force at A when the beam moves will be 50 lb. along BA.
Let the direction of motion of B make an angle α with AB, then the frictional force at B will be 50 lb. in the opposite direction. Let T be the tension in the rope.
Resolving along and parallel to AB and taking moments about G, we have

$$50 - 50\sqrt{2} + 50\cos\alpha = 0, \quad (1)$$
$$T - 50\sqrt{2} - 50\sin\alpha = 0, \quad (2)$$
$$M - 5T - 50 \times 5\sin\alpha = 0. \quad (3)$$

Hence
$$\cos\alpha = \sqrt{2} - 1$$
$$M = 500\sin\alpha + 250\sqrt{2}$$
$$= 808 \text{ ft.lb. approximately.}$$

EXERCISES 9 (*b*)

1. A non-uniform heavy rod of length l is in equilibrium with its end A resting against a rough vertical wall and its end B connected by a light string of length l to a point C in the wall vertically above A, the plane ABC being normal to the wall. If the inclination of the rod to the horizontal is θ, show that the distance of its centre of mass from its lower end cannot be less than $2l\tan\theta/(\mu + \tan\theta)$, where μ is the coefficient of friction. (L.U., Pt. I)

2. A uniform solid hemisphere of radius a rests with its curved surface touching a rough horizontal floor and a rough vertical wall, the co-

efficient of friction for both points of contact being 1/4. If the solid is on the point of slipping, show that the reactions at the floor and at the wall intersect at a point distant $12a/17$ from the wall. Hence, or otherwise, prove that the plane base is inclined to the horizontal at an angle $\sin^{-1}(40/51)$. (L.U., Pt. I)

3. Two uniform ladders AB, BC each of length l and weights W, nW ($n < 1$) are hinged together at the top B and stand on rough ground, the coefficient of friction at A and C being μ. The angle ABC is gradually increased until slipping occurs. State, with reasons, which ladder slips. If the angle ABC is then 2θ show that

$$(1 + 3n)\mu = (1 + n)\tan\theta,$$

and that the total reaction at the contact where slipping does not occur is $\frac{1}{4}W\{(3 + n)^2 + \mu^2(1 + 3n)^2\}^{1/2}$. (L.U., Pt. I)

4. Two equal uniform rods AB, BC smoothly jointed at B, are in equilibrium with the end C resting on a rough horizontal plane and the end A freely pivoted at a point above the plane. Prove that if α and β are the inclinations of CB and BA to the horizontal, the coefficient of friction must exceed $2/(\tan\beta + 3\tan\alpha)$.

5. A light clamp consists of a ring with two arms at right-angles to each other. The ring surrounds a fixed rod and the arms touch the rod at the points S and T as shown in Fig. 204. A force P is applied at V in the plane STV. If the clamp is just on the point of sliding to the right, show that $\tan\theta = \mu(a - \mu b)/\{a - \mu(b + 2c)\}$, where μ is the coefficient of limiting friction. (Q.E.)

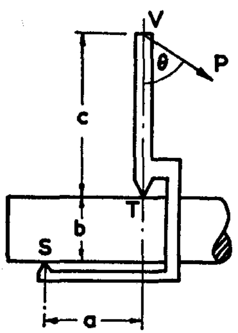

Fig. 204

6. The banks of a frozen pond are inclined to the horizontal at an angle of 30°. A uniform plank is placed with one end on the ice and the other end on a bank, the vertical plane through the plank cutting the bank along a line of greatest slope. Assuming the ice to be perfectly smooth and the plank to be just on the point of slipping, find the coefficient of limiting friction between the plank and the bank.

If the plank weighs 25 lb. and a man weighing 150 lb. stands on it at a distance from the bank equal to $\frac{1}{4}$ of the length of the plank, find the reactions of the two ends. (Q.E.)

7. A uniform ladder weighing 25 lb. and 25 ft. long is placed on level ground against a vertical wall, making an angle of 30° with the wall. If the coefficient of friction at each point is $\frac{1}{3}$, find, graphically or otherwise, how near to the top of the ladder a man weighing 150 lb. can proceed before the ladder slips. (Q.E.)

8. A uniform vertical sliding door, 8 ft. high and 2 ft. wide, runs between well-fitting guides at its top and bottom. The door weighs 10 lb., and the coefficient of friction is $\frac{1}{3}$. Calculate the horizontal force, applied halfway up the door, required to move it. Assume that contact takes place at diagonally opposite corners only. (Q.E.)

9. A bar is fixed to a boss by means of a cotter as shown in Fig. 205. The coefficient of friction between the metals is 0·14. Show that the cotter will pull out when the rod is pulled if the angle θ exceeds 16°. (Q.E.)

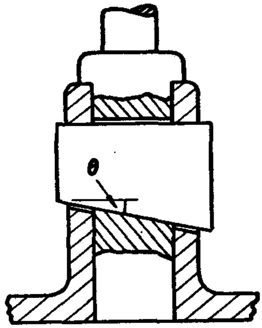

Fig. 205

10. A circular hoop is held in a vertical plane, and a uniform straight rod, whose length is equal to the radius of the hoop, is laid in the hoop with its ends on the inner circumference, which is rough. The hoop is then rotated slowly in the vertical plane until the rod slips. Show that when this happens the angle which the rod makes with the horizontal is $\tan^{-1} 4\mu/(3 - \mu^2)$, where μ is the coefficient of friction. (Q.E.)

11. Two similar uniform cylinders, each weighing 10 lb., lie on a plane inclined at an angle of 10° to the horizontal. Their axes are horizontal, they are in contact with each other throughout their length and the coefficient of friction between them is $\frac{1}{4}$. What force, parallel to the line of greatest slope of the plane, must be applied to the lower cylinder through its centre of gravity to cause it to move up the plane? The coefficient of friction between the plane and the cylinders is sufficient to prevent sliding. (Q.E.)

12. The shaft of a flywheel rests horizontally in two V-grooves, the angle of each being 60°. A rope round the wheel supports a weight P. The coefficient of friction is μ and the wheel and shaft weigh W. Show that the least value of P which will cause the shaft to rotate in the grooves is given by

$$P = \frac{2\mu d}{(1 + \mu^2)D - 2\mu d} W,$$

where d is the diameter of the shaft and D the diameter of the wheel. (Q.E.)

9.10 Centre of Gravity

Consider a number of particles in a plane whose weights are w_1, w_2, . . ., w_n, and whose coordinates with respect to rectangular axes OXY (Fig. 206) are (x_1, y_1), (x_2, y_2), . . ., (x_n, y_n). We may think of the weights as acting in a direction perpendicular to the plane OXY, in which case the total weight which is their resultant will act through the point $(\bar{x}, \bar{y})$, where

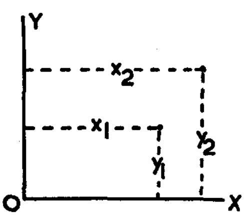

Fig. 206

$$\bar{x} = \frac{\Sigma wx}{\Sigma w}, \quad \bar{y} = \frac{\Sigma wy}{\Sigma w}.$$

The point $(\bar{x}, \bar{y})$ is in fact independent of the direction in which the forces act and is the centre of mass of the particles. If the particles are the constituent particles of a body the point is the centre of mass of the body

and is usually found by methods of calculus. It is also the centre of gravity on the assumption that the gravity forces of the particles act in parallel directions. The centroid, or centre of area, is calculated in the same way with elements of area replacing elements of weight and for a uniform area the centroid coincides with the centre of mass. If the body is three-dimensional the third coordinate $\bar{z} = \Sigma wz/\Sigma w$, is found in the same way.

If a body has a centre of symmetry G, such that for every particle A there is a corresponding particle A' of equal mass and G is the mid-point of AA', the resultant of the weights of A and A' may be taken as acting at G, and G is the centre of mass of the body. Thus the centre of gravity of any body which has a centre of symmetry is at the centre of symmetry. Similarly, if a body has an axis of symmetry the centre of gravity will lie on the axis. The positions of the centres of gravity of the following bodies are easily found by elementary methods:

Triangular lamina	at the intersection of the medians.
Triangle of three uniform rods	at the centre of the inscribed circle.
Solid tetrahedron or pyramid	at the point which divides the join of the vertex to the centroid of the base in the ratio 3 : 1.
Solid circular cone	on the axis at $\frac{1}{4}h$ above the base, h being the height.
Hollow circular cone	on the axis at $\frac{1}{3}h$ above the base, h being the height.
Circular arc	on axis of symmetry at $\dfrac{r \sin \alpha}{\alpha}$ from the centre, r being the radius and 2α the angle subtended by the arc at the centre.
Circular sector	on the axis of symmetry at $\dfrac{2r \sin \alpha}{3\alpha}$ from the centre r being the radius and 2α the angle included by the bounding radii.
Semicircle	at $\dfrac{4r}{3\pi}$ from the centre.
Quadrant of a circle	at $\dfrac{4r}{3\pi}$ from each bounding radius.
Solid hemisphere	on the axis of symmetry, at $\frac{3}{8}r$ from the centre.
Hemispherical shell	on the axis of symmetry at $\frac{1}{2}r$ from the centre.

9.11 Centre of Gravity of Composite Bodies

If a body has two or more parts the weight and centre of gravity of each of which is known, the centre of gravity of the whole is easily found by finding the line of action of the resultant of the weights of the parts. If a part is taken away from a body the centres of gravity and weights of the whole and of the part being known, the result is found in the same way by considering the weight of the part taken away as being an added negative weight.

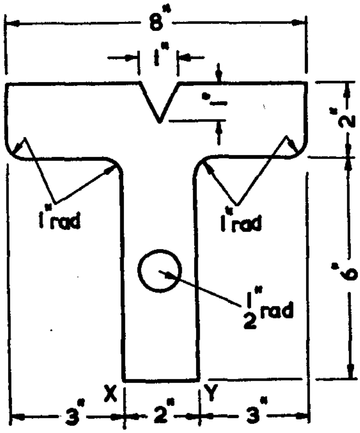

Fig. 207

Example 7. *Find the position of the centre of gravity of the T-section shown in Fig.* 207.

The T-section may be built up by additions and subtractions from two rectangles as shown in Fig. 208 (*a*), (*b*) and (*c*).
The elements shown in (*b*) are added to (*a*) and those shown in (*c*) are subtracted and are therefore considered as negative areas. The areas of the elements and the distances of the centroids from XY are tabulated and the sum of moments of the areas about XY computed.

Element	*Area*	*CG from XY*	*Moment*
A	16	7	112
B	12	3	36
C	$\frac{\pi}{2}$	$7 - \frac{4}{3\pi}$	10·33
D	2	5·5	11
E	$-\frac{1}{2}$	$7\frac{2}{3}$	−3·83
F	−2	6·5	−13
G	$-\frac{\pi}{2}$	$5 + \frac{4}{3\pi}$	−8·52
H	$-\frac{\pi}{4}$	3	−2·36
Total	$27 \cdot 5 - \frac{\pi}{4}$ $= 26 \cdot 71$		$169 \cdot 33 - 27 \cdot 71$ $= 141 \cdot 62$

Hence the distance of the centre of gravity from XY is $\frac{141 \cdot 62}{26 \cdot 71} = 5 \cdot 30$ in.

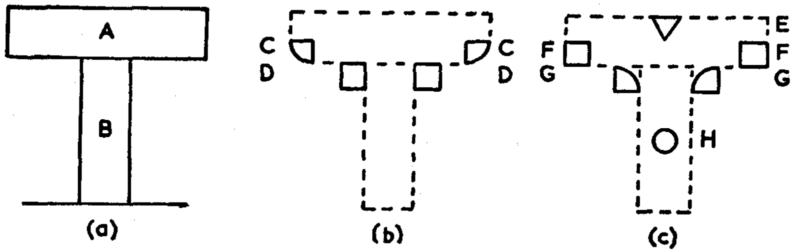

Fig. 208

9.12 Theorem of Pappus

If a plane area is revolved about an axis in its plane, the volume generated is the product of the area and the length of the path described by the centroid of the area. Let A be the area, OX the axis of rotation (Fig. 209) and let a small element of area be distant y from OX. Then if the area be rotated through an angle α the volume generated by the element δA is $\alpha y \delta A$, to the first order. The volume described by the whole area is

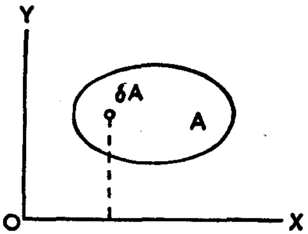

Fig. 209

$$\Sigma \alpha y \delta A = \alpha \Sigma y \delta A = \alpha A \bar{y},$$

where $\bar{y}$ is the distance of the centroid from OX. The length of the path of the centroid is $\alpha \bar{y}$ and so the theorem is proved.

If the axis divides the area into two parts of areas A_1 and A_2 whose centroids are distant $\bar{y}_1$ and $\bar{y}_2$ respectively from the axis, the volumes generated by the parts are $\alpha \bar{y}_1 A_1$ and $\alpha \bar{y}_2 A_2$. The *difference* of these volumes is

$$\begin{aligned} &\alpha(\bar{y}_1 A_1 - \bar{y}_2 A_2) \\ &= \alpha A \bar{y}, \end{aligned}$$

where $\bar{y}$ is the distance of the centroid of the whole from the axis.

If a semicircle of radius a is rotated through an angle 2π about its bounding diameter it generates a sphere. Then if $\bar{y}$ be the distance of the centroid of the semicircle from the diameter

$$\frac{1}{2}\pi a^2 \times 2\pi \bar{y} = \frac{4}{3}\pi a^3,$$

$$\bar{y} = \frac{4a}{3\pi}.$$

Thus the theorem of Pappus can be used to find the position of the centroid.

Example 8. *Use Pappus' theorem to find the centre of gravity of the uniform sheet of metal shown in Fig.* 210. (L.U., Pt. I)

If the figure is rotated through 2π about AC the volume generated consists of a cylinder and the difference between two hemispheres, that is

$$2 \times 6^2 \times \pi + \frac{2}{3}\pi \times 6^3 - \frac{2}{3}\pi \times 4^3 = \frac{520\pi}{3}.$$

The area $= 12 + 5\pi = 27{\cdot}71$.
Therefore if $\bar{y}$ be the distance of the centroid from AC,

$$2\pi\bar{y} \times 27{\cdot}71 = \frac{520\pi}{3},$$

$$\bar{y} = \frac{260}{83{\cdot}13} = 3{\cdot}13 \text{ in.}$$

If the figure is rotated about BD the volumes generated by the two parts are $304\pi/3$ and 24π and their difference is $232\pi/3$.
Therefore if $\bar{x}$ be the distance of the centroid from BD,

$$2\pi\bar{x} \times 27{\cdot}71 = \frac{232\pi}{3},$$

$$\bar{x} = \frac{116}{83{\cdot}13} = 1{\cdot}395 \text{ in.}$$

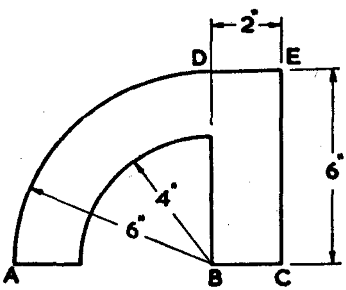

Fig. 210

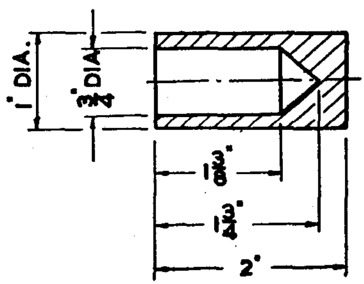

Fig. 211

EXERCISES 9 (*c*)

1. Fig. 211 shows a central section through a cylinder which has an axial hole, ending in a cone, extending part way along it. Find the distance of the centre of gravity from the left-hand end. (Q.E.)

2. A uniform circular disc can rotate without friction about a horizontal axis through its centre perpendicular to the disc. At four equidistant points on the circumference masses of 3, 10, 5, 7 lb. are fixed, in the order given. Find the angle which the radius to the 10 lb. mass makes with the vertical when the disc has turned to its position of stable equilibrium.

   A fifth mass is to be fixed on the circumference so that the disc will rest in any angular position. Find the magnitude and position of this mass, stating the quadrant in which it is placed, and the angular distance between it and the 10 lb. mass. (Q.E.)

3. A uniform plate is in the form of an equilateral triangle ABC, of side 8 in. Two of the corners have been cut off by circular arcs of radii 1 in. and 3 in. having centres at A and C respectively. Find the area of the plate and the distance of its centroid from the edge AB. (L.U., Pt. I)

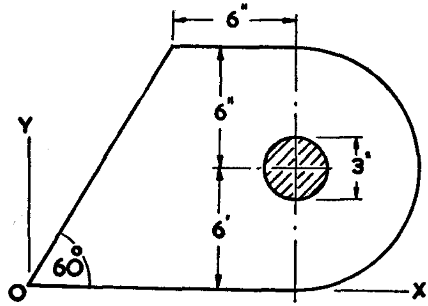

Fig. 212

4. A sheet of metal of uniform thickness is shaped as shown in Fig. 212. The shaded area represents a circular hole of 3 in. diameter. Find the position of the centre of gravity stating its distances from OX and OY respectively. (Q.E.)

5. A solid, of uniform material, consists of a cylinder of length 10 in. and diameter 4 in., with one hemispherical end, also of diameter 4 in. From the other end a concentric hole, 7 in. deep and 2 in. in diameter is bored into the cylinder. Find the distance of the mass-centre from the flat end. (L.U., Pt. I)

6. Find the centroid of the area of a sector of a circle of radius r, the angle between the bounding radii being a.

By using Pappus' theorem, or otherwise, find the volume swept out by rotating the plane figure $ABCDE$ (consisting of a rectangle and a sector of a circle) (Fig. 213), through one revolution about the side AE. (L.U Pt. I)

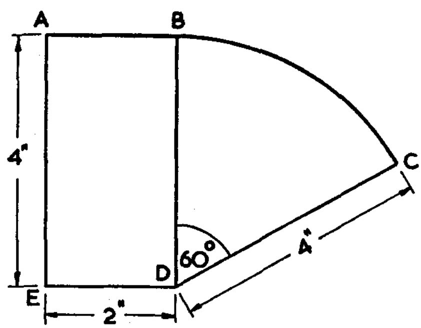

Fig. 213

7. Show that the centroid of a segmental area of a circle of radius r which subtends an angle $2a$ at the centre is at a distance $\frac{4r}{3}\sin^3 a/(2a - \sin 2a)$ from the centre.

The smaller segment cut off from a circle of radius 5 cm. by a chord of length 6 cm. is rotated about the chord through an angle of 360°. Find, by Pappus' theorem or otherwise, the volume of the spindle so formed. (L.U., Pt. I)

8. A uniform solid of revolution is formed by joining the base of a right circular cone of height H to the equal base of a right circular cylinder of height h. Calculate the distance of the centre of mass of the solid from its plane face when $H = 12$ in., $h = 3$ in.

The solid is placed with its plane face on a rough inclined plane and

the inclination to the horizontal of the plane is gradually increased. Show, that if the radius r of the base is 2 in., and the coefficient of friction μ is 0·5, the body will topple over before it begins to slide.

If the heights are so chosen that the centre of mass of the solid is at the centre of the common base, show that, if $\mu H < r\sqrt{6}$, the solid will slide before it topples. (L.U., Pt. I)

9. A circular quadrant of radius $a/2$ and centre B is removed from a square $ABCD$ of side a. If the remaining portion is rotated through an angle of 360° about the side AD, prove that the volume generated is $\pi a^3(13/12 - \pi/8)$. (L.U., Pt. I)

10. The diagram (Fig. 214) represents the section of a girder; find the distances of its centre of gravity from AB and AC. (L.U., Pt. I)

11. A casting, uniform in material and thickness, has the form given in the sketch (Fig. 215). Find the distance of its mass centre from AB. (L.U., Pt. I)

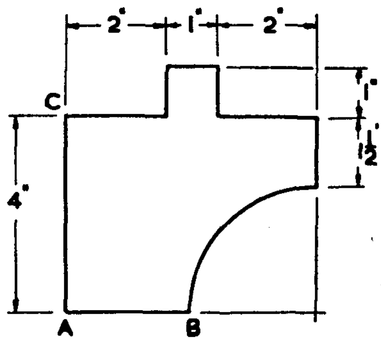

Fig. 214

12. The diagram (Fig. 216) shows a rectangular T-piece, of uniform material, with rounded corners at E and F, these being each a quadrant of a circle of radius 1 in. Using the dimensions shown, calculate the distance of the centre of gravity from AB.

(L.U., Pt. I)

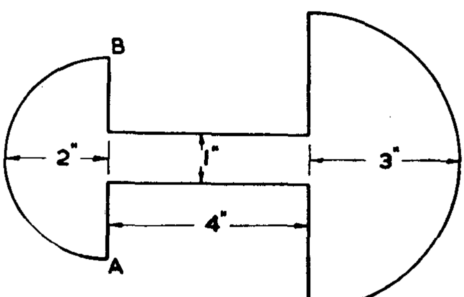

Fig. 215

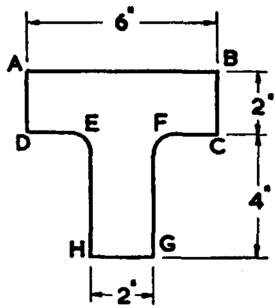

Fig. 216

9.13 Funicular Polygon

The magnitude and direction of the resultant of a number of coplanar forces acting on a body can be found by drawing the polygon of forces. The force vectors are placed end to end in any order and the line which with the vectors forms a closed polygon represents the resultant of the forces in magnitude and direction. The line of action of the resultant

is found by drawing, in addition, what is called a funicular polygon.

Let P, Q, R, S be the given forces and let their lines of action be as shown in Fig. 217 (*a*). The force polygon is shown in Fig. 217 (*b*). The force T is the resultant of the forces P, Q, R, S and its magnitude and direction is determined.

Choose a point o, called the pole, in the plane of the polygon and join o to the intersections of the forces P, Q, R, S, T. It is convenient to label the triangles formed by o with the forces as p, q, r, s, t, then we may denote, for example, the join of o to the intersection of P and Q by pq.

To draw the funicular polygon, select a point p on the line of action of the force P in Fig. 217 (*a*). Through p draw pt and pq parallel to the

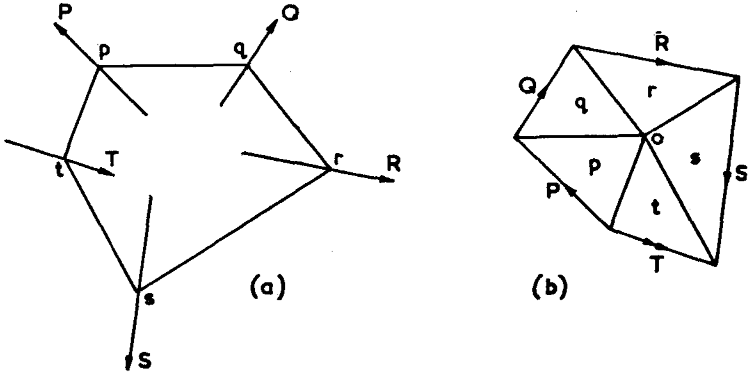

Fig. 217

lines pt and pq in the force polygon and let pq meet the line of action of Q in q. Through q draw qr parallel to the line qr in the force polygon to meet the line of action of R in r.

Through r draw rs parallel to the line rs in the force polygon to meet the line of action of S in s.

Through s draw st parallel to the line st in the force polygon to meet pt in the point t.

Then t is a point on the line of action of the resultant T and the resultant is completely determined.

The result is proved by considering the triangles p, q, r, s, t in the force polygon as triangles of forces. Thus tp and pq are equivalent to the force P, pq in the opposite sense and qr are equivalent to Q, qr in the opposite sense and rs are equivalent to R, rs in the opposite sense and st are equivalent to S.

Thus in the funicular polygon the forces P, Q, R, S may be considered as resolved along the directions tp, pq, qr, rs, st. The resolved parts along pq, qr, rs balance out and the forces P, Q, R, S are, therefore,

equivalent to the components along pt and st, hence their resultant passes through t.

If the forces are in equilibrium the polygon of forces will close. The converse is not true since the forces may reduce to a couple. This couple is not zero unless the funicular polygon also closes. Thus if $T = 0$ (Fig. 217) the points $pqrs$ form the funicular polygon, but the line ps drawn from p may not coincide with the line sp drawn from s and the system reduces to equal parallel forces along these lines.

9.14 Resultant of Parallel Forces

When the forces acting on a body are parallel the polygon of forces, having all its sides parallel, reduces to a straight line, but if a pole is taken outside the polygon a funicular polygon can be drawn and the line of action of the resultant determined.

Consider the case where parallel forces P, Q, R act vertically on a simply supported beam. It is desired to find graphically the end reactions X and Y (Fig. 218).

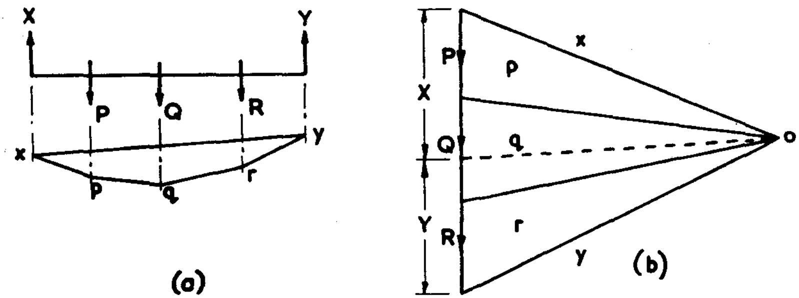

Fig. 218

Having drawn the polygon with the forces P, Q, R a pole is selected at o and o is joined to the extremities of P, Q, R, the triangles formed being labelled p, q, r and the spaces above and below the polygon x and y. A point p is chosen on the line of action of P and the funicular polygon $pqryx$ is drawn, x and y being the points in which the lines px and py meet the lines of action of X and Y respectively. In the force diagram the line xy is drawn parallel to the line xy in the funicular polygon; this line divides the total force $P + Q + R$ into the forces X and Y.

The result follows from the fact that the forces P, Q, R are equivalent to the component of P along xp and the component of R along yr.

The resultant of the component along xp and X must be equal and opposite to the resultant of the component along yr and Y, therefore both resultants must act along the line xy.

Therefore in the force diagram the force along px and the force along xy with X form a triangle of forces.

The funicular polygon represents a possible configuration of light inextensible string supporting weights P, Q, R at p, q, r respectively, and in this case the lines xp, pq, etc., in the force diagram would give the tensions in the portions of the string. Hence the name funicular polygon.

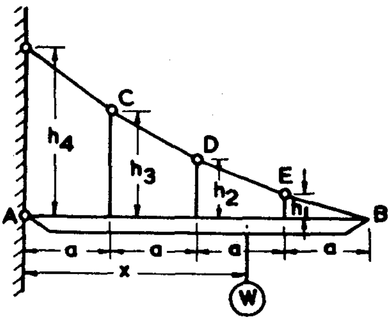

Fig. 219

Example 9. *A rigid beam AB, hinged at one extremity A, carries a load W and is supported horizontally by a linkwork of rods in the manner shown in Fig. 219. Neglecting the weight of the beam and the rods show that H, the horizontal component of the tension in the chain, is given by $H = Wx/h_4$. Determine expressions for the tensions in the three vertical rods.* (C.U.)

The forces at B are H, a vertical weight T_0 and the tension in BE, and the triangle of forces has vertical side h_1 and horizontal side a (Fig. 220). Similarly if T_1 be the tension of the rod at E, this force with the tensions in EB and ED forms a triangle of forces, the slope of DE being $\tan^{-1}(h_2 - h_1)/a$. Similarly for the tensions T_2 and T_3 in the rods at D and C respectively.

Thus
$$\begin{aligned} T_0 &= Hh_1/a, \\ T_0 + T_1 &= H(h_2 - h_1)/a, \\ T_0 + T_1 + T_2 &= H(h_3 - h_2)/a, \\ T_0 + T_1 + T_2 + T_3 &= H(h_4 - h_3)/a. \end{aligned}$$

Taking moments about A we have
$$4aT_0 + 3aT_1 + 2aT_2 + aT_3 = Wx,$$
and from the previous equations this gives
$$H = \frac{Wx}{h_4}.$$

Then
$$\begin{aligned} T_1 &= Wx(h_2 - 2h_1)/ah_4 \\ T_2 &= Wx(h_3 - 2h_2 + h_1)/ah_4 \\ T_3 &= Wx(h_4 - 2h_3 + h_2)/ah_4. \end{aligned}$$

9.15 Stresses in Frameworks

A framework or truss is normally built up with triangles since this shape cannot be deformed without altering the lengths of the bars. The bars of a triangulated framework are assumed to be pin-jointed at their ends with the centre of the joint in the line of the bars and it is usually assumed that the

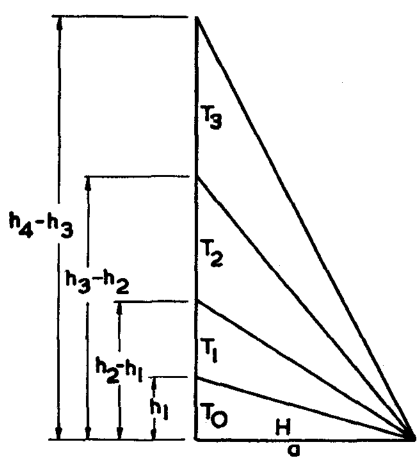

Fig. 220

external loads are applied through the pins. The *primary stresses* in a framework are found by working on these assumptions and if necessary secondary stresses due to departure from these assumptions may be found.

The forces in a triangulated framework may be found when the external forces are known. For a two-dimensional framework, if b is number of bars and j the number of joints, we have $2j$ equations between the forces acting at the joints which are the forces of tension or compression in the bars and the external forces. Since there are 3 equations for the equilibrium of the forces on the framework as a whole, we are left with $2j - 3$ independent equations and, therefore, loads in the bars can be determined if the number of bars is $2j - 3$. A framework in which $b = 2j - 3$ is said to be *simply stiff*. If the number of bars exceeds this number the loads in the bars are determined by their elasticity and cannot be found by the methods of statics of a rigid body.

The formula $b = 2j - 3$ may also be seen from the way in which a triangulated framework is built up. Three bars are required for the first three joints and two more for each additional joint.

Thus
$$\begin{aligned} b &= 3 + 2(j - 3) \\ &= 2j - 3. \end{aligned}$$

9.16 Bow's Notation

The primary stresses in the members of a simply stiff framework are found graphically by application of the triangle or polygon of forces to the forces acting at each of the pin-joints. The procedure is simplified by the use of Bow's notation by which each member is specified by two capital letters and the force in this member in the force diagram is specified by the same letters in small type. This is done by lettering the spaces between the members and between the lines of action of the forces in the space diagram, that is, the diagram showing the relative positions of the members. The minimum number of letters must be used to ensure that each force is represented by two letters and that each pair of letters represents a force. In drawing the force diagram the forces acting at each point should be drawn in the order of the letters describing the point, beginning with the known forces and going around each point in the same sense.

Example 10. *The framework shown in Fig.* 221 *consists of equilateral triangles. It is freely supported at its lower extremities and carries loads of* 4 *tons and* 8 *tons at the upper joints. Find the stresses in the members.*

The force at each support is found by taking moments about the other support and we have $Q = 7$ tons, $P = 5$ tons. The forces at the supports may also be found graphically by the methods of § 9.14 by taking a pole o and drawing a funicular polygon to determine ad on the force diagram. We may call the left-hand support the point AED and we draw the triangle of

forces *aed* for the forces acting at the point (Fig. 222). Proceeding to the point $EABF$, the forces *ea* and *ab* are known and we construct the polygon of forces *eabf*. The points $FBCG$ and $DEFG$ are dealt with in the same way and the force diagram is then complete, and by measurement the stresses in the members are

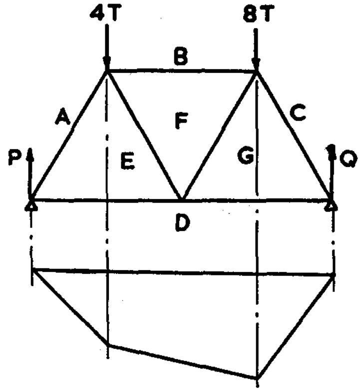

Fig. 221

AE	5·77 tons,
BF	3·46 ,,
CG	8·08 ,,
GF	1·15 ,,
ED	2·89 ,,
EF	1·15 ,,
GD	4·04 ,,

By considering the triangle of forces it is seen that the force in EA acts towards the support and the force in ED away from it. Hence EA is in compression and ED in tension. Similarly the members EF and GD are in tension and all the others in compression.

9.17 Method of Sections

The forces in the members of a framework may be calculated by considering a portion of the framework as being in equilibrium under the action of the external forces which act on it and the forces in the members joining this portion to the remainder of the framework. A line of section is drawn cutting not more than three members in which the forces are unknown; if the lines of these three members are not concurrent we have three conditions of equilibrium involving the external forces and the forces in the three members. Hence these forces can be found and by repeated sections the forces in all the members can be calculated.

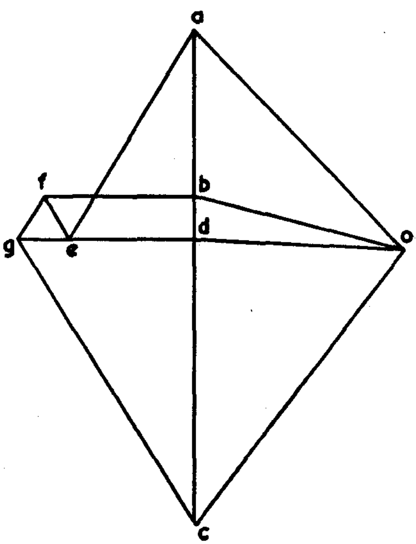

Fig. 222

Example 11. *Calculate the stress in the member CD of the Warren girder shown in Fig.* 223.

By taking moments we find the reactions at the supports are $P = 15$ tons, $Q = 19$ tons. Let T_1, T_2, T_3 denote the *tensions* in the members BD, DC, CE respectively.
The portion ABC of the girder

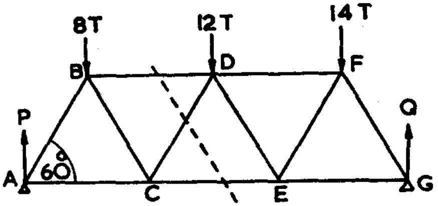

Fig. 223

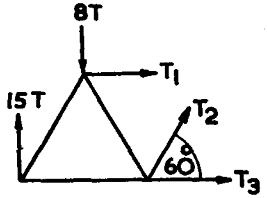

Fig. 224

is in equilibrium under the action of the forces P, 8 tons, T_1, T_2, T_3 (Fig. 224).

Hence $\qquad T_2 \sin 60° + 15 = 8$

$$T_2 = -\frac{14}{\sqrt{3}} = -8{\cdot}08 \text{ tons.}$$

Thus the member CD is in compression and the force is 8·08 tons.

EXERCISES 9 (*d*)

1. Forces 13, 2, 3, 8 lb.wt., act along the sides AB, BC, CD, AD respectively of a square $ABCD$ in the sense indicated by the order of the letters. Find graphically the magnitude and direction of their resultant and show that its line of action passes through the mid-point of AB.

2. A light horizontal beam AB is 20 ft. long and is supported at 5 ft. from A and 3 ft. from B. Loads of 5, 7, 9, 6, 4 lb.wt., are carried at 1, 4, 7, 12, 16 ft. respectively from A. Find graphically the reactions at the supports.

3. A light rigid beam, hinged at one extremity, carries a load W and is supported horizontally by a linkwork of rods as shown in Fig. 225. Find the lengths of the vertical rods if the tensions in them are to be equal. Find this tension and the horizontal component of the reaction at the hinge.

4. A pin-jointed Warren girder in which the panels are all equilateral triangles is end-supported and has $2n$ bays in the lower boom. Find in magnitude and sense the forces in the three members enclosing the central panel due to a load W at $r(r < n)$ bays from one end. (L.U., Pt. I)

5. By means of a force diagram, or otherwise, find the forces in all the members of the given frame (Fig. 226) due to the given load

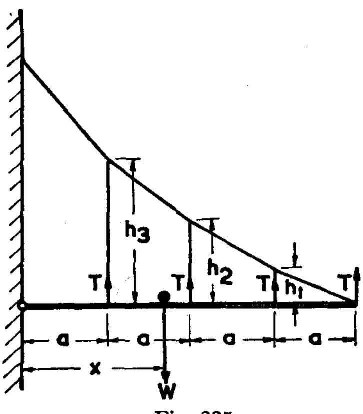

Fig. 225

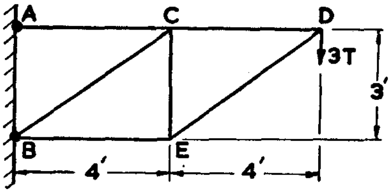

Fig. 226

stating whether the members are in tension or compression. The framework is pinned to a rigid support at A and B.

(L.U., Pt. I)

6. Find the forces in all members of the frame shown in Fig. 227, loaded as shown and simply supported at A and D. AC and BD are not jointed together where they cross.

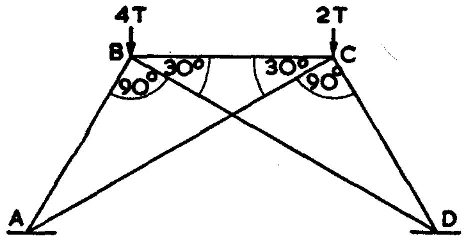

Fig. 227

7. The diagram (Fig. 228) represents a framework of ten smoothly jointed light bars hinged at A and G to points in a vertical wall. Find by a graphical construction or otherwise the forces in the bars FE and BE and the reaction of the hinge at A.

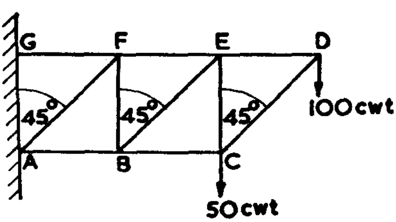

Fig. 228

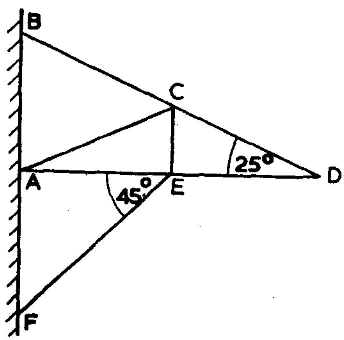

Fig. 229

8. Find the stresses in the bars of the wall crane shown in Fig. 229 due to a load of 5 tons hanging from a chain at D. The chain passes over pulleys at D and E; it is fixed to the wall at F, and $AE = ED$.

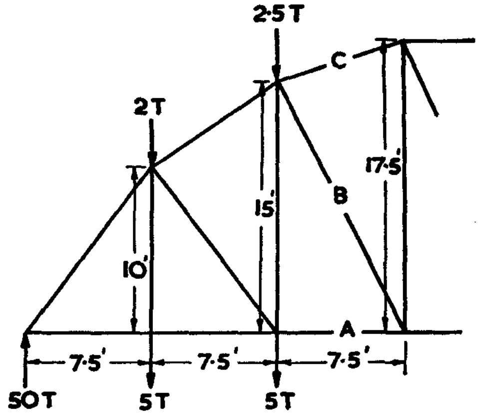

Fig. 230

9. Fig. 230 shows a pin-jointed bridge-truss. Find the forces in the members A, B and C distinguishing between tension and compression.

10. The jointed framework in Fig. 231 is supported on a hinge at E, AE being kept horizontal by a vertical force at A. The seven members

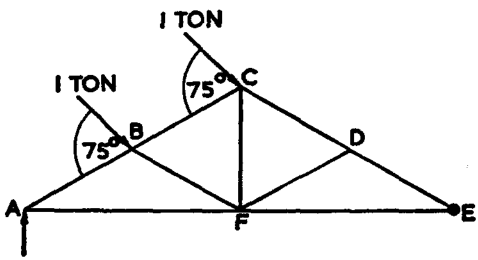

Fig. 231

AB, BC, CD, DE, CF, BF, DF are all equal. Two loads, each of 1 ton, are applied as shown. Find the reactions at A and at E, and the stress in the member BF. (Q.E.)

11. For the truss shown in Fig. 232 determine the stresses in the members

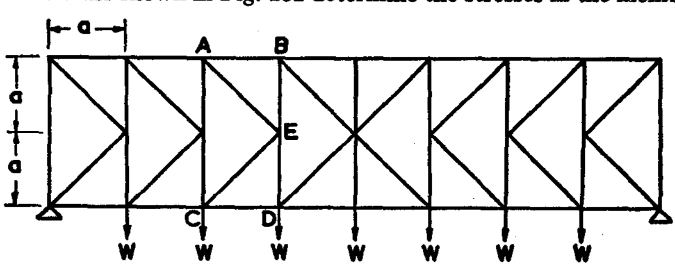

Fig. 232

AB and CD by taking a section which cuts the members AB, BE, ED and CD. Find also the stresses in AE and CE.

12. Fig. 233 represents a roof truss hinged at A. The end E is freely sup-

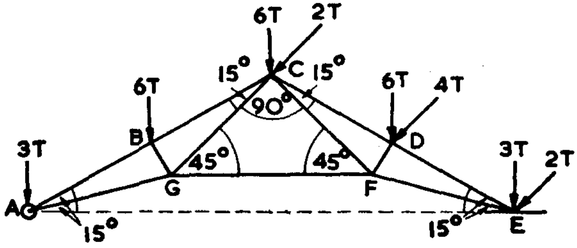

Fig. 233

ported at the same level. The loads due to the dead weight and to wind thrust are as shown. Draw a stress diagram and find the tensions in the members GC, CF and FG.

9.18 Work

The work done by a force in a small displacement of its point of application is defined as the product of the force and the projection of the displacement on the line of action of the force. Thus if δW be the work done by a force P in a displacement δs (Fig. 234) we have

Fig. 234

$$\delta W = P \cos \theta . \delta s,$$

where θ is the angle between the displacement and the direction of P. The work done in a finite displacement s_0 to s_1 is then

$$W = \int_{s_0}^{s_1} P \cos \theta ds.$$

The units of work are those of a product of a force and a distance, such as foot pounds, foot poundals, dyne centimetres, etc. In vector notation (see § 7.11) work is expressed as the scalar product of the force vector $\mathbf{P}$ and the displacement vector $\delta\mathbf{s}$,

that is $$\delta W = \mathbf{P}.\delta\mathbf{s}.$$

If $\mathbf{P} = iX + jY + kZ$ and $\delta\mathbf{s} = i\delta x + j\delta y + k\delta z$, we have

$$\delta W = X\delta x + Y\delta y + Z\delta z,$$

and this shows that the work done by a force is equal to the work done by its components.

The work done by a couple of moment G turning through a small angle $\delta\theta$ is $G\delta\theta$. If the forces of the couple be P acting at points A and B (Fig. 235) it is evident that for equal linear displacements of A and B no work is done. If AB turns through a small angle $\delta\theta$ so that B moves to B' the work done is

$$P.BB' = P.AB.\delta\theta = G.\delta\theta.$$

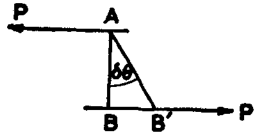

Fig. 235

If the moment of the couple remains constant the work done in a finite rotation through an angle θ is $G.\theta$.

The work done in stretching an elastic string or a helical spring is the product of the extension and the mean of the initial and final tensions.

If λ be the modulus, l the unstretched length and x the extension, the tension is $\lambda x/l$. The work done in stretching a further distance δx is therefore $\dfrac{\lambda}{l}x\delta x$. The work done in increasing the extension from x_1 to x_2 is therefore

$$\int_{x_1}^{x_2}\frac{\lambda}{l}x\,dx,$$

$$= \frac{\lambda}{2l}(x_2^2 - x_1^2),$$

$$= (x_2 - x_1)\frac{1}{2}\left(\frac{\lambda}{l}x_2 + \frac{\lambda}{l}x_1\right),$$

and this is the extension multiplied by the mean of the initial and final tensions.

The work done in raising a system of particles from one position to another position is Wh, where W is their total weight and h is the distance through which their centre of gravity is raised.

Let the weights of the particles be $w_1, w_2, \ldots, w_n$ and let their heights above some standard position before and after raising be $x_1, x_2, \ldots, x_n$, and $x_1', x_2', \ldots, x_n'$ respectively and $\bar{x}$ and $\bar{x}'$ the heights of their centres of gravity.

Then
$$W\bar{x} = w_1x_1 + w_2x_2 + \ldots + w_nx_n,$$
$$W\bar{x}' = w_1x_1' + w_2x_2' + \ldots + w_nx_n',$$
$$W(\bar{x}' - \bar{x}) = w_1(x_1' - x_1) + w_2(x_2' - x_2) + \ldots + w_n(x_n' - x_n).$$

The right-hand side in this equation is the total of the work done in raising the individual particles and the left-hand side is Wh, hence the theorem is proved. The particles may be constituent particles of a rigid body. It should be noted that the work done depends only on the initial and final positions of the centre of gravity, and is independent of the path followed by the particles.

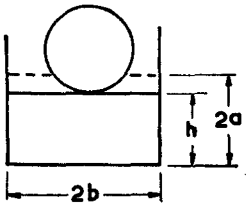

Fig. 236

Example 12. *A sphere of weight W and radius a lies at the bottom of a cylinder of radius b. Water is poured into the cylinder until the sphere is just covered. W′ is the weight of water displaced by the sphere. Find the work done in lifting the sphere just clear of the water.*

The volume of water

$$= 2ab^2\pi - \frac{4}{3}\pi a^3.$$

Its height h (Fig. 236) $= 2a - \dfrac{4a^3}{3b^2}$.

The work done on the sphere $= W\left(2a - \dfrac{4a^3}{3b^2}\right)$.

The weight of the water $= W'\dfrac{(2ab^2\pi - \frac{4}{3}\pi a^3)}{\frac{4}{3}\pi a^3}$.

The fall of its centre of gravity $= \dfrac{2a^3}{3b^2}$.

The work done by gravity $= W'\left(a - \dfrac{2a^3}{3b^2}\right)$.

Therefore the total work done $= (2W - W')\left(a - \dfrac{2a^3}{3b^2}\right)$.

9.19 Potential Energy and the Work Function

When the work done by a force on a particle as it moves from a position A to a position B depends only on the positions of A and B and not on the path by which the particle travels from A to B the force is said to be *conservative.* We have proved that the force of gravity acting on a system of particles or on a rigid body is conservative. An example of a non-conservative force is the force of friction. Thus the work done by friction on a particle sliding down an inclined plane is increased if the particle first moves up the plane and then down.

If the forces which act on a body are conservative the work which can be done by the forces as the body moves from its position to some standard position is called its potential energy. Thus the potential energy of a particle of weight w at height h above the ground is wh. If the particle rises through a further height δh, the increase of potential energy is $w\delta h$ and the work done by gravity is $-w\delta h$. If the tension in an elastic string or helical spring is T and it is extended a distance δx the potential energy stored in the spring is increased by $T\delta x$ and the work done by the tension is $-T\delta x$. The same reasoning applies to other conservative forces. Thus if it is possible to write down the potential energy V of a body in any position the work done when V increases by δV is $-\delta V$. If we write $W = -V$, W is called the *work*

function, it varies with the position of the body and the work done as the body moves from a position A to a position B is

$$W_B - W_A.$$

9.20 Machines

An apparatus in which a force, called the effort, is applied to overcome a resistance or move a weight is called a machine. Familiar examples are the lever, systems of pulleys, wheels and axles and the screw.

The *velocity ratio* of a machine is the ratio of the distance moved by the point of application of the effort to the distance moved by the point of application of the resistance or weight. If P is the effort and W the resistance or weight, we have

$$\text{velocity ratio} = \frac{\text{distance moved by } P}{\text{distance moved by } W}.$$

The *mechanical advantage* is the ratio W/P. Thus

$$\text{mechanical advantage} = \frac{\text{resistance}}{\text{effort}} = \frac{W}{P}.$$

If the machine were perfectly frictionless and its moving parts weightless the work done by the effort would be equal to the work done against the resistance and hence the velocity ratio would be equal to the mechanical advantage. This is shown to follow from the principle of virtual work enunciated in the following section. In all machines, however, some part of the work done by the effort is consumed in overcoming friction or moving parts of the machine, and hence the mechanical advantage is less than the velocity ratio.

The ratio of the mechanical advantage to the velocity ratio is called the *efficiency* of the machine, and this is also the ratio of the work done against the resistance to the work done by the effort.

If the efficiency is from frictional causes less than 50 per cent the machine will not overhaul when the effort ceases, since the work which the resistance can do will be less than the work required to overcome the reversed friction.

The velocity ratio depends only on the arrangement of the parts of the machine. The mechanical advantage, and therefore the efficiency, may vary with the load applied and may be found experimentally for different values of the effort P and the resistance W. It is usually found that P and W are connected by a linear relation of the form $P = a + bW$, where a and b are constants. Such a relation is called the *law of the machine*. Thus the efficiency varies with the load and efficiency is calculated relative to a particular load.

EXERCISES 9 (*e*)

1. A uniform log is in the form of a right prism whose cross-section is a triangle of sides $1\frac{1}{2}$ ft., 2 ft., $2\frac{1}{2}$ ft., and its weight is 6 cwt. It rests with its smallest rectangular face on a horizontal plane. Find in foot-tons the least work needed to raise it on one edge so that it may fall over on to its largest rectangular face. (L.U.)
2. A uniform shaft of length 3 ft., diameter 6 in., and modulus of rigidity 12×10^6 lb./in.^{2}, is given a twist of 5 degrees. Find the work done.
3. A circular cone of base diameter 1 ft., and height 2 ft., is totally immersed in water contained in a vertical cylindrical drum 1·5 ft., in diameter. If the density of the cone equals that of the water (62·5 lb./cu. ft.), how much work must be done to lift the cone by its apex until it is just clear of the water. (Q.E.)
4. A wheel and axle working with vertical loads has a frictional torque resisting motion of μ times the total load. If R and r are the radii of wheel and axle respectively, find expressions for the mechanical advantage and the efficiency of the machine and show that, if μ is greater than r, the machine is irreversible. (L.U., Pt. I)
5. A bucket is lowered into a well by means of a windlass and pulley. The end of the rope of the windlass is attached to the frame of the windlass, and the pulley, with bucket attached, slides in the loop of the rope, the hanging parts of the rope being vertical. Neglecting friction and the weight of the rope, determine by the principle of work, or otherwise, the force that must be applied at the arm of the windlass to maintain the bucket in equilibrium, having given the weight W of the bucket and its load, b the diameter of the barrel of the windlass and a the length of the arm. What is the efficiency if the force that will just raise the bucket is nW? (L.U.)
6. A screw-jack is to be used to lift two wheels of a motor-car off the ground. The load on each of these wheels is 500 lb., and the springs deflect 4 in. under this load, the jack is applied to the frame of the car at a point midway between the wheels. The velocity ratio between the handle and the lifting point of the jack is 200 : 1, and the friction in the jack is such that the applied force is 200 per cent greater than it would be without friction. Find the greatest force which has to be applied to the handle, and the total work done, in lifting the car until there is 1 in. clearance between the wheels and the road. The weight of the wheels and axle parts may be neglected. (Q.E.)

9.21 Forces which do no Work

(1) *The internal forces of a rigid body*

These forces are mutual actions and reactions between the particles of the body. Let P be the force acting between two particles at A and B (Fig. 237) and let the particles be displaced to A' and B'. Since the

body is rigid $AB = A'B'$. Let the angle between $A'B'$ and AB be $\delta\theta$ and let M and N be the perpendiculars from A' and B' respectively to AB. The work done in the displacement

$$= P.AM - P.BN,$$
$$= P.(AB - MN),$$
$$= P.AB(1 - \cos\delta\theta),$$
$$= P.AB(\tfrac{1}{2}\delta\theta^2 + \text{higher powers of } \delta\theta).$$

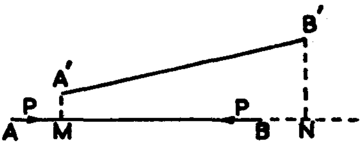

Fig. 237

Thus the work is zero to the first order of small quantities.

(2) *The tension in an inextensible string*

The string being taut any displacement of one end relative to the other can only be motion in a circle. If the string turns through a small angle $\delta\theta$ (Fig. 238) the displacement along the line of the string is $l(1 - \cos\delta\theta)$, where l is the length of the string, and this is of the second order of small quantities. If both ends of the string move in the same direction the work done is zero.

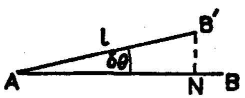

Fig. 238

(3) *The normal reaction to a smooth surface*

If the body moves on the surface the reaction will be normal to the surface and the displacement of the point where it acts can only be in a perpendicular direction.

(4) *The reaction at the point of contact of a body rolling on a plane surface*

Let a be the radius of curvature of the rolling body at the point of contact and let the body turn through an angle $\delta\theta$ (Fig. 239). The displacement of the centre of curvature is $a\delta\theta$ parallel to the surface. The relative displacement of the point of contact is $-a\sin\delta\theta$ parallel to the surface and $a(1 - \cos\delta\theta)$ perpendicular to it. Therefore the point of contact has displacements

$$a(\delta\theta - \sin\delta\theta) \text{ and } a(1 - \cos\delta\theta)$$

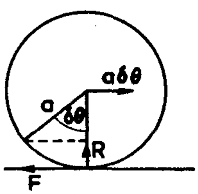

Fig. 239

parallel and perpendicular to the surface. These quantities are of the third and second order of small quantities respectively and hence to the first order no work is done by the forces at the point of contact.

(5) *The reaction at the point of contact of two bodies rolling on each other*

The motion of each body is initially one of rolling on the common tangent plane at the point of contact and hence the work done in a small displacement of either body is of the second order of small quantities.

9.22 The Principle of Virtual Work

If a body or system of bodies is in equilibrium and is given an arbitrary small displacement, the work done by the external forces acting on the system in this displacement is zero.

Such a displacement is called a *virtual* or imagined displacement. By equating to zero the work done in such displacements, relations between the external forces holding the system in equilibrium are obtained.

First, consider a particle at a point whose coordinates are (x, y, z) acted on by a force with components (X, Y, Z) parallel to the axes. In a displacement of the particle in which x increases by δx, y by δy and z by δz, the forces X, Y and Z will also increase infinitesimally, but the work done in the displacement is to the first order of small quantities $X\delta x + Y\delta y + Z\delta z$. If the particle is in equilibrium $X = Y = Z = 0$, and hence the work done in the virtual displacement is zero.

Now consider a rigid body in equilibrium. The work done on each of the particles of the body by the forces, both internal and external, which act on it is zero in a virtual displacement of the body, therefore the work done on all the particles of the body in this displacement is zero. But the work done by the internal forces of a rigid body in any displacement is zero, therefore the work done by the *external* forces acting on the rigid body in any virtual displacement is zero.

When a system contains more than one body the mutual reactions between separate bodies may be considered as internal forces and the work done by them in a virtual displacement will be zero provided that the displacement does not alter the nature of these forces, that is, it must not be such that it makes a reaction vanish, or makes friction change its direction or cease to act. Within these limitations the work done in any virtual displacement of a system of bodies in equilibrium will be zero.

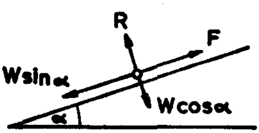

Fig. 240

For example, consider a particle of weight W at rest on a rough inclined plane of inclination α (Fig. 240). The forces acting on the particle are $W \sin \alpha$ and the friction F parallel to the plane, $W \cos \alpha$ and the reaction R perpendicular to the plane. Consider a virtual displacement δx down the plane. The virtual work is

$$(W \sin \alpha - F)\delta x,$$

and, since this is zero, we have $F = W \sin \alpha$. Consider a virtual displacement δy *into* the plane. The virtual work is

$$(W \cos \alpha - R)\delta y,$$

and, since this is zero, we have $R = W \cos \alpha$.

These equations can, of course, be obtained in the usual way by resolving the forces.

If, however, we consider a displacement δy in the opposite direction *out of* the plane, the reaction R would cease to act and the work done $(-W \cos \alpha)\delta y$ would not be zero.

The principle of virtual work is used to find the forces which hold a body or system of bodies in equilibrium by equating to zero the work done in different virtual displacements. No information is obtained which cannot also be obtained by using the ordinary conditions of equilibrium, but there is a considerable simplification due to the fact that those forces which do no work need not be considered.

It was pointed out in § 9.20 that the work done by the effort in a frictionless machine is equal to the work done against the resistance. This follows from the fact that when the effort is just sufficient to balance the resistance the system is in equilibrium. In a virtual displacement δx of the resistance the effort moves a distance $\varrho\delta x$ where ϱ is the velocity ratio and the work done

$$P\varrho\delta x - W\delta x = 0.$$

Thus the mechanical advantage is equal to the velocity ratio. If there is friction in the machine the work done by the effort will balance the work done against the resistance and friction, and the mechanical advantage will be less than the velocity ratio.

Example 13. *A uniform rod rests with its ends on two smooth planes inclined to the horizontal at angles α and β. If the planes intersect in a horizontal line and the rod is in equilibrium, find the inclination of the rod to the horizontal.*

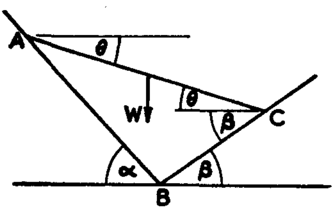

Fig. 241

Let AC (Fig. 241) be the rod, inclined at an angle θ to the horizontal. The rod may be assumed to be in a vertical plane containing lines of greatest slope of either plane, otherwise equilibrium would be impossible.

If the rod is given a small angular displacement in this plane, its ends remaining in contact with the plane, the virtual work will be zero. The only force doing work in such a displacement is the force of gravity.

Let y be the height of the centre of gravity of the rod above B and $2a$ the length of the rod. Then

$$y = AB \sin \alpha - a \sin \theta.$$

Also
$$\frac{AB}{\sin (\beta + \theta)} = \frac{2a}{\sin (\alpha + \beta)},$$

therefore
$$y = \frac{2a \sin \alpha \sin (\beta + \theta)}{\sin (\alpha + \beta)} - a \sin \theta.$$

The virtual work

$$= W\delta y$$

$$= W\left\{\frac{2a \sin \alpha \cos (\beta + \theta)}{\sin (\alpha + \beta)} - a \cos \theta\right\}\delta\theta$$

$$= \frac{Wa}{\sin (\alpha + \beta)}\{2 \sin \alpha \cos \beta \cos \theta - 2 \sin \alpha \sin \beta \sin \theta - \sin (\alpha + \beta) \cos \theta\}\delta\theta$$

$$= \frac{Wa}{\sin (\alpha + \beta)}\{\sin (\alpha - \beta) \cos \theta - 2 \sin \alpha \sin \beta \sin \theta\}\delta\theta.$$

The virtual work must be zero, therefore

$$\tan \theta = \frac{\sin (\alpha - \beta)}{2 \sin \alpha \sin \beta}.$$

Example 14. *Two uniform rods AB and BC of equal lengths and of weights W and W' respectively are freely jointed at B. They rest in a vertical plane with the ends A and C joined by a light inextensible string which is taut. If the angle ABC is 2α find the tension in the string.*

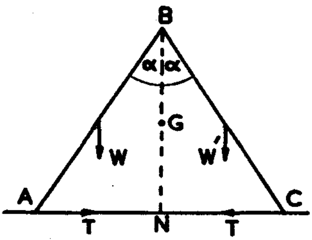

Fig. 242

We shall consider a virtual displacement in which A and C (Fig. 242) move farther apart. In this displacement the only forces which do work are the weights of the rods and the tension in the string. We can describe this displacement as due to a small increase $\delta\alpha$ of α.

If $AB = 2a$, $AC = 4a \sin \alpha$.

The height of the centre of gravity of the rods above the ground is $h = a \cos \alpha$. Corresponding to an increase $\delta\alpha$ of α, the increase of AC is $\delta(4a \sin \alpha) = 4a \cos \alpha . \delta\alpha$, the increase of h is $\delta h = -a \sin \alpha . \delta\alpha$. The work done in this displacement is

$$(-T)4a \cos \alpha . \delta\alpha - (W + W')(-a \sin \alpha . \delta\alpha).$$

Since this is zero we have

$$T = \tfrac{1}{4}(W + W') \tan \alpha.$$

Example 15. *Four equal uniform rods each of length $2a$ and weight W are freely jointed at their ends to form a rhombus $ABCD$. The rhombus is suspended from the corner A and is kept in the form of a square by a light rod joining the mid-points of the rods BC and CD. Find the thrust in this rod.*

If the angle DAB is 2α (Fig. 243) the centre of gravity of the rods is at a depth $2a \cos \alpha$ below A and the length of the rod EF is $2a \sin \alpha$.

Consider a virtual displacement in which α increases by $\delta\alpha$. The only forces doing work are the weights of the rods and the thrust T in the rod EF.

The virtual work

$$= 4W\delta(2a \cos \alpha) + T\delta(2a \sin \alpha)$$

$$= -4W . 2a \sin \alpha . \delta\alpha + T . 2a \cos \alpha . \delta\alpha.$$

Since this is zero $\quad T = 4W \tan \alpha,$

$$= 4W, \text{ when } \alpha = 45°.$$

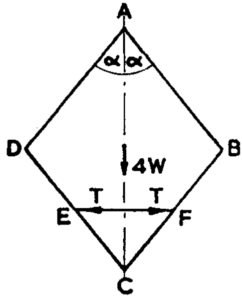

Fig. 243

Example 16. *The symmetrical framework ABCDE (Fig. 244) consists of four uniform rods each of weight w per unit length. AC and BD are each of length 2l, and DE and EC are each of length l, while the angle OAB is 30°, and O is the mid-point of AC. The rods are pin-jointed at O, D, E and C, and the frame rests in a vertical plane, A and B being connected by a light string and resting on a smooth horizontal floor. If each of the four joints is capable of exerting a frictional torque of magnitude* $\frac{1}{2}wl^2$ *show, by using the principle of virtual work, that the least value of the tension in the string is* $\frac{1}{2}wl(9\sqrt{3} - 8)$. (L.U., Pt. II)

Let the angle OAB be θ.
We have a weight $4wl$ at O at height $l \sin \theta$ and $2wl$ at height $(5/2)l \sin \theta$ above AB and the centre of gravity is at height $(3/2)l \sin \theta$.
The length of AB is $2l \cos \theta$.
As θ increases by $\delta\theta$ the angle between each pair of rods increases by $2\delta\theta$ and the work done by friction at the four joints is $4(\frac{1}{2}wl^2)2\delta\theta$. If T be the tension in the string the total virtual work is

$$-6wl.\delta\left(\frac{3}{2}l \sin \theta\right) - T.\delta(2l \cos \theta) + 4\left(\frac{1}{2}wl^2\right)2\delta\theta = 0,$$

that is
$$-9wl \cos \theta + 2Tl \sin \theta + 4wl^2 = 0,$$
$$T = wl\frac{9 \cos \theta - 4}{2 \sin \theta}$$
$$= \frac{1}{2}wl(9\sqrt{3} - 8).$$

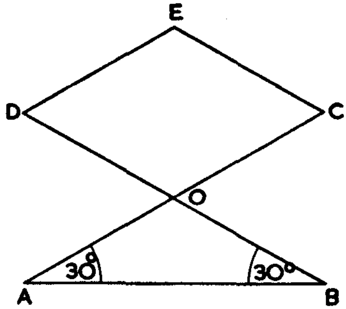

Fig. 244

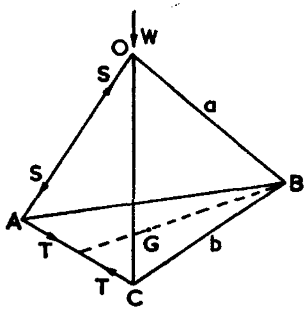

Fig. 245

Example 17. *Three light rods OA, OB, OC each of length a are freely jointed at O and rest with the ends A, B and C on smooth horizontal ground joined by three light inextensible strings each of length b (Fig. 245). A weight W is suspended from the point O. Find the tension in the strings and the thrust in the rods.*

To find the tension in the strings we use a virtual displacement which increases the length of the strings. The ends A, B, C are the vertices of an equilateral triangle and O will be vertically above the centroid G of the triangle.

We have
$$GB = \frac{2}{3}.b \cos 30° = \frac{b}{\sqrt{3}},$$
$$OG = \sqrt{(a^2 - \frac{1}{3}b^2)}.$$

If b increases by δb, the corresponding displacement of O is

$$\delta(OG) = \frac{-b}{3\sqrt{(a^2 - \frac{1}{3}b^2)}}\delta b.$$

The virtual work is
$$-W\delta(OG) - 3T\delta b,$$
$$= \frac{\frac{1}{3}Wb}{\sqrt{(a^2 - \frac{1}{3}b^2)}}\delta b - 3T\delta b = 0.$$

Therefore
$$T = \frac{\frac{1}{9}Wb}{\sqrt{(a^2 - \frac{1}{3}b^2)}}.$$

To find the thrust S in a rod we use a virtual displacement in which a increases by δa.

Then
$$\delta(OG) = \frac{a\delta a}{\sqrt{(a^2 - \frac{1}{3}b^2)}}.$$

The virtual work is
$$-W\delta(OG) + 3S.\delta a,$$
$$= -\frac{Wa}{\sqrt{(a^2 - \frac{1}{3}b^2)}}.\delta a + 3S.\delta a = 0.$$

Therefore
$$S = \frac{\frac{1}{3}Wa}{\sqrt{(a^2 - \frac{1}{3}b^2)}}.$$

EXERCISES 9 (*f*)

1. Find the work done in stretching an elastic string from its natural length l to the length $l + x$, the modulus of elasticity (in gravitational units) being λ. A thin elastic ring of weight W has a radius a when unstretched. The ring is placed horizontally over a smooth cone of vertical angle 90° which is fixed with its axis vertical and vertex upwards. The ring is allowed to slide slowly down the cone. Find the depth below the vertex at which the ring will rest in equilibrium. (L.U., Pt. I)

2. A light straight rigid rod $ABCD$, where $2AB = 2BC = CD$, is hung up by three equal light parallel elastic strings attached to A, C and D so that $ABCD$ is horizontal. If a weight W is then fixed to the rod at B, prove that the potential energy of the system is
$$V = \frac{1}{8}\frac{\lambda}{a}(5x^2 + 5z^2 + 2xz) - \frac{1}{4}W(3x + z),$$
where x and z are the small increases in length of the strings at A and D respectively, λ is the modulus and a the natural length of each of the strings. Hence show that in equilibrium the tensions in the strings are $7W/12$, $W/3$, $W/12$. (L.U., Pt. I)

3. Four equal uniform rods, each of weight W, are freely jointed to form a rhombus $ABCD$. The rhombus rests with A on a horizontal plane and AC vertical being kept in shape by a light string joining B and D. Show that the tension in the string is $2W \tan BAC$. If the string joined the mid-points of BC and CD show that the tension would be doubled.

4. Four equal uniform rods, each of weight W, are freely jointed to form a rhombus $ABCD$. The rhombus is suspended from A and kept in

shape by a string joining A and C. Show that the tension in the string is $2W$.

5. Six equal uniform rods, each of weight W, are freely jointed to form a hexagon $ABCDEF$. The hexagon rests in a vertical plane with AB fixed in a horizontal position, and is kept in the form of a regular hexagon by a light string joining C and F. Find the tension in the string.

6. A rod whose centre of gravity divides it into lengths a and b rests with its ends on two smooth planes each inclined to the horizontal at an angle α. If the planes intersect in a horizontal line show that the inclination of the rod to the horizontal is

$$\tan^{-1}\{(b - a)\cot \alpha/(a + b)\}.$$

7. A uniform ladder of weight W and length $2a$ rests in a vertical plane with one end on horizontal ground and the other against a smooth vertical wall and is inclined at an angle α to the vertical. Use the principle of virtual work to show that the coefficient of friction at the ground must be at least $\frac{1}{2}\tan\alpha$. If the coefficient of friction μ is less than this show that the couple required to hold the ladder in position is $Wa(\sin\alpha - 2\mu\cos\alpha)$.

8. A triangle ABC made of light rods smoothly jointed rests in a vertical plane on two smooth pegs at the same level at a distance $2d$ apart. A weight W is suspended from A which is below the pegs. If $AB = AC = b$ and the angle $BAC = 2\alpha$, show that the stress in the rod BC is $Wd/(b\sin\alpha\sin 2\alpha)$.

9. Four equal uniform heavy rods each of weight W are freely pivoted at their ends to form a closed frame. The frame hangs in the form of a square with one diagonal vertical, supported by two smooth pegs at the same level in contact with the two upper rods. Find the reactions at the corners of the frame and show that the distance between the pegs must be $\frac{1}{2}\sqrt{2}l$, where l is the length of one rod. (L.U., Pt. II)

10. The two parts of a step ladder are of equal length and uniform and are of weights W_1 and W_2 respectively. The ladder rests on smooth horizontal ground with a weight W placed on top of it and the mid-points of the parts joined by a light string of length $2d$. If the top is at a height h above the ground show that the tension in the string is $(2W + W_1 + W_2)d/h$.

11. A quadrilateral $ABCD$ of four uniform smoothly jointed rods is freely suspended from A and a light string joins A to C so that the angle ABC is a right-angle. If AB and AD are each of length a, BC and CD each of length b and all the rods are of weight w per unit length find the tension in the string.

12. A uniform circular disc of weight W and radius a is held in a horizontal plane by four vertical strings each of length l attached to points at the ends of two perpendicular diameters. Find the couple required to hold the disc in position when it has been turned through an angle θ from its equilibrium position.

13. A mechanism for raising and lowering a platform is shown in Fig. 246. The bars AB and CD are equal and hinged together at their middle points. AB is hinged to a point on the ground at A and pressed against the under side of the platform at B: CD is hinged to one end of the platform at D and rests on a horizontal surface at C. The co-efficient of friction between the surfaces in contact at B and C is 0·25 and the centre of gravity of the platform is at a horizontal distance from D equal to half the length of AB or CD. The platform is raised by means of a horizontal force applied at C. The weight of the bars and the friction at the hinges may be ignored.

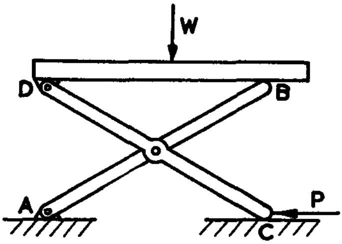

Fig. 246

Determine the instantaneous efficiency of the mechanism when the platform is rising and the bars are at an angle of 45° to the horizontal.

14. Four equal particles A, B, C, D each of weight W are connected by light strings AB, BC, CD, DA each of length l and then placed at rest symmetrically on a smooth sphere of radius $R(l < \pi R/2)$. Prove by virtual work or otherwise that the tension in each string is $\frac{1}{2}W \sin (l/R) \sec (l/R)$. (L.U., Pt. II)

15. A tetrahedron $OABC$ formed of 6 equal uniform rods each of weight W and freely jointed at their ends is suspended from the point O. Find the stresses in the rods OA and AB.

16. A square $ABCD$ of equal smoothly jointed light rods each of length a lies on a smooth horizontal table; OA, OB, OC, OD are four equal light rigid rods, each of length $2a$, smooth jointed at O, A, B, C, D, so that all the rods form the edges of a pyramid. A weight W is suspended in equilibrium at O. By the method of virtual work, or otherwise, find the stress in a rod such as OA, and a rod such as AB. (L.U., Pt. II)

9.23 Stability of Equilibrium

When a body is in equilibrium, the equilibrium may be stable or unstable or neutral. It is *stable* if when the body is given any small displacement consistent with the constraints it tends to return to the equilibrium position. It is *unstable* if it tends to move away from the equilibrium position and *neutral* if it rests in equilibrium in the displaced position.

Thus a cone resting on its base is stable and balanced on its vertex is unstable; a sphere resting on a plane is in neutral equilibrium.

The displacements considered are assumed to be infinitesimal. We have seen that in a small displacement from an equilibrium position the work done is zero to the first order of small quantities and thus a

position of equilibrium is a turning value of the work function and of the potential energy if the forces are conservative. The stability of the position of equilibrium depends on the second-order quantities in the expression for the work done.

9.24 Energy Test for Stability

The energy test for stability provides a simple method of finding positions of equilibrium and discussing their stability when the forces acting on a body or system of bodies are conservative. In this case an expression may be found for the potential energy V of the system in terms of the coordinates which determine its position. A position of equilibrium is then a turning value for the potential energy since the work done, which is the change of potential energy, in a small displacement is zero.

The sum of the kinetic and potential energies of a system is constant, therefore, if a system begins to move from a displaced position it must be gaining kinetic energy and losing potential energy.

If a position gives a maximum of the potential energy and the system is given a small displacement it will move in such a way that the potential energy decreases, that is, away from the position of equilibrium and the position is unstable.

If the position gives a minimum of the potential energy and the system is given a small displacement it will move in such a way that the potential energy decreases, that is, back towards the position of equilibrium.

Therefore the potential energy of a system is a maximum in positions of unstable equilibrium and a minimum in positions of stable equilibrium. This may be remembered from Fig. 247, showing a sphere on a corrugated surface. At positions such as A and B the potential energy is a maximum and equilibrium is unstable; at C and E the potential energy is a minimum and there is stability. At D the potential energy has a point of inflexion, a displacement towards E will cause instability and, therefore, the position must be considered unstable.

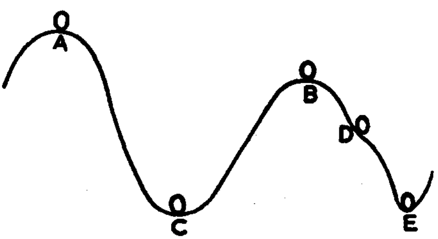

Fig. 247

It is shown in § 6.3 that when a system has one degree of freedom, so that the potential energy V is a function of one coordinate θ, the system will oscillate about the position of equilibrium if $\dfrac{d^2V}{d\theta^2} > 0$,

which is the condition for a minimum, and the period of the small oscillations is

$$2\pi\left\{2f(\theta) \div \frac{d^2V}{d\theta^2}\right\}^{1/2},$$

where $f(\theta)\dot{\theta}^2$ is the kinetic energy of the system.

9.25 Examples

The solution of problems involving the energy test of stability involves finding an expression for the potential energy in any position. When this has been done it is an elementary exercise in maxima and minima to discuss stability. It should be remembered that if the second differential coefficient of the potential energy vanishes a maximum or minimum may occur if the third differential coefficient also vanishes and the fourth is negative or positive. We shall consider problems in which there is only one degree of freedom. When there are two or more degrees of freedom the potential energy is a function of two or more coordinates and the problem is one of finding maxima or minima values of a function of two or more variables.

Example 18. *A cylinder of radius r has its centre of gravity at a distance h from the axis. Show that the inclination α to the horizontal of the steepest rough plane on which it can rest with its axis horizontal is given by $h = r \sin \alpha$. Show that for planes less steep there is a position of stable equilibrium.*

(L.U., Pt. I)

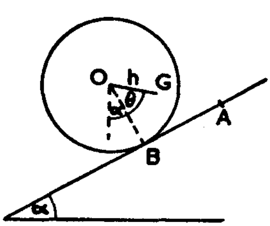

Fig. 248

To find the potential energy we need the height of the centre of gravity above some standard position. Let A (Fig. 248) be this position, assuming that when the point of contact of the cylinder with the plane is at A the radius OG passes through A. Let B be the point of contact when the cylinder has turned through an angle θ so that the angle $BOG = \theta$.

Since the cylinder has rolled from A to B, $AB = r\theta$. The height of G above A is the sum of the projections of AB, BO and OG on the vertical, that is

$$-r\theta \sin \alpha + r \cos \alpha - h \cos (\alpha + \theta).$$

Therefore the potential energy at B is, m being the mass of the cylinder,

$$V = mg\{-r\theta \sin \alpha + r \cos \alpha - h \cos (\alpha + \theta)\}.$$

We have $$\frac{dV}{d\theta} = mg\{-r \sin \alpha + h \sin (\alpha + \theta)\}.$$

For equilibrium we must have

$$\sin (\alpha + \theta) = \frac{r}{h} \sin \alpha,$$

and this determines a position of equilibrium if and only if $r \sin \alpha \leqslant h$. The steepest slope on which equilibrium is possible is therefore given by $r \sin \alpha = h$, and then $\alpha + \theta = \frac{\pi}{2}$.

If $r \sin \alpha < h$, $\qquad \dfrac{d^2V}{d\theta^2} = mgh \cos (\alpha + \theta).$

Now since $\sin (\alpha + \theta) = \dfrac{r \sin \alpha}{h}$, there are two values of $\alpha + \theta$ for which equilibrium is possible, one value being the supplement of the other. Taking the value of $\alpha + \theta$ which is less than $\frac{1}{2}\pi$, $\cos (\alpha + \theta)$ is positive, and the potential energy has a minimum, therefore the position is stable. If $\alpha + \theta > \frac{1}{2}\pi$ the position is unstable.

Example 19. *A uniform plank of thickness* $2d$ *rests in equilibrium on a fixed rough cylinder of radius* a, *the plank being horizontal and perpendicular to the axis of the cylinder. Discuss the stability of the system.*

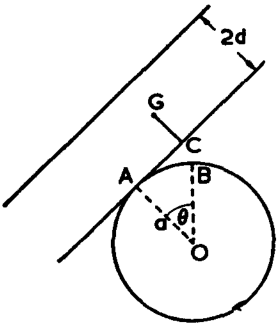

Fig. 249

If the plank has rolled through an angle θ from the horizontal position so that the point of contact has moved from C to A (Fig. 249) we have $AC = a\theta$. If O be the centre of the cylinder and G the centre of gravity of the plank the height of G above O is the sum of the projections of OA, AC and CG on the vertical, that is $(a + d) \cos \theta + a\theta \sin \theta$.

Therefore
$$V = mg\{(a + d) \cos \theta + a\theta \sin \theta\},$$
$$\frac{dV}{d\theta} = mg\{a\theta \cos \theta - d \sin \theta\},$$
$$\frac{d^2V}{d\theta^2} = mg\{-a\theta \sin \theta + (a - d) \cos \theta\}.$$

When $\theta = 0$,
$$\frac{dV}{d\theta} = 0,$$
$$\frac{d^2V}{d\theta^2} = mg(a - d).$$

Hence, the horizontal position is stable if $a > d$.

If $a = d$,
$$\frac{d^2V}{d\theta^2} = 0 \text{ for } \theta = 0,$$
$$\frac{d^3V}{d\theta^3} = mga\{-\sin \theta - \theta \cos \theta\},$$
$$\frac{d^4V}{d\theta^4} = mga\{-2 \cos \theta + \theta \sin \theta\}.$$

For $\theta = 0$, $\dfrac{d^3V}{d\theta^3} = 0$, $\dfrac{d^4V}{d\theta^4}$ is negative and the position is unstable.

There is also a position of equilibrium obtained by equating $\dfrac{dV}{d\theta}$ to zero, that is
$$a\theta \cos \theta = d \sin \theta,$$
$$\frac{\tan \theta}{\theta} = \frac{a}{d}.$$

This gives a position of equilibrium only if $a > d$, since $(\tan \theta)/\theta > 1$. This position will be unstable since maxima and minima of a continuous function separate each other.

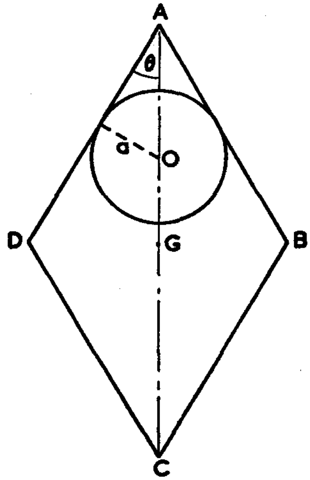

Fig. 250

Example 20. *ABCD is a rhombus formed by four equal thin uniform smoothly jointed rods each of length $4\sqrt{(3)}a$. The system hangs over a smooth circular cylinder of radius a and centre O whose axis is horizontal. Find the angle BAO in the position of equilibrium in which AOC is a vertical straight line and show that it is a stable position for displacements in which A and C move vertically.*

(L.U., Pt. II)

Let m be the mass of the rods and let the angle $BAO = \theta$ (Fig. 250). Then $AO = a \operatorname{cosec} \theta$.

The centre of gravity G of the rods is at a depth $4\sqrt{3}a \cos\theta$ below A, that is $4\sqrt{3}a\cos\theta - a\operatorname{cosec}\theta$ below O.

Therefore

$$V = mga(\operatorname{cosec}\theta - 4\sqrt{3}\cos\theta).$$

$$\frac{dV}{d\theta} = mga(-\operatorname{cosec}\theta\cot\theta + 4\sqrt{3}\sin\theta),$$

$$\frac{d^2V}{d\theta^2} = mga(\operatorname{cosec}\theta\cot^2\theta + \operatorname{cosec}^3\theta + 4\sqrt{3}\cos\theta).$$

When $\frac{dV}{d\theta} = 0$, we have $\cos\theta = 4\sqrt{3}\sin^3\theta,$

and this is satisfied by the value $\theta = 30°$. For this value of θ, $\frac{d^2V}{d\theta^2}$ is positive and the equilibrium is stable.

Example 21. *A uniform rod AB of length 2a and weight W is freely hinged to a fixed point at A. A light inextensible string attached to B passes over a smooth peg C at a height 2b (< 2a) vertically above A and carries a weight nW hanging freely. Find the condition that there should be a position of equilibrium with the rod inclined to the vertical and discuss its stability.*

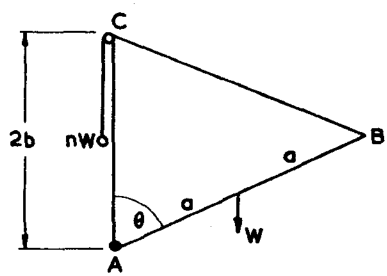

Fig. 251

If the angle $BAC = \theta$ (Fig. 251)

$BC = 2\{a^2 + b^2 - 2ab\cos\theta\}^{1/2}$.

If l be the length of the string the heights of the weights above A are $a\cos\theta$ and $2b - (l - BC)$.

Omitting the constant terms we have

$$V = Wa\cos\theta + 2nW\{a^2 + b^2 - 2ab\cos\theta\}^{1/2},$$

$$\frac{dV}{d\theta} = -Wa\sin\theta + 2nW\frac{ab\sin\theta}{(a^2 + b^2 - 2ab\cos\theta)^{1/2}}.$$

For equilibrium with $\theta \neq 0$, we have

$$(a^2 + b^2 - 2ab\cos\theta)^{1/2} = 2nb,$$

$$\cos\theta = \frac{a^2 + b^2 - 4n^2b^2}{2ab}.$$

This is a possible position if

$$-1 < \frac{a^2 + b^2 - 4n^2b^2}{2ab} < 1,$$

that is

$$(a - b)^2 < 4n^2b^2 < (a + b)^2,$$
$$a - b < 2nb < a + b,$$
$$(2n - 1)b < a < (2n + 1)b.$$

If this condition is satisfied

$$\frac{d^2V}{d\theta^2} = -Wa\cos\theta + \frac{2nWab\cos\theta}{(a^2 + b^2 - 2ab\cos\theta)^{1/2}} - \frac{2nWa^2b^2\sin^2\theta}{(a^2 + b^2 - 2ab\cos\theta)^{3/2}}.$$

Putting the surd equal to $2nb$ we have

$$\frac{d^2V}{d\theta^2} = -\frac{Wa^2\sin^2\theta}{4n^2b},$$

and since this is negative the position is unstable.

9.26 Stability of Rocking Stones

If a body rests in equilibrium on another fixed rough body, the portions of the two bodies near the point of contact being spherical with radii r and R respectively and the centre of gravity of the upper body being distant h from the point of contact, the equilibrium is stable if $\dfrac{1}{h} > \dfrac{1}{R} + \dfrac{1}{r}$.

Let the radius to the point of contact of the lower body be inclined at θ to the vertical and let the upper body have turned through an angle $\theta + \phi$ (Fig. 252).

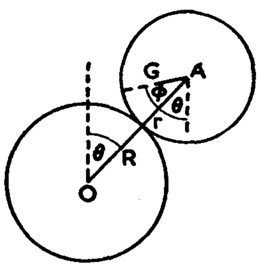

Fig. 252

Then

$$r\phi = R\theta,$$

$$\theta + \phi = \frac{R + r}{r}\theta.$$

The potential energy is proportional to the height of the centre of gravity of the rolling body above the centre of curvature of the fixed body. If m be its mass we have

$$V = mg\left\{(R + r)\cos\theta - (r - h)\cos\frac{R + r}{r}\theta\right\},$$

$$\frac{dV}{d\theta} = (R + r)mg\left\{-\sin\theta + \frac{r - h}{r}\sin\frac{R + r}{r}\theta\right\},$$

$$\frac{d^2V}{d\theta^2} = (R + r)mg\left\{-\cos\theta + \frac{(r - h)(r + R)}{r^2}\cos\frac{R + r}{r}\theta\right\}.$$

The position $\theta = 0$ is stable if

$$\frac{(r-h)(r+R)}{r^2} > 1,$$

that is

$$rR - rh - Rh > 0,$$

$$\frac{1}{h} > \frac{1}{R} + \frac{1}{r}.$$

If $\frac{1}{h} = \frac{1}{R} + \frac{1}{r}$, it is easily verified that for $\theta = 0$, $\frac{d^2V}{d\theta^2} = \frac{d^3V}{d\theta^3} = 0$,

$$\frac{d^4V}{d\theta^4} = -mg\frac{R(R+r)(R+2r)}{r^2},$$

and hence the equilibrium is unstable.

EXERCISES 9 (*g*)

1. Two equal particles are joined by a light inextensible string and rest in equilibrium hung over a smooth fixed cylinder whose axis is horizontal, the length of the string being less than half the circumference of the cylinder. Prove that the equilibrium is unstable.

2. A uniform rectangular block rests in equilibrium with one face in contact with the highest point of a fixed rough sphere of radius R. Show, from first principles, that the equilibrium will be stable for rolling displacements if the height h of the centre of gravity of the block above thc point of contact is less than R.

   Examine the stability of the equilibrium if h is equal to R. (L.U., Pt. II)

3. A straight rod AB slides by means of small smooth rings at the ends, A on a fixed wire OX inclined upwards to the horizontal at angle α, and B on a fixed wire OY inclined upwards to the horizontal at angle $\pi/2 - \alpha$, the whole system being in a vertical plane, with A and B on opposite sides of the vertical through O. The centre of gravity G of the rod is such that $AG = a$ and $BG = b$, where $a < b$. Prove that as the rod moves, G describes an ellipse. Show that in the position of equilibrium the rod is inclined at an angle θ to the vertical where

$$\cot\theta = \frac{a\cot\alpha - b\tan\alpha}{a+b},$$

   and examine the stability of this position. (L.U., Pt. I)

4. A uniform beam rests with its ends on two smooth planes inclined to the horizontal at angles α and β respectively and intersecting in a horizontal line. Find the inclination of the beam to the horizontal and show that the equilibrium is unstable.

5. A uniform rod AB, of length $2a$ and weight $2W$, can turn freely about A which is fixed. An inelastic string attached to B passes over a small pulley at C at a distance b ($> 2a$) vertically above A and carries a

weight W at its free end. Obtain the potential energy of the system in terms of θ, the angle AB makes with the upward vertical. Show that there are three distinct positions of equilibrium and examine their stability. (L.U., Pt. II)

6. A light uniform flexible inextensible string, of length h, is attached at its ends to the ends of a thin uniform rod of length $2a$ ($h > 2a$) and the string is supported by a smooth small peg. Show that the position of equilibrium in which the rod is horizontal is unstable. (L.U., Pt. II)

7. A uniform rod AB, of mass m and length $2a$, can turn freely about a horizontal axis at A. An elastic string of unstretched length a and modulus λ is attached to B and to a point C vertically over A, where $AC = 2a$. If $3\lambda > mg$, prove that there is a stable position of equilibrium given by

$$\sin\frac{\alpha}{2} = \frac{\lambda}{4\lambda - mg},$$

where α is the angle the rod makes with the upward vertical. (L.U., Pt. II)

8. A uniform rod AB of mass M and length a is freely hinged at A to a fixed point. It is supported by a spring attached to B and to a point C which is at a distance a vertically above A. When the rod is horizontal the spring is just unstretched and its strength is such that the weight of the mass would elongate it a length e. Show that there is a position of stable equilibrium when ϕ, the inclination of AB to the vertical, is given by $\cos\frac{1}{2}\phi = a\sqrt{2}/(2a - e)$.

9. A uniform rod, of weight $2w$ and of length $2\lambda a$, is free to turn about one end, and is supported horizontally in equilibrium by a vertical string attached to the other end. The string passes over a small smooth peg, at a height a above the end of the rod to which the string is attached, and carries a weight w at its other extremity. If the rod is given an upward angular displacement θ, show that, neglecting the fifth and higher powers of $\sin\theta$, the weight w descends a distance $2a\lambda\sin\theta - 8a\lambda^2\sin^4\frac{1}{2}\theta$. Hence, prove that the rod is in stable equilibrium. (L.U., Pt. II)

10. A uniform solid cone rests with its axis vertical in a smooth circular hole in a horizontal table. The radius of the hole is one-quarter of the radius of the base of the cone. Show that the equilibrium is stable if the semi-vertical angle of the cone is greater than $\tan^{-1}(1/\sqrt{(2)})$.

11. A uniform solid sphere rests inside a rough fixed hollow sphere of twice its radius. A weight is attached at the highest point of the smaller sphere in its equilibrium position. Show that the position of equilibrium is stable.

12. A uniform solid hemisphere of radius a rests with its plane face horizontal and its curved surface in contact with a fixed rough sphere of radius $2a$. Prove that the hemisphere can be rolled into a position of

equilibrium with the plane face inclined to the horizontal and find the angle between the common normal and the vertical in this position. Determine if this position is stable or unstable. (L.U., Pt. II)

13. A uniform hemisphere of radius a has a uniform cone of radius a and height h attached to its base. The hemisphere rests on a fixed rough sphere of radius $2a$. Show that equilibrium will be stable if
$$h/a < (\sqrt{7} - 2)/3.$$

14. A body rests in equilibrium on a fixed rough concave surface whose radius of curvature at the point of contact is R. The body has radius of curvature r at the point of contact and its centre of gravity is distant h from the point of contact. Show that the equilibrium is stable if $\dfrac{1}{h} > \dfrac{1}{r} - \dfrac{1}{R}$ and that if $\dfrac{1}{h} = \dfrac{1}{r} - \dfrac{1}{R}$ it is stable if $R > 2r$.

15. A uniform square lamina rests in equilibrium with its plane vertical and adjacent sides in contact with two smooth pegs, A and B, in the same horizontal line, one corner C being below AB. The diagonals of the square are of length $2a$ and $AB = 2b$, $(b < a)$. If the diagonal through C makes an angle θ with AB, prove that equilibrium is stable when $\theta = 90°$ and $4b > a$, and that two unstable positions of equilibrium then exist defined by $\sin\theta = a/4b$, provided $a > 2b\sqrt{2}$. (L.U.)

16. A light rigid wire framework in the form of a square $ABCD$ is free to rotate in a vertical plane about A, which is fixed. P is a point vertically above A at a distance equal to AC $(= a)$ from it and P is joined to C by an elastic string of natural length $\frac{1}{2}a\sqrt{3}$ and modulus $w\sqrt{3}$. If two particles, each of weight w, are fixed at B and D, find the potential energy of the system when AC makes an angle θ with the downward vertical. Show that there are positions of equilibrium when $\theta = 0$, and $\theta = 60°$ and determine their stability. (L.U.)

17. A uniform square plate of side $2a$ is placed with two adjoining sides resting on two smooth parallel rails which are distant $2b$ apart in the same horizontal plane, the plane of the plate being vertical and perpendicular to the rails. Express the potential energy of the plate for displacements in its plane in terms of the angle θ which a diagonal makes with the vertical. Hence, show that positions of equilibrium occur when $\theta = 0$ and $\cos\theta = \sqrt{2}a/4b$. Show that the second condition gives positions of unstable equilibrium. (C.U.)

CHAPTER 10

STATICS OF ROPES AND CHAINS

10.1 Friction of a Rope around a Post

Consider a rope AB in contact with a cylindrical surface (Fig. 253) the coefficient of friction being μ. Assuming that the pull T_1 at A is greater than the pull T_0 at B we shall call A the taut end and B the slack end of the rope. Let T be the tension in the rope at P and $T + \delta T$ the tension at Q, and consider the equilibrium of the portion PQ of the rope of length $r\delta\theta$ where r is the radius of the cylinder. The tensions at P and Q are along the tangents at P and Q which are inclined to each other at an angle $\delta\theta$. If R be the normal pressure per unit length the normal reaction is $Rr\,\delta\theta$ and the friction is $\mu Rr\,\delta\theta$. The directions of these forces are to the first order of small quantities perpendicular to and along the tangent at P respectively.

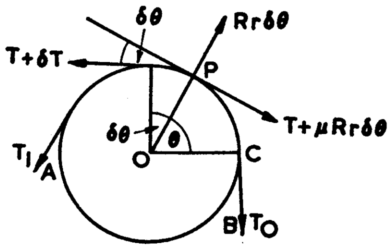

Fig. 253

We have, resolving along and perpendicular to the tangent at P,

$$T + \mu Rr\,\delta\theta = (T + \delta T)\cos\delta\theta,$$
$$Rr\,\delta\theta = (T + \delta T)\sin\delta\theta.$$

Neglecting second-order quantities we have

$$\delta T = \mu Rr\,\delta\theta,$$
$$T\delta\theta = Rr\,\delta\theta,$$

therefore

$$\delta T = \mu T\delta\theta.$$

In the limit

$$\frac{dT}{d\theta} = \mu T,$$

$$\int\frac{dT}{T} = \int\mu d\theta,$$

$$\log T = \mu\theta + \text{constant},$$

and hence

$$T = Ae^{\mu\theta}.$$

If θ is measured from C where the slack end leaves the cylinder,

$$T = T_0 e^{\mu\theta}.$$

If the post is not circular the equation $\dfrac{dT}{d\theta} = \mu T$ is obtained in the

same way taking r as the radius of curvature of the surface at P, and hence provided there are no sudden changes of curvature the equation $T = T_0e^{\mu\theta}$ follows as before.

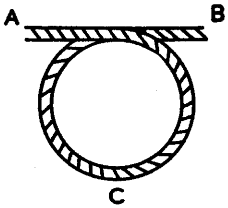

Fig. 254

Example 1. *A rope AB makes one complete turn round a circular post and the ends A and B are pulled in opposite directions with a force of* 1000 *lb. Find the least tension in the rope if the coefficient of friction between the rope and the post is* 0·4.

The least tension T_0 will obviously be at C (Fig. 254) and we have

$$1000 = T_0e^{\mu\pi},$$
$$T_0 = 1000e^{-0\cdot4\pi},$$
$$= 285 \text{ lb.}$$

10.2 Effect of the Weight of the Rope

Consider a rope or chain of weight w per unit length hanging over a horizontal cylinder of radius r so that lengths l hang freely at either side and let the coefficient of friction be μ. We shall find the force F which must be applied to one end of the chain to make it move. Let T be the tension at P (Fig. 255) where the angle $POA = \theta$ is measured from A where the slack end leaves the cylinder. The forces on the element PQ of length $r\delta\theta$ now include its weight $wr\delta\theta$ and we have

$$\delta T = \mu Rr\,\delta\theta + wr\cos\theta\,\delta\theta,$$
$$T\delta\theta = Rr\,\delta\theta - wr\sin\theta\,\delta\theta.$$

Eliminating R we obtain the differential equation

$$\frac{dT}{d\theta} - \mu T = wr(\cos\theta + \mu\sin\theta).$$

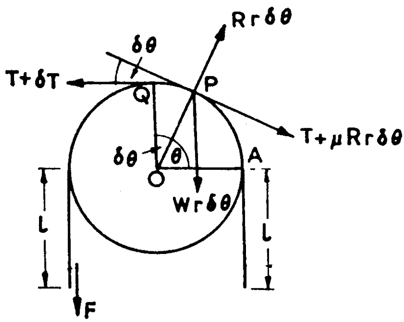

Fig. 255

A particular integral of this differential equation is

$$\frac{wr}{D-\mu}(\cos\theta + \mu\sin\theta) = \frac{wr}{D^2-\mu^2}(D+\mu)(\cos\theta + \mu\sin\theta)$$
$$= -\frac{wr}{1+\mu^2}(D+\mu)(\cos\theta + \mu\sin\theta)$$
$$= \frac{wr}{1+\mu^2}\{-2\mu\cos\theta + (1-\mu^2)\sin\theta\}.$$

The complete solution is

$$T = Ae^{\mu\theta} + \frac{wr}{1+\mu^2}\{-2\mu\cos\theta + (1-\mu^2)\sin\theta\}.$$

The arbitrary constant A is determined by the condition that $T = wl$ when $\theta = 0$, and we have

$$T = \left(wl + \frac{2\mu wr}{1+\mu^2}\right)e^{\mu\theta} + \frac{wr}{1+\mu^2}\{-2\mu\cos\theta + (1-\mu^2)\sin\theta\}.$$

The tension T_1 at B is found by putting $\theta = \pi$, and we have

$$T_1 = wle^{\mu\pi} + \frac{2\mu wr}{1+\mu^2}(e^{\mu\pi} + 1).$$

The force F at the taut end is therefore

$$F = wl(e^{\mu\pi} - 1) + \frac{2\mu wr}{1+\mu^2}(e^{\mu\pi} + 1).$$

10.3 Power Transmitted by a Belt Drive

Consider a belt running on a wheel of radius r in contact over a length $r\alpha$ of the circumference (Fig. 256). Let T_1 be the tension at the taut end and T_0 the tension at the slack end. On an element PQ of the belt of length $r\delta\theta$ we have the forces $T, T + \delta T$, $Rr\delta\theta$ and $\mu Rr\,\delta\theta$ as before. In addition we have the reversed effective force $\frac{(wr\delta\theta)}{g}.\frac{v^2}{r}$, where w is the mass per unit length of the belt and v is its velocity. We have on resolving

$$\delta T = \mu Rr\,\delta\theta,$$

$$T\delta\theta = Rr\,\delta\theta + \frac{wv^2}{g}\delta\theta.$$

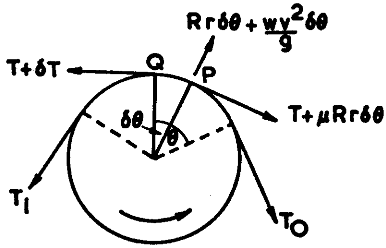

Fig. 256

Eliminating R we have the differential equation

$$\frac{dT}{d\theta} - \mu T = -\frac{\mu wv^2}{g}.$$

Hence $$T = Ae^{\mu\theta} + \frac{wv^2}{g}, \text{ where } A \text{ is a constant.}$$

Since $T = T_0$ when $\theta = 0$ we have

$$T = T_0 e^{\mu\theta} - \frac{wv^2}{g}(e^{\mu\theta} - 1),$$

and $$T_1 = T_0 e^{\mu\alpha} - \frac{wv^2}{g}(e^{\mu\alpha} - 1).$$

If the units are lb., ft. and seconds, the horse-power transmitted is

$$H = \frac{(T_1 - T_0)v}{550}.$$

Now $$T_1(1 - e^{\mu a}) = (T_0 - T_1)e^{\mu a} - \frac{wv^2}{g}(e^{\mu a} - 1),$$

and therefore $$H = \frac{(1 - e^{-\mu a})}{550}\left(T_1 v - \frac{wv^3}{g}\right).$$

For a given value of T_1 the horse-power transmitted is a maximum as v varies when

$$T_1 - \frac{3wv^2}{g} = 0,$$

that is $$v = \left(\frac{gT_1}{3w}\right)^{1/2}.$$

Example 2. *The ideal cable for a barrage balloon would taper so that the stress across every section had the same value. If w is the weight per unit volume of the material of the cable, f the constant stress, and r the radius at a distance x below the balloon, show that*

$$\frac{dr}{dx} = -\frac{wr}{2f}.$$

Find the formula for r in terms of w, f, and the tension T at $x = 0$, and show that the total weight p of the cable, of length L, is given by

$$p = T(1 - e^{-wL/f}). \qquad \text{(L.U., Pt. II)}$$

Let $r + \delta r$ be the radius at distance $x + \delta x$ below the balloon (Fig. 257). The tension at x below the balloon is $\pi r^2 f$, at $x + x\delta$ it is $\pi(r + \delta r)^2 f$. δr is negative and the second tension is less than the first by the weight of cable in between, which to the first order is $w\pi r^2 \delta x$. That is, to the first order of small quantities,

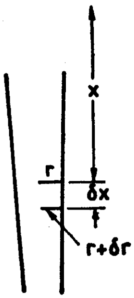

Fig. 257

$$\pi r^2 f - (\pi r^2 f + 2\pi r\,\delta r\,.f) = w\pi r^2\,\delta x,$$
$$-2f\delta r = wr\,\delta x,$$

so that $$\frac{dr}{dx} = -\frac{wr}{2f}.$$

Hence $$\int\frac{dr}{r} = -\int\frac{w}{2f}dx,$$

$$\log r = -\frac{wx}{2f} + \log r_0,$$

where r_0 is the radius when $x = 0$, $$r = r_0 e^{-wx/2f}.$$

We have $$T = \pi r_0^2 f,$$

therefore $$r = \left(\frac{T}{\pi f}\right)^{1/2} e^{-wx/2f}.$$

The total weight of the cable is

$$\int_0^L w\pi r^2 dx = \frac{wT}{f}\int_0^L e^{-wx/f}dx$$
$$= T(1 - e^{-wL/f}).$$

10.4 Chain of Heavy Particles

Consider the equilibrium of a light inextensible string with its ends attached to fixed points and with weights attached to the string at points along its length. Let there be n weights $W_1, W_2, \ldots, W_n$, and let the lengths into which the string is divided by the points of attachment of the weights be $a_1, a_2, \ldots, a_{n+1}$ (Fig. 258). Let $\theta_1, \theta_2, \ldots, \theta_{n+1}$ be the inclinations of the portions of the string to the horizontal and $T_1, T_2, \ldots, T_{n+1}$ the tensions in these portions. We shall suppose that the end Y is distant l horizontally and k vertically from the end X.

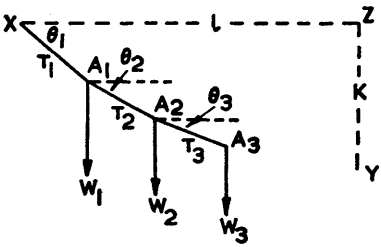

Fig. 258

For the equilibrium of each of the weights we have

$$T_1 \cos\theta_1 = T_2 \cos\theta_2 = T_3 \cos\theta_3 = \ldots = T_{n+1}\cos\theta_{n+1} = H \text{ (say)}, \quad (1)$$

$$\left.\begin{aligned} T_1 \sin\theta_1 - T_2 \sin\theta_2 &= W_1, \\ T_2 \sin\theta_2 - T_3 \sin\theta_3 &= W_2, \\ \ldots\ldots&\ldots\ldots \\ T_n \sin\theta_n - T_{n+1}\sin\theta_{n+1} &= W_n. \end{aligned}\right\} \quad (2)$$

We have, therefore, $2n$ equations involving the tensions and inclinations of the strings. We have in addition two geometrical equations, namely

$$a_1 \cos\theta_1 + a_2 \cos\theta_2 + \ldots + a_{n+1}\cos\theta_{n+1} = l, \quad (3)$$

$$a_1 \sin\theta_1 + a_2 \sin\theta_2 + \ldots + a_{n+1}\sin\theta_{n+1} = k. \quad (4)$$

We thus have in all $2n + 2$ equations to determine the $n + 1$ unknown tensions and the $n + 1$ unknown inclinations.

Dividing each of the equations (2) by H we have

$$\tan\theta_1 - \tan\theta_2 = \frac{W_1}{H},$$

$$\tan\theta_2 - \tan\theta_3 = \frac{W_2}{H},$$

$$\ldots\ldots\ldots$$

$$\tan\theta_n - \tan\theta_{n+1} = \frac{W_n}{H}.$$

Thus $$\tan\theta_r = \tan\theta_1 - \frac{W_1 + W_2 + \ldots + W_{r-1}}{H}.$$

Substitution for $\theta_2, \theta_3, \ldots, \theta_{n+1}$ in equations (3) and (4) gives two equations involving θ_1 and H. The solution of these equations in a particular case is possible but not easy.

If the inclinations to the horizontal θ_1 and θ_{n+1} of the end portions of the string are known, since the total weight is supported by these portions of the string, we can draw a triangle of forces $O\alpha_1 a_{n+1}$ (Fig. 259), the vertical representing the sum of the weights. We thus find the tensions in the end portions of the string and if the vertical $a_1 a_{n+1}$ is divided into lengths representing the individual weights at the points

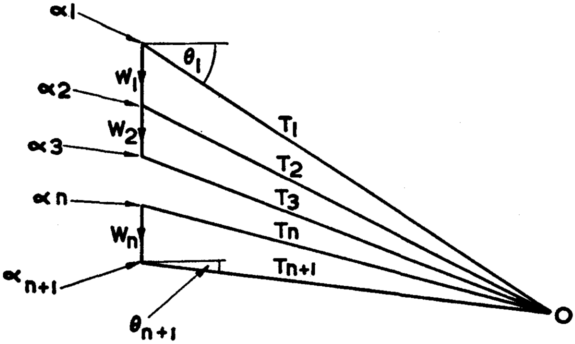

Fig. 259

a_2, a_3, etc., the lines Oa_2, Oa_3, etc., represent the tensions T_2, T_3, etc., in magnitude and direction.

It is evident that the point O can be found if the magnitudes or directions of any two of the tensions are known.

10.5 Weights Equally Spaced Horizontally

If the weights are equally spaced horizontally we have

$a_1 \cos\theta_1 = a_2 \cos\theta_2 = \ldots = a_{n+1}\cos\theta_{n+1} = h$ (say), and $(n+1)h = l$.

We have also
$$a_r \sin\theta_r = h\tan\theta_r,$$
and hence
$$\tan\theta_1 + \tan\theta_2 + \ldots + \tan\theta_{n+1} = \frac{k}{h}.$$
Now
$$\tan\theta_2 = \tan\theta_1 - \frac{W_1}{H},$$
$$\tan\theta_3 = \tan\theta_1 - \frac{W_1 + W_2}{H},$$
$$\cdots\cdots\cdots\cdots$$
therefore
$$(n+1)\tan\theta_1 = \frac{k}{h} + \frac{nW_1 + (n-1)W_2 + \ldots + W_n}{H}.$$
Thus if one length of string, a_1 for instance, is known,
$$\tan\theta_1 = \left(\frac{a_1^2}{h^2} - 1\right)^{1/2},$$
and the other unknowns are easily found.

We shall show that if the weights are all equal their points of attachment lie on a parabola.

Let $W_1 = W_2 = \ldots = W_n = W$, and let x be the horizontal and y the vertical displacement of the rth weight from the point X (Fig. 258).

Then
$$x = rh,$$
$$y = h(\tan\theta_1 + \tan\theta_2 + \ldots + \tan\theta_r).$$

Now since the weights are equal

$$\tan\theta_r = \tan\theta_1 - \frac{(r-1)W}{H},$$
$$= h\left[r\tan\theta_1 - \frac{W}{H}\{1 + 2 + \ldots + (r-1)\}\right]$$
$$= hr\tan\theta_1 - \frac{Wh}{H}\frac{r(r-1)}{2}$$
$$= x\tan\theta_1 - \frac{W}{2hH}x(x-h).$$

Thus x and y are connected by this equation for all values of r and this is the equation of a parabola.

In a suspension bridge the vertical tie rods carrying the roadway are of lengths such that the tensions in them are equal and hence the curve of the chain from which they are suspended is a parabola.

Example 3. *Five equal weights W are attached to a light inextensible string which hangs from two points A and B at the same level. The horizontal projections of the six intervals of string are each h and the depth of the lowest weight below AB is $3h$. Find the inclinations to the horizontal of the portions of the string and the greatest string tension. Find also the equation of the parabola on which the points of attachment lie.*

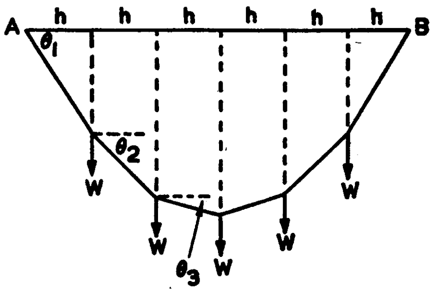

Fig. 260

If $\theta_1, \theta_2, \theta_3$ be the inclinations of the first three lengths of string (Fig. 260),

$$h(\tan\theta_1 + \tan\theta_2 + \tan\theta_3) = 3h,$$
$$\tan\theta_1 + \tan\theta_2 + \tan\theta_3 = 3.$$

If the corresponding tensions be T_1, T_2, T_3, we have

$$T_1 \cos\theta_1 = T_2 \cos\theta_2 = T_3 \cos\theta_3 = H \text{ (say)},$$
$$T_1 \sin\theta_1 - T_2 \sin\theta_2 = W,$$
$$T_2 \sin\theta_2 - T_3 \sin\theta_3 = W,$$
$$T_3 \sin\theta_3 = \frac{1}{2}W.$$

Hence,
$$\tan\theta_3 = \frac{W}{2H},$$
$$\tan\theta_2 = \frac{3W}{2H},$$
$$\tan\theta_1 = \frac{5W}{2H},$$

and
$$\frac{5W}{2H} + \frac{3W}{2H} + \frac{W}{2H} = 3,$$
$$H = \frac{3W}{2},$$
$$\tan\theta_1 = \frac{5}{3}, \quad \tan\theta_2 = 1, \quad \tan\theta_3 = \frac{1}{3}.$$
$$T_1 = \frac{H}{\cos\theta_1} = \frac{3}{2}W\left(1 + \frac{25}{9}\right)^{1/2},$$
$$= \frac{1}{2}W\sqrt{(34)},$$

and this is the greatest tension.

From § 10.5 the equation of the parabola referred to horizontal and vertical axes through A is

$$y = \frac{5}{3}x - \frac{1}{3h}x(x - h),$$
$$3hy = x(6h - x).$$

EXERCISES 10 (*a*)

1. In the guy-rope runner shown in Fig. 261 the string turns through 45° at each bend and A is a smooth pulley. Prove that the runner will not slip under steady tension between A and B, provided that μ is not less than 0·22 approximately. (L.U., Pt. II)

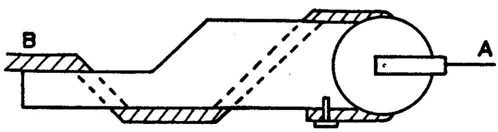

Fig. 261

2. A uniform chain of weight W is of length equal to half the circumference of a fixed rough vertical circle. The chain rests on the upper edge of the circle, extending from one end of the horizontal diameter to the other. A vertically downward force is applied at one end of the chain. Prove that the chain will not move unless the force is greater

than $2\mu W(1 + e^{\mu\pi})/\pi(1 + \mu^2)$, where μ is the coefficient of friction between the chain and circle. (L.U., Pt. I)

3. A rope AB 10 ft. long makes one complete turn around a circular post of diameter 1 ft., and the ends A and B are pulled in opposite directions with a force of 1000 lb. If the coefficient of friction of the rope on the post is 0·3, find the least tension in the rope. If the strain of a unit length of the rope due to a pull of 1000 lb. is 0·05, find the total increase in length of the rope due to the pull of 1000 lb.

4. When a rope is passed around a bollard there is an increase of stress in some fibres due to bending of the rope. Assuming that the maximum fibre stress due to bending is EY/ϱ, where ϱ is the radius of curvature, and that the diminution of stress due to friction is $f_0(1 - e^{-\mu\psi})$, where ψ is the change of inclination, show that the maximum stress should remain constant if the rope is led on to a surface which is such that the s, ψ equation of the rope is of the form $s = \dfrac{EY}{\mu f_0} \log_e (e^{\mu\psi} - 1)$.

5. A belt weighing 2 lb. per ft. run, in which the safe tension is 1000 lb., drives a wheel and is in contact over half of its circumference. If the coefficient of friction between the belt and the wheel is 0·2, find the greatest horse-power that can be transmitted to the wheel.

6. Four equal weights are attached to a light inextensible string which hangs from two points A and B at the same level. The horizontal projection of each of the five portions of the string is equal to h and the lowest point of the string is $6h$ below AB. Find the inclinations to the horizontal of the portions of the string and the tension in the portion which is horizontal.

7. A suspension bridge has a span of 72 ft. and the weight of the roadway is 90 tons, 15 tons of which is carried by the piers and the remainder by five pairs of tie rods equally spaced. Light chains hanging between towers 55 ft. high at either pier support the tie rods and the length of a middle tie rod is 10 ft. Find the lengths of the other tie rods and the greatest tension in the chains.

8. Two light beams, AC and BC, are freely hinged together at C and to fixed supports at A and B, as shown in Fig. 262. They are supported by a suspension chain DE and seven equally spaced tie rods, the lengths of which are so arranged that each tie rod has the same tension. The dip at the centre of the chain is equal to $\frac{2}{5}AC$.

If a weight W is placed at C, show that the greatest tension in the suspension chain is $\dfrac{\sqrt{149}}{8}W$. (C.U.)

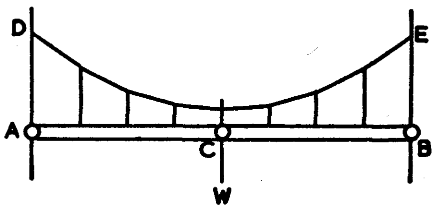

Fig. 262

10.6 The Catenary

A heavy cable or chain suspended between two points hangs in the form of a curve which is called a catenary.

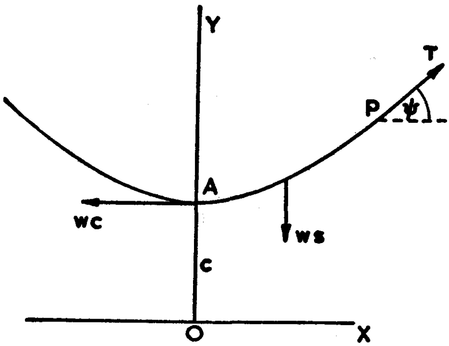

Fig. 263

Let A (Fig. 263) be the lowest point of the curve, ψ the inclination to the horizontal of the tangent to the curve at any point P, s the length of cable between A and P and w its weight per unit length. The length of cable AP is in equilibrium under the action of its weight ws and the tensions at A and P. The tension at A acts along the tangent at A and is denoted by wc, that is we assume it to be equal to the weight of an unknown length c of cable. The tension at P is along the tangent at P and is denoted by T. We have, therefore,

$$T \sin \psi = ws,$$
$$T \cos \psi = wc,$$

and hence
$$s = c \tan \psi. \tag{1}$$

This is the intrinsic equation of the catenary; c is called the parameter of the catenary and is determined by the length of the span and the dip at the centre.

To obtain the cartesian coordinates of a point on the catenary we use the relations

$$\frac{dx}{ds} = \cos \psi, \frac{dy}{ds} = \sin \psi.$$

For the catenary
$$\frac{dx}{d\psi} = \frac{dx}{ds} \cdot \frac{ds}{d\psi}$$
$$= \cos \psi . c \sec^2 \psi,$$
$$x = \int c \sec \psi d\psi$$
$$= c \log (\sec \psi + \tan \psi).$$

If x is measured from the lowest point an additive constant is not required. Also

$$\frac{dy}{d\psi} = \frac{dy}{ds} \cdot \frac{ds}{d\psi}$$
$$= \sin \psi . c \sec^2 \psi,$$

$$y = c\int \frac{\sin \psi}{\cos^2 \psi} d\psi$$

$$= c \sec \psi + \text{constant}.$$

If y is measured from the lowest point of the catenary where ψ is zero the added constant is $-c$. It is customary, however to take the origin of coordinates at a depth c below the lowest point of the catenary so that $y = c$ when $\psi = 0$, and then no added constant is required.

The equations

$$x = c \log (\sec \psi + \tan \psi), \tag{2}$$

$$y = c \sec \psi, \tag{3}$$

give the equation of the curve in parametric form.

We have for the tension at any point,

$$T = wc \sec \psi$$

$$= wy. \tag{4}$$

Thus the tension at any point is proportional to the height of the point above the origin.

We have also

$$s^2 = c^2 \tan^2 \psi$$

$$= c^2 \sec^2 \psi - c^2$$

$$= y^2 - c^2,$$

$$y^2 = s^2 + c^2. \tag{5}$$

Most problems on the catenary can be solved by using the five formulae given in this section.

10.7 Determination of the Parameter of a Catenary

The parameter c of a catenary is known if the weight per unit length of the cable and the horizontal component of the tension at any point is known. In other cases it is found from a knowledge of two of the dimensions of the catenary.

If the ends of the catenary are at the same level, let $2l$ be the span, d the dip at the centre, $2S$ the length of the cable and ψ_1 the slope at the ends of the cable.

Given ψ_1 and S, l or d, the value of c can be found directly from equation (1), (2) or (3), remembering when d is given that $y = d + c$ when $\psi = \psi_1$.

Given S and d, we have from (5)

$$(d + c)^2 = S^2 + c^2,$$

$$c = \frac{S^2}{2d} - \frac{d}{2}.$$

Given l and d we have

$$l = c \log (\sec \psi_1 + \tan \psi_1),$$
$$c + d = c \sec \psi_1.$$

The value of c can be found from these equations only by numerical or graphical methods. The solution of the equations is also more complicated when l and S are given.

Example 4. *A chain* 40 *ft. long hangs between two points* 30 *ft. apart at the same level. Find the dip at the centre.*

If c be the parameter of the catenary and ψ_1 the slope at the ends we have

$$15 = c \log (\sec \psi_1 + \tan \psi_1),$$
$$20 = c \tan \psi_1,$$
$$0{\cdot}75 = \frac{\log (\sec \psi_1 + \tan \psi_1)}{\tan \psi_1}.$$

We obtain the value of the right-hand side for various values of ψ_1.

ψ_1	*tan* ψ_1	*sec* ψ_1 + *tan* ψ_1	$\log_e$ (*sec* ψ_1 + *tan* ψ_1)	*ratio*
40°	0·8391	2·1445	0·7629	0·909
50°	1·1918	2·7475	1·0107	0·848
60°	1·7321	3·7321	1·3169	0·761
61°	1·8040	3·8667	1·3524	0·750

Hence

$$\psi_1 = 61°,$$
$$c = \frac{20}{1{\cdot}804} = 11{\cdot}08 \text{ ft.}$$
$$d = c(\sec \psi_1 - 1)$$
$$= 11{\cdot}08 \times 1{\cdot}063$$
$$= 11{\cdot}78 \text{ ft.}$$

Example 5. *A uniform chain* 100 *ft. long is suspended from two points at the same level and at each point the inclination of the chain to the horizontal is* 45 *degrees. The chain weighs* 10 *lb. per foot. Find the span, the dip at the centre and the tension at the points of support.*

Since

$$s = c \tan \psi,$$
$$50 = c \tan 45°,$$
$$c = 50 \text{ ft.}$$
$$x = c \log (\sec \psi + \tan \psi),$$
$$2l = 100 \log_e (1 + 1{\cdot}414)$$
$$= 88{\cdot}12 \text{ ft.}$$
$$y = c \sec \psi,$$
$$d + 50 = 50 \sqrt{2},$$
$$d = 20{\cdot}7 \text{ ft.}$$
$$wc = 500 \text{ lb.},$$

and since the horizontal and vertical components of tension at the supports are equal

$$T = 500 \sqrt{2}$$
$$= 707 \text{ lb.}$$

Example 6. *A uniform chain has one end fixed and rests in equilibrium passing over a small smooth peg with its other end hanging freely. The length of the vertical portion is* 8 *ft. The fixed end and the lowest point of the catenary are at depths* 5·5 *ft., and* 3·5 *ft., respectively above the free end. Find the angles which the chain makes with the horizontal at the fixed end and at the peg, and the total length of the chain.*

The tension at the peg B (Fig. 264) supports 8 ft. of chain hanging vertically and is therefore $8w$. But with the usual axes of coordinates this is wy, therefore at B, $y = 8$. It follows that $c = 3{\cdot}5$.
The slope at B is given by $y = c \sec \psi$,

$$8 = 3{\cdot}5 \sec \psi_B,$$
$$\psi_B = 64^\circ\ 3'.$$

At A,

$$5{\cdot}5 = 3{\cdot}5 \sec \psi_A,$$
$$\psi_A = 50^\circ\ 29'.$$

The total length of the chain is

$$S = 8 + 3{\cdot}5 \tan \psi_A + 3{\cdot}5 \tan \psi_B$$
$$= 19{\cdot}44 \text{ ft.}$$

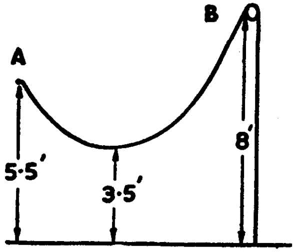

Fig. 264

Example 7. *A chain* 500 *ft. long and weighing* 100 *lb. per ft. run is attached to the bows of a ship,* 50 *ft. of the chain hangs vertically and the remainder is laid out on the ground in a horizontal line running away from the ship. If the coefficient of friction between the chain and the ground is* 0·4, *find how far the ship will have moved backwards along the line of the chain before the chain begins to slip and the horizontal force which the chain then exerts on the ship.*

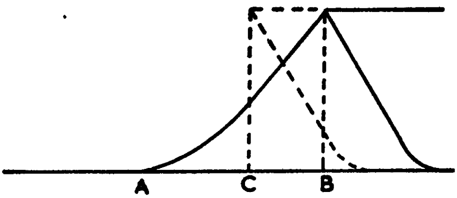

Fig. 265

Let a ft. (Fig. 265) be the length of chain hanging in a catenary when the chain is about to slip. The tension at the lowest point is then the frictional drag and

$$wc = 0{\cdot}4\,(500 - a)100,$$
$$c = 200 - 0{\cdot}4a.$$

For the catenary

$$(c + 50)^2 = c^2 + a^2,$$
$$a^2 = 2500 + 100(200 - 0{\cdot}4a),$$
$$a^2 + 40a - 22{,}500 = 0,$$
$$a = 131{\cdot}3 \text{ ft.}$$
$$c = 147{\cdot}48 \text{ ft.}$$

The horizontal drag is then 40(500 − 131·3) lb.

$$= 6{\cdot}58 \text{ tons.}$$

At the bows,

$$\begin{aligned} c + 50 &= c \sec \psi, \\ 197\cdot48 &= 147\cdot48 \sec \psi, \\ \psi &= 41^\circ\ 41'. \\ AB = x &= 147\cdot48 \log_e (\sec \psi + \tan \psi) \\ &= 147\cdot48 \times 0\cdot802 \\ &= 118\cdot3 \text{ ft.} \end{aligned}$$

The ship was originally at C where $AC = 81\cdot3$ ft., therefore it has moved 37 ft.

10.8 Use of Hyperbolic Functions

The formulae for the catenary may be expressed in terms of hyperbolic functions.

We have

$$x = c \log (\sec \psi + \tan \psi),$$
$$e^{x/c} = \sec \psi + \tan \psi.$$

Hence, since $\sec^2 \psi - \tan^2 \psi = 1$,

$$e^{-x/c} = \sec \psi - \tan \psi,$$

$$\sec \psi = \cosh \frac{x}{c},$$

$$\tan \psi = \sinh \frac{x}{c}.$$

Thus we have

$$y = c \cosh \frac{x}{c},$$

$$s = c \sinh \frac{x}{c}.$$

When the span $2l$ and the length of cable $2S$ are given the equation to be solved to find the parameter c is

$$\frac{S}{c} = \sinh \frac{l}{c},$$

and this equation may sometimes be solved by inspection of the tables of sinh.

In Example 3, $l = 15$, $S = 20$ and we have

$$\frac{20}{c} = \sinh \frac{15}{c}.$$

$c = 10$	$20/c = 2$	$\sinh 15/c = 2\cdot129$
$c = 11$	$20/c = 1\cdot818$	$\sinh 15/c = 1\cdot829$
$c = 12$	$20/c = 1\cdot667$	$\sinh 15/c = 1\cdot604.$

Hence, by interpolation, $c = 11\cdot1$ approximately.

Example 8. *One end of a uniform chain of length l is attached to a fixed point A; B is the rounded edge of a rough horizontal table, AB being a horizontal line of length $2a$. The chain lies partly on the table at right-angles to the edge and partly hangs as a festoon between A and B.*

Prove that when the length on the table has the least value consistent with equilibrium, the parameter c of the catenary is given by

$$\mu e^{\theta}\left(\frac{xl}{a} - 2\sinh x\right) = \cosh x,\ \theta = \mu \tan^{-1}(\sinh x),$$

where $x = \dfrac{a}{c}$ and μ is the coefficient of friction. (L.U., Pt. II)

Let b be the length of chain on the table and ψ the slope of the chain at B (Fig. 266). The friction force at B is μbw, and therefore at the highest

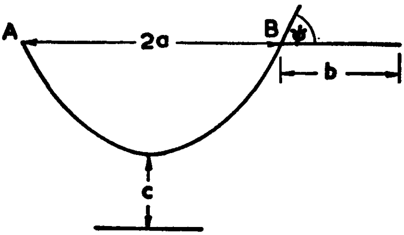

Fig. 266

point of the catenary the chain having turned about the rough surface through an angle ψ the tension is

$$T = e^{\mu\psi}\mu bw.$$

Now
$$T = wy = wc\cosh\frac{x}{c},$$

therefore
$$e^{\mu\psi}\mu b = c\cosh\frac{a}{c}.$$

Also
$$\frac{1}{2}(l - b) = c\sinh\frac{a}{c}.$$

Writing $\mu\psi = \theta$ and $a/c = x$, we have

$$\begin{aligned} \mu b e^{\theta} &= c\cosh x, \\ l - b &= 2c\sinh x, \\ \mu e^{\theta}(l - 2c\sinh x) &= c\cosh x, \\ \mu e^{\theta}\left(\frac{xl}{a} - 2\sinh x\right) &= \cosh x. \end{aligned}$$

Also
$$\tan\psi = \sinh\frac{a}{c} = \sinh x,$$

therefore
$$\frac{\theta}{\mu} = \tan^{-1}(\sinh x).$$

Example 9. *A uniform cable is suspended between two points at the same level 2a apart. Find the dip at the centre which will make the tension at the ends a minimum and the length of cable required for this.*

Let c be the parameter of the catenary and w the weight of cable per unit length.

The end tension is $$T = wc\cosh\frac{a}{c}.$$

For this to be a minimum we have

$$\frac{dT}{dc} = w\cosh\frac{a}{c} - w\frac{a}{c}\sinh\frac{a}{c} = 0,$$

that is $$\frac{a}{c}\tanh\frac{a}{c} = 1.$$

For $\dfrac{a}{c} = 1{\cdot}20$, $\dfrac{a}{c}\tanh\dfrac{a}{c} = 1{\cdot}0004$ and the equation is satisfied approximately.

If d be the dip, $$d = c\left(\cosh\frac{a}{c} - 1\right),$$

$$\frac{d}{a} = \frac{1}{1{\cdot}20}(\cosh 1{\cdot}20 - 1)$$
$$= \frac{0{\cdot}811}{1{\cdot}20}$$
$$= 0{\cdot}676.$$

Hence, for minimum tension the ratio of dip to span is approximately $\dfrac{1}{3}$.

For length of cable $2S$ we have

$$S = c\sinh\frac{a}{c},$$

$$\frac{S}{a} = \frac{c}{a}\sinh\frac{a}{c}$$
$$= \frac{1{\cdot}5095}{1{\cdot}20}$$
$$= 1{\cdot}258.$$

Hence, the length of cable must be about 25 per cent greater than the span.

EXERCISES 10 (*b*)

1. A chain 100 ft. long is suspended from two points at the same level and the ends are inclined at 60° to the horizontal. Determine the span and the sag at the lowest point. (L.U., Pt. II)

2. An endless uniform string hangs in equilibrium over a smooth pulley and is in contact with it over three-quarters of the circumference: show that the length of the free portion is $\sqrt{2}/\log(1+\sqrt{2})$ times the radius of the pulley. (L.U., Pt. II)

3. A chain of length L has its ends attached to light rings which slide on a rough horizontal rod. If μ is the coefficient of friction, show that the greatest distance apart in which the rings can rest in equilibrium is $\mu L\log_e[\{1 + \sqrt{(1 + \mu^2)}\}/\mu]$. (L.U., Pt. II)

4. A uniform chain of length 30 ft. hangs over two smooth pegs and the parts which hang vertically are 10 ft. and 11 ft. long respectively. Find the parameter of the catenary in which the central portion hangs and the distance between the pegs.

5. A heavy uniform chain of length l is lying in a straight line on a horizontal floor. One end of the chain is slowly raised vertically until one half of the chain is clear of the floor, the remainder being on the point of slipping. If the coefficient of friction is $\frac{1}{3}$, show that the height above the floor of the raised end is $\frac{1}{6}l(\sqrt{10} - 1)$, and that the horizontal distance of the mid-point of the chain from that end is $\frac{1}{6}l \log (\sqrt{10} + 3)$. (L.U., Pt. I)

6. One end of a uniform flexible chain of length a and weight W is attached to a fixed point and the other end is drawn aside by a horizontal force. If this force is equal to W, show that the horizontal deflection of the lower end is $a \log_e (1 + \sqrt{2})$ and find the tension at the support. (L.U., Pt. II)

7. A heavy uniform flexible string has one end fixed and rests in equilibrium, passing over a small smooth peg, with its other end hanging freely. The length of the vertical portion of the string is 2 ft. The fixed end and the lowest point of the catenary are respectively $2/\sqrt{3}$ ft. and 1 ft. above the free end. Find the angles which the string makes with the horizontal at the fixed end and at the peg, and also find the total length of the string. (L.U., Pt. II)

8. A uniform flexible chain hangs under gravity from two points with its lowest point between the ends. Obtain expressions for the horizontal and vertical distances of any point on the chain from the lowest point in terms of the parameter of the catenary and the inclination at the point.

If the horizontal distance between the ends is 22 ft. and the tangents at the ends are inclined at 45° and 60° to the horizontal, find approximate values of the length of the chain and the vertical distance between ends. (L.U., Pt. II)

9. One end of a uniform chain of weight W, length l, slides on a smooth straight wire OA inclined at 45° below the horizontal. The other end is attached to a small ring of weight W which slides on a smooth vertical wire OB. Prove that the catenary formed by the chain has a parameter $2l$, and find the difference in height of the ends. (L.U., Pt. II)

10. A chain consists of two uniform portions AC, CB, of equal weight and of respective lengths a, b.

The chain is suspended from A and B so that C is its lowest point. Prove that the tangents at A and B make the same angle with the horizontal and that if this angle is β, the difference in level of A and B is $(a - b) \tan \frac{1}{2}\beta$. Prove also, that the horizontal distance apart of A and B is $(a + b) \cot \beta \log (\sec \beta + \tan \beta)$. (L.U., Pt. II)

11. Prove that in a uniform catenary, with the usual notation $y^2 = s^2 + c^2$. A uniform chain hangs over two smooth pegs at different levels so

that there is a length l of chain between the pegs and lengths l_1, l_2 hanging vertically on the two sides. Prove that the lowest point of the curved part of the chain divides this part in the ratio $l^2 + l_1^2 - l_2^2 : l^2 + l_2^2 - l_1^2$, and that the parameter c of the catenary is given by $4l^2c^2 = (l + l_1 + l_2)(l + l_2 - l_1)(l + l_1 - l_2)(l_1 + l_2 - l)$. (L.U., Pt. II)

12. A uniform flexible string hangs in equilibrium symmetrically over two small smooth pegs at the same level, distance $2a$ apart. Find the relation between the length $2l$ of the string and the parameter c of the catenary in which the portion of string between the pegs hangs. Hence show that equilibrium is not possible unless $l \geqslant ae$ and find, when $l = ae$, the acute angle which a tangent to the string at a peg makes with the horizontal. (L.U., Pt. II)

10.9 Taut Wires

If the horizontal tension in a wire which hangs in the form of a catenary is increased, since this is equal to wc, the parameter c is increased. We shall show that when c is large so that $(1/c)^2$ and higher powers of $1/c$ may be neglected the catenary approximates to a parabola.

The equation of a catenary is

$$y = c \cosh \frac{x}{c}$$

$$= c\left(1 + \frac{x^2}{2c^2} + \frac{x^4}{24c^4} + \ldots\right),$$

that is $$y - c = \frac{x^2}{2c}, \text{ approximately.}$$

If the origin now be taken at the lowest point of the curve (Fig. 267)

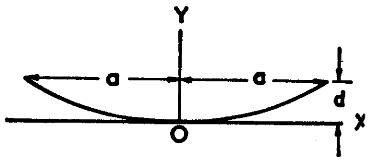

Fig. 267

where $y = c$, the equation becomes

$$x^2 = 2cy,$$

and this is the equation of a parabola. If $2a$ be the span and d the central dip we have

$$a^2 = 2cd,$$

and hence $$x^2 = (a^2/d)y.$$

The tension at the lowest point

$$= wc = \frac{wa^2}{2d}.$$

The greatest tension

$$= wc \cosh \frac{a}{c}$$

$$= wc + \frac{wa^2}{2c}, \text{ approximately,}$$

$$= w\left(\frac{a^2}{2d} + d\right).$$

For the length of the wire we have

$$s = c \sinh \frac{x}{c}$$

$$= x + \frac{x^3}{6c^2}, \text{approximately.}$$

The total length of the wire is therefore $2a$ to a first approximation, and to a second approximation,

$$S = 2a + \frac{a^3}{3c^2}$$

$$= 2a + \frac{4d^2}{3a}$$

$$= l + \frac{8d^2}{3l}, \text{ where } l = 2a.$$

10.10 Uniformly Distributed Load

When a wire is stretched tightly between two points its weight may be considered to be uniformly distributed across the span. This is also approximately true for suspension bridges where the main load is the weight of the roadway.

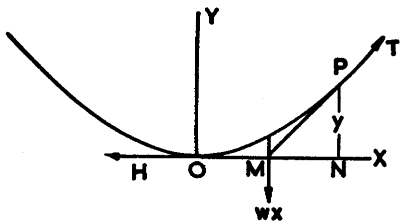

Fig. 268

Let O (Fig. 268) be the lowest point of such a wire with axes OX horizontal and OY vertical. Let w be the weight per unit length, measured horizontally, and let $2a$ be the span and d the central dip.

If P be a point (x, y) on the wire the weight of OP is wx and acts through the point $M(\frac{1}{2}x, 0)$. The tension H at the lowest point is horizontal and also acts through this point.

The tension T at P must, therefore, pass through this point and the triangle PNM is a triangle of forces for the forces on OP.

Therefore $$\frac{wx}{PN} = \frac{H}{MN} = \frac{T}{MP},$$

that is $$\frac{wx}{y} = \frac{H}{\frac{1}{2}x} = \frac{T}{(y^2 + \frac{1}{4}x^2)^{1/2}}.$$

Thus writing $H = wc$, we have

$$\begin{aligned} x^2 &= 2cy, \\ &= \left(\frac{a^2}{d}\right)y. \end{aligned}$$

The approximation, therefore, leads to the parabolic curve obtained in § 10.7. The horizontal tension is

$$\begin{aligned} H &= \frac{wa^2}{2d} \\ &= \frac{Wl}{8d}, \end{aligned}$$

where W is the total weight and $l = 2a$.

Example 10. *A uniform chain is stretched between two points at the same level 22 yds. apart; if the sag at the centre is 3 ft prove that the parameter of the catenary is approximately 182 ft. If the chain is tightened so that the sag is 1 ft., find the new parameter, and determine the ratio of the tensions at the supports in the two cases, and the ratio of the lengths of chain between the supports.* (L.U., Pt. II)

Assuming that the catenary approximates to a parabola, we have from § 10.9,

$$c = \frac{a^2}{2d},$$

where $2a$ is the span and d is the dip.

Thus in case (1) $$c = \frac{33^2}{6} = 181{\cdot}5 \text{ ft.}$$

In case (2) $$c = \frac{33}{2} = 544{\cdot}5 \text{ ft.}$$

The greatest tension in each case is $w(c + d)$, and the ratio of the tensions

$$= \frac{181{\cdot}5 + 3}{544{\cdot}5 + 1} = 0{\cdot}34.$$

For the length of the chain,

$$\begin{aligned} s &= l + \frac{8d^2}{3l}, \\ s_1 &= 66 + \frac{72}{198} = 66{\cdot}36 \text{ ft.}, \\ s_2 &= 66 + \frac{8}{198} = 66{\cdot}04 \text{ ft.}, \\ s_1/s_2 &= 1{\cdot}005. \end{aligned}$$

Example 11. *A suspension bridge is supported by two steel cables. The span is* 120 *yds., the cables are* 20 *ft. above the roadway at the centre and* 60 *ft. at the ends and the loading is* 1200 *lb. per ft. run. If the safe tension in a cable whose circumference is c in. is* $9c^2$ *cwt., find the diameter of the cables required. Find also the height of a cable above the roadway at* 20 *yds. from one end of the bridge. The vertical pillars to which the cables are attached are stayed by guys each fixed to the top of a pillar and secured to the ground so as to make an angle of* 60° *with the pillar. Find the tension in each of the guys.*

The load carried by one cable over half the span

$$= 180 \times 600 \text{ lb.}$$
$$= 964{\cdot}3 \text{ cwt.}$$

If H cwt. be the tension at the lowest point and T cwt. the tension at the pillars, from the triangle of forces (Fig. 269) we have

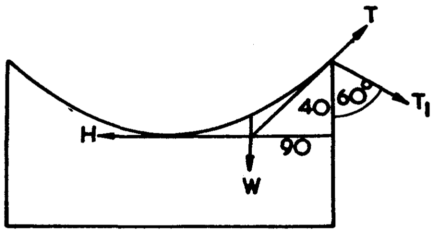

Fig. 269

$$\frac{T}{(40^2 + 90^2)^{1/2}} = \frac{H}{90} = \frac{964{\cdot}3}{40},$$

hence
$$H = 2170 \text{ cwt.},$$
$$T = 2375 \text{ cwt.}$$

The circumference of the cable is given by

$$9c^2 = 2375,$$
$$c = 16{\cdot}25 \text{ in.},$$

and the diameter of the cable is 5·17 in.
If T_1 be the tension in the guy,

$$T_1 \sin 60 = H = 2170 \text{ cwt.},$$
$$T_1 = 125{\cdot}3 \text{ tons.}$$

The height of the cable above the roadway at x ft. from the centre is given by the equation of the parabola,

$$y = 20 + 40\left(\frac{x}{180}\right)^2.$$

At 20 yds. from one end $x = 120$,

$$y = 20 + 40\left(\frac{120}{180}\right)^2$$
$$= 37{\cdot}8 \text{ ft}$$

10.11 Variation due to a Small Increase of Span

From the formulae found in § 10.9 we can obtain the increase of the dip and the tension consequent on a small increase of the span.

From the formula

$$S = l + \frac{8d^2}{3l}$$

if S remains constant and l increases by δl and d by δd, we have

$$0 = \delta l - \frac{8d^2}{3l^2}.\delta l + \frac{16d}{3l}.\delta d.$$

Therefore, assuming that d^2/l^2 is small compared with unity, we have

$$\delta d = -\frac{3}{16}\frac{l}{d}.\delta l.$$

The horizontal tension is $H = \dfrac{Wl}{8d}$, and if H is increased by δH when l is increased by δl, we have

$$\begin{aligned} \delta H &= \frac{W}{8d}.\delta l - \frac{Wl}{8d^2}.\delta d \\ &= \frac{W}{8d}.\delta l + \frac{Wl}{8d^2}\left(\frac{3}{16}\frac{l}{d}.\delta l\right) \\ &= \frac{3}{128}\frac{Wl^2}{d^3}.\delta l, \text{ approximately.} \end{aligned}$$

10.12 Variation due to a Small Increase of Length of Wire

Let δd be the increase in dip due to a small increase δS in the length of the wire, the span l remaining constant.

We have
$$S = l + \frac{8d^2}{3l},$$

$$\delta S = \frac{16}{3}\frac{d}{l}.\delta d,$$

$$\delta d = \frac{3}{16}\frac{l}{d}.\delta S.$$

If, in consequence of a rise in temperature τ, the wire stretches an amount $k\tau$ per unit length we have

$$\frac{\delta S}{S} = k\tau,$$

$$\begin{aligned} \delta d &= \frac{3}{16}k\tau\frac{l^2}{d}\left(1 + \frac{8}{3}\frac{d^2}{l^2}\right) \\ &= \frac{3}{16}k\tau\frac{l^2}{d}, \text{ approximately.} \end{aligned}$$

10.13 Variation due to a Small Increase of Weight

If the weight w per unit length is increased the shape of the curve in which the wire hangs is unaltered unless account is taken of the elasticity of the material.

Let E be the modulus of elasticity of the wire and A the cross-sectional area and consider the increase δd of the dip consequent on a small increase δw of w.

Let δH be the increase of the horizontal tension and δS the increase in the length of the wire.

We have
$$H = \frac{wl^2}{8d},$$
$$\delta H = \frac{l^2}{8d}.\delta w - \frac{wl^2}{8d^2}.\delta d.$$

The stress-strain equation gives
$$\frac{\delta H}{A} = E\frac{\delta S}{S}$$
$$= \frac{E\dfrac{16}{3}\dfrac{d}{l}.\delta d}{l + \dfrac{8d^2}{3l}}.$$

Consequently
$$\frac{l^2}{8d}\delta w = \frac{wl^2}{8d^2}.\delta d + \frac{\dfrac{16}{3}AE\dfrac{d}{l}}{l + \dfrac{8}{3}\dfrac{d^2}{l}}\delta d,$$
$$\frac{\delta w}{w} = \frac{\delta d}{d}\left(1 + \frac{128d^3AE}{3wl^4(1 + 8d^2/3l^2)}\right).$$
$$\frac{\delta d}{d} = \frac{1}{1 + 128d^3AE/3wl^4}.\frac{\delta w}{w}, \text{ approximately.}$$

10.14 Cable with Ends at Different Levels

If a wire is tightly stretched between two points which are not at the same level the weight may be considered to be uniformly distributed horizontally and the curve is, therefore, an arc of a parabola. Let A and B be the two points (Fig. 270) and let $AB = l$ and the inclination of AB to the horizontal be α.

Let w be the weight per unit length horizontally and let the total weight be $W = wl \cos \alpha$.

Let $x^2 = 2cy$ be the parabola referred to axes OX and OY through its vertex and let A be the point (ξ, η).

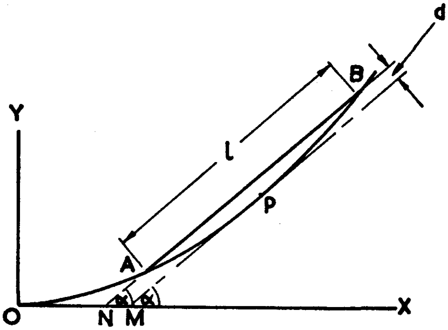

Fig. 270

Then
$$\xi^2 = 2c\eta,$$
$$(\xi + l\cos\alpha)^2 = 2c(\eta + l\sin\alpha),$$
$$\xi = c\tan\alpha - \frac{1}{2}l\cos\alpha,$$
$$\eta = (c\tan\alpha - \frac{1}{2}l\cos\alpha)^2/2c.$$

Since $\dfrac{dy}{dx} = \dfrac{x}{c}$, the tangent is parallel to AB at the point $P(c\tan\alpha, \frac{1}{2}c\tan^2\alpha)$ and this tangent meets OX at M where $OM = \frac{1}{2}c\tan\alpha$. The chord AB meets OX at N where $ON = \xi - \eta\cot\alpha$, that is $ON = \frac{1}{2}c\tan\alpha - \dfrac{l^2}{8c}\cos^2\alpha\cot\alpha$.

Thus
$$MN = \frac{l^2}{8c}\cos^2\alpha\cot\alpha.$$

The perpendicular distance between the chord and the tangent is thus $MN\sin\alpha$. Denoting this length by d, which we may call the dip of the wire, we have
$$d = \frac{l^2\cos^3\alpha}{8c}.$$

The tension at O is
$$H = wc$$
$$= \frac{wl^2\cos^3\alpha}{8d}$$
$$= \frac{Wl\cos^2\alpha}{8d}.$$

The tension at P is $\qquad T = H \sec \alpha$

$$= \frac{Wl \cos \alpha}{8d}.$$

The slope of the tangents at B and A are given by

$$\tan \theta = \tan \alpha \pm \frac{l}{2c} \cos \alpha$$

$$= \tan \alpha \pm \frac{4d}{l} \sec^2 \alpha.$$

$$\sec \theta = \left(1 + \tan^2 \alpha + \frac{16d^2}{l^2} \sec^4 \alpha \pm \frac{8d}{l} \sec^2 \alpha \tan \alpha\right)^{1/2},$$

$$= \sec \alpha \left(1 + \frac{16d^2}{l^2} \sec^2 \alpha \pm \frac{8d}{l} \tan \alpha\right)^{1/2}$$

$$= \sec \alpha \left(1 \pm \frac{4d}{l} \tan \alpha\right), \text{ approximately.}$$

The tension T at B or A is given by

$$T = H \sec \theta$$

$$= \frac{Wl \cos \alpha}{8d}\left(1 \pm \frac{4d}{l} \tan \alpha\right), \text{ approximately,}$$

$$= \frac{Wl \cos \alpha}{8d} \pm \frac{1}{2} W \sin \alpha.$$

The length of the arc AB is found by integration.

We have $\qquad S = \displaystyle\int_{\xi}^{\xi + l\cos\alpha} \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{1/2} dx$

$$= \int_{\xi}^{\xi + l\cos\alpha} \sqrt{\left(1 + \frac{x^2}{c^2}\right)} dx.$$

Let $x = c \tan \alpha + z$,

$$S = \int_{-1/2l\cos\alpha}^{1/2l\cos\alpha} \left(1 + \tan^2 \alpha + 2 \tan \alpha . \frac{z}{c} + \frac{z^2}{c^2}\right)^{1/2} dz$$

$$= \sec \alpha \int_{-1/2l\cos\alpha}^{1/2l\cos\alpha} \left(1 + 2 \sin \alpha \cos \alpha \frac{z}{c} + \cos^2 \alpha \frac{z^2}{c^2}\right)^{1/2} dz$$

$$= \sec \alpha \int_{-1/2l\cos\alpha}^{1/2l\cos\alpha} \left(1 + \sin \alpha \cos \alpha \frac{z}{c} + \frac{1}{2} \cos^4 \alpha \frac{z^2}{c^2} + \ldots\right) dz.$$

Neglecting cubes and higher powers of z/c, we have

$$S = \sec a\left[z + \frac{1}{2}\sin a\cos a.\frac{z^2}{c} + \frac{1}{6}\cos^4 a.\frac{z^3}{c^2}\right]_{-1/2l\cos a}^{1/2l\cos a}$$

$$= \sec a\left(l\cos a + \frac{1}{24}\frac{l^3\cos^7 a}{c^2}\right)$$

$$= l + \frac{l^3}{24}\frac{\cos^6 a}{c^2}.$$

Therefore, since $c = l^2\cos^3 a/(8d)$, we have

$$S = l + \frac{8d^2}{3l}.$$

Thus the length of the arc is given in terms of the span and the dip by the same approximate formula that is used when the supports are at the same level.

EXERCISES 10 (c)

1. The distance between the posts of a tennis net is 40 ft. and the net weighs 30 lb. If the wire carrying the net is tightened until the drop at the centre is only 2 in. and the diameter of the wire is 0·2 in., find the maximum stress in the wire, which is of steel weighing 480 lb./ft.^{3}.
2. The tensile stress in each of the two cables of a suspension bridge is not to exceed 7 tons/in.^{2} and the circumference of the cable is to be 30 in. Find the maximum permissible tension in a cable, and the weight per foot run of the heaviest roadway which can be supported over a span of 450 ft. allowing a dip at the centre of 80 ft.
3. If the tension in the cable of a suspension bridge is not to vary by more than 10 per cent, calculate the maximum ratio of the dip to the span and find the corresponding slope of the cable at each end.
4. If the dip at the centre of a steel wire may not exceed 20 in. and the stress in it may not exceed 16 tons/in.^{2}, find the greatest possible span for the wire the weight of the steel being 500 lb./ft.^{3}.
5. In a suspension bridge of 200 ft. span the total load of 2·2 tons per foot run is carried equally by two chains. The maximum tension in a chain must not exceed 250 tons. Calculate the minimum dip of the chains.
6. A suspension bridge has a span of 500 ft. Owing to the position of the anchorages the direction of each cable near the piers must make an angle of 28° with the horizontal. Calculate the dip of the cables at mid-span and quarter-span.

The cable of a suspension bridge carries a load of half a ton per horizontal foot run, the span is 90 ft., and the dip at the centre 8 ft. Find the greatest tension in the cable. Prove that the angle θ which the cable makes with the horizontal at a pier is given by $\tan\theta = 4d/l$, where d is the dip and l the span, and find θ in this case.

8. A telegraph pole A carries 16 lines of wires and the poles on either side are 80 yards distant. The lines make an angle of 120° at A. A wire weighs 0·9 oz. per ft. and its tensile strength is 1000 lb. Using a factor of safety of 4, find the least safe dip for the wires. If a stay on the pole A is inclined at 45° to the horizontal and is equally inclined to the lines of the wires, find the tension in the stay and the downward thrust on the pole A.

9. The cable of a suspension bridge has a dip of 10 ft. on a span of 120 ft. and carries a load of 0·375 tons per foot run horizontally. Find the greatest tension in the cable. If the top of one pier deflects a distance of 1 ft. towards the centre of the span find the increase of the dip and the decrease of the tension in the cable.

10. A wire runs between points at the same level 120 ft. apart, and the dip at the centre is 4 ft. Find the length of the wire and the amount by which it must be shortened to reduce the dip by 1 ft.

11. A steel wire, $\frac{1}{16}$ in. diameter, is stretched tightly between two points at the same level 100 ft. apart. When the dip at the centre is 8 in., find the tensile stress in the wire. The steel weighs 480 lb./ft.^{3}.

Due to a deposit of ice the weight per foot run is increased by 10 per cent. Find the consequent increase in dip. E(steel) $= 3 \times 10^7$ lb./in.^{2}.

12. A cable is fixed to two points A and B. B is 100 ft. from A and 10 ft. higher than A and the dip in the cable is 1 ft. The cable weighs 8 lb. per foot. Find the tension in the cable at A and at B.

13. A steel wire acts as a stay on a vertical flagpost. It joins a point on the post 30 ft. above the ground to a point on the ground 40 ft. from the base of the post. The steel weighs 480 lb./ft.^{3}. Show that if the dip of the wire from the straight line joining its ends is 2 in. the greatest stress in the wire will be about 5050 lb./in.^{2}.

14. The tops of two vertical towers, of equal height and 1000 feet apart, are joined by a cable whose weight is 4000 lb. The length of the cable is such that for a span of 1000 feet the central dip is 50 feet. If the towers are slightly flexible so that a horizontal pull of 10,000 lb. causes a tower to deflect six inches out of the vertical, neglecting alterations in the length of the cable due to alterations of stress, calculate approximately its central dip. (C.U.)

CHAPTER 11

STRUCTURAL STATICS

11.1 Bending Moments and Shearing Forces

In this chapter we consider how the internal forces of a body may be measured by the calculation of shearing forces and bending moments, and how we can depart from the concept of a rigid body to measure the deformation of a body under a load.

Consider a beam of negligible weight built into a wall and projecting horizontally with a weight W attached to its free end (Fig. 271). If the beam were cut across any vertical section AB, the outer portion $ABCD$ would fall away. When it has not been cut the portion $ABCD$ is held in position by the internal forces across the section. These forces may be reduced to a force and a couple and must balance the external forces acting on $ABCD$. Therefore, the force must act vertically upwards and be equal to W. The couple must balance the couple formed by the two forces W and its moment must be $W \times AD$. We thus have found the internal force and couple at the section AB. The *shearing force* at any section of a loaded beam is defined as the algebraic sum of the components of the external forces parallel to the section on one side of the section. This force is balanced by an internal force in the plane of the section which is called the *shear resistance.* We shall use the convention of signs for normal sections of horizontal beams that in calculating the shearing force the positive direction for forces to the right of the section is upwards, and is downwards for forces to the left of the section. Thus, in Fig. 271, the shearing force at the section AB is $-W$.

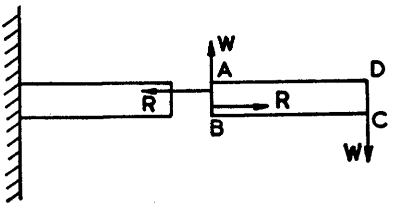

Fig. 271

The *bending moment* at any section of a loaded beam is defined as the sum of the moments of the external forces on one side of the section about the section. We shall use the convention of signs for horizontal beams that moments which tend to stretch the upper fibres of a beam are positive, and these are called hogging moments. Moments which tend to stretch the lower fibres are called sagging moments and are negative. Thus the bending moment at the section AB (Fig. 271) is $+W \times AD$. The moment about the section of a beam means the moment about a line in the section through its centroid which is called

the neutral axis of the section. This line divides the section into two parts in one of which the stress is tensile and in the other compressive. For bending moments caused by vertical forces on a horizontal beam the moment is the same about any horizontal line in the section. The bending moment is balanced by the couple exerted by the internal forces in the plane of the section and this couple is called the *moment of resistance* at the section.

Bending moments and shearing forces are shown graphically by plotting their values as ordinates along the beam.

11.2 Standard Cases

In the following cases let l be the length of the beam, W the total load, w the load per unit length if it is uniformly distributed over the length, S_x the shearing force and M_x the bending moment at x from the left-hand end of the beam.

(i) *Light Cantilever with Load at End* (Fig. 272).

$$S_x = -W,$$
$$M_x = W(l - x).$$

The maximum bending moment is Wl at $x = 0$.

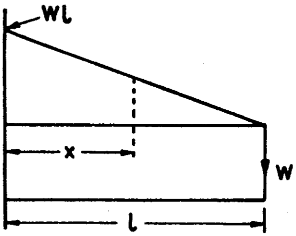

Fig. 272

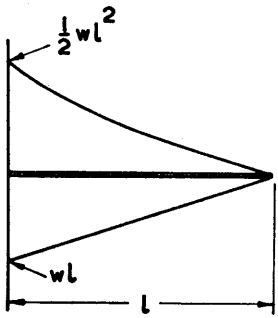

Fig. 273

(ii) *Cantilever with Uniformly Distributed Load* (Fig. 273).

$$S_x = -w(l - x),$$

$$M_x = \frac{1}{2}w(l - x)^2.$$

The maximum bending moment is $\frac{1}{2}wl^2$ at $x = 0$.

(iii) *End-supported Beam with Single Load* (Fig. 274).

Let the load be at $x = a$. The reaction at $x = 0$ is $R = \frac{Wb}{l}$,

$$S_x = -\frac{Wb}{l}, \qquad x < a,$$

$$= \frac{Wa}{l}, \qquad x > a.$$

$$M_x = -\frac{Wb}{l}x, \qquad x < a,$$

$$= -\frac{Wa}{l}(l - x), \quad x > a.$$

The maximum bending moment is $-\frac{Wab}{l}$ at $x = a$.

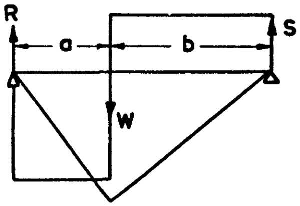

Fig. 274

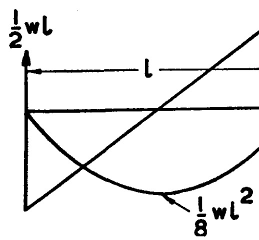

Fig. 275

(iv) *End-supported Beam with Uniform Load* (Fig. 275).

The end reaction is $\frac{1}{2}wl$ and we have

$$S_x = wx - \frac{1}{2}wl,$$

$$M_x = wx \times \frac{x}{2} - \frac{1}{2}wl \times x$$

$$= \frac{1}{2}wx(l - x).$$

The bending-moment graph is a parabola and the maximum bending moment is $\frac{1}{8}wl^2$ at $x = \frac{1}{2}l$.

11.3 Relation of Bending Moment, Shearing Force and Load

Let w be the load per unit length along a horizontal beam. The quantity w, called the intensity of load, may include the weight of the

beam itself and may vary with x the distance along the beam. Let S be the shearing force and M the bending moment at distance x along the beam. Consider a small further length AB of the beam (Fig. 276) of length δx. Let $S + \delta S$ be the shearing force and $M + \delta M$ the bending moment at B. In the figure S and M are shown as positive, that is, the forces to the left of A add up to a force S downwards and those to the right of B to a force $S + \delta S$ upwards, while the moment tends to stretch the upper fibres of the beam.

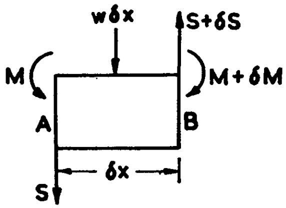

Fig. 276

From the equilibrium of AB we have

$$\delta S = w\delta x,$$

and taking moments about B,

$$M + S\delta x + w\delta x \times \frac{1}{2}\delta x = M + \delta M.$$

Hence in the limit as δx tends to zero

we have
$$\frac{dS}{dx} = w,$$

$$\frac{dM}{dx} = S,$$

and hence
$$\frac{d^2M}{dx^2} = w.$$

It follows that the expression for the bending moment has a maximum or minimum at points at which the shearing force is zero.

The shearing force and bending moment may thus be derived from the loading by integration.

We have
$$S = \int_0^x w dx + \text{constant},$$

$$M = \int_0^x S dx + \text{constant},$$

the constant being the values of S and M at $x = 0$.

A concentrated load at a point of the beam or the upthrust of a support causes a discontinuity of the shearing force. Thus if there is a load W at $x = a$ and a reaction X at the support where $x = 0$, we have

$$S = -X + \int_0^x w dx, \text{ for } x < a,$$

$$S = -X + \int_0^x w dx + W, \text{ for } x > a.$$

These two equations may be written as one by using square brackets to indicate the discontinuity, thus

$$S = -X + \int_0^x w dx + \Big[W\Big]_{x>a}.$$

The discontinuity may be carried over to the next integration giving

$$M = G - Xx + \int_0^x dx \int_0^x w dx + \Big[W(x-a)\Big]_{x>a}.$$

The shearing force and bending moment for a beam with any number of concentrated loads may be found in the same way.

Example 1. *A uniform beam AB, 10 ft. long and weighing 12 cwt., is supported at its ends and carries a load whose intensity varies uniformly from 0 at A to 4 cwt. per foot run at B. Draw the bending-moment and shearing-force diagrams for the beam and find the maximum bending moment.*

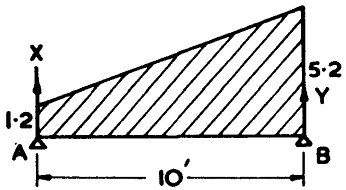

Fig. 277

The intensity of loading at x ft., from A (Fig. 277) is $1{\cdot}2 + 0{\cdot}4x$ cwt./ft.

$$S = \int_0^x (1{\cdot}2 + 0{\cdot}4x)dx - X$$

$$= 1{\cdot}2x + 0{\cdot}2x^2 - X.$$

$$M = 0{\cdot}6x^2 + \frac{0{\cdot}2x^3}{3} - Xx + G.$$

$M = 0$ when $x = 0$ and when $x = 10$, therefore $G = 0$,

and $$60 + \frac{200}{3} - 10X = 0,$$

$$X = 12\tfrac{2}{3}.$$

The maximum bending moment occurs when $S = 0$, that is

$$1{\cdot}2x + 0{\cdot}2x^2 - 12{\cdot}67 = 0,$$
$$x = 5{\cdot}51.$$

For this value of x,

$$M = 0{\cdot}6(5{\cdot}51)^2 + 0{\cdot}2\frac{(5{\cdot}51)^3}{3} - \frac{38}{3}(5{\cdot}51),$$
$$= -40{\cdot}39.$$

The shearing-force and bending-moment diagrams are drawn from the expressions for S and M (Fig. 278). The greatest shearing force is 19·33 cwt. at B and the greatest bending moment is 40·39 ft.cwt.

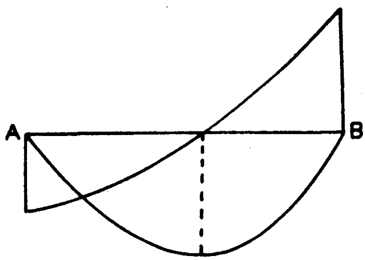

Fig. 278

Example 2. *A beam AB, 12 ft. long, is supported at points* 1 *ft. from A, and* 3 *ft. from B* (Fig. 279). *It carries a uniformly distributed load of* 2 *cwt./ft. and a concentrated load of*

12 cwt. at the mid-point of AB. Draw the shearing-force and bending-moment diagrams for the beam and find the maximum bending moment.

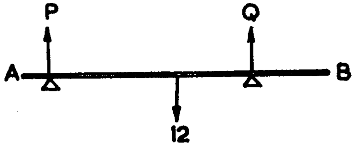

Fig. 279

Let P cwt. and Q cwt. be the reactions at the supports and let x be measured from A in ft.

$$\int_0^x w dx = 2x \text{ cwt.}$$

$$S = 2x - \Big[P\Big]_{x>1} + \Big[12\Big]_{x>6} - \Big[Q\Big]_{x>9}.$$

When $x = 12$, $S = 0$, therefore

$$24 - P + 12 - Q = 0,$$
$$P + Q = 36.$$

$$M = x^2 - \Big[P(x-1)\Big]_{x>1} + \Big[12(x-6)\Big]_{x>6} - \Big[Q(x-9)\Big]_{x>9}.$$

When $x = 12$, $M = 0$, therefore

$$144 - 11P + 72 - 3Q = 0.$$

Hence $$P = 13{\cdot}5,\ Q = 22{\cdot}5.$$

$$S = 2x - \Big[13{\cdot}5\Big]_{x>1} + \Big[12\Big]_{x>6} - \Big[22{\cdot}5\Big]_{x>9}.$$

$$M = x^2 - \Big[13{\cdot}5(x-1)\Big]_{x>1} + \Big[12(x-6)\Big]_{x>6} - \Big[22{\cdot}5(x-9)\Big]_{x>9}.$$

The shearing-force graph (Fig. 280) consists of a number of parallel lines with three discontinuities. The bending-moment graph is made up of arcs of the parabolas

$$y = x^2,$$
$$y = x^2 - 13{\cdot}5x + 13{\cdot}5,$$
$$y = x^2 - 1{\cdot}5x - 58{\cdot}5,$$
$$y = x^2 - 24x + 144.$$

None of these parabolas has a turning value within the range of values for which it is valid and so the maximum bending moment is $-31{\cdot}5$ ft.cwt. at the centre of the beam.

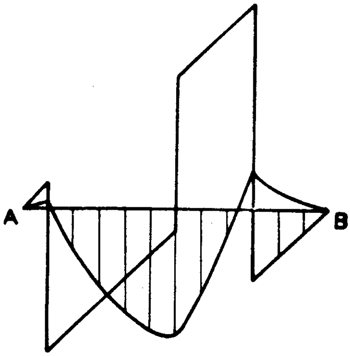

Fig. 280

11.4 Graphical Construction for Bending Moments

When a beam carries a number of loads at points along its length, a funicular polygon may be drawn (§ 9.14) to determine the reactions at the supports, and this polygon is also a bending-moment diagram showing the magnitude of the bending moment at each point.

Consider an end-supported beam AB (Fig. 281) carrying loads P, Q, R at points C, D, E. A funicular polygon is drawn in the usual way to determine the reactions X and Y at A and B.

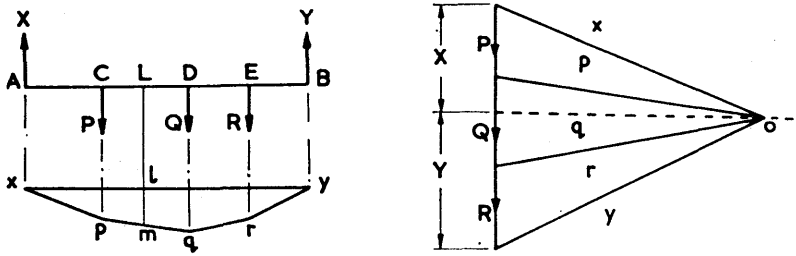

Fig. 281

Let L be any point of the beam and let the vertical through L meet xy in l and pq in m. Then the bending moment at L is proportional to the length lm.

Suppose the point L lies in CD. The bending moment at L is the sum of the moments of the forces P and X about L. P is the sum of forces xp and pq in the force diagram, X is the sum of forces px and xy, therefore P and Q are together equivalent to forces in the directions pq and xy which may be taken to act at m and l respectively. From the force diagram the horizontal component of each of these forces is equal to h, the perpendicular from o to the vertical PQR. Hence the difference of their moments about L is $h \times lm$. Similarly, the bending moment at any point of the beam is the product of the force h and the intercept made on the vertical through the point by the funicular polygon.

EXERCISES 11 (*a*)

1. A uniform steel bar, of weight 32 lb. and length 16 in., rests horizontally on two supports at its ends. A weight of 8 lb. is suspended from the bar at a point 4 in. from one end. Find by calculation at what point of the bar the shearing force is zero and evaluate the bending moment at this point. Draw a diagram showing the distribution of shearing force along the bar. (L.U.)

2. A light 12-ft. beam AB is supported at its ends, and carries weights of 2 tons at C and 4 tons at D, where $AC = 4$ ft., $CD = 2$ ft. Draw the shearing-stress and bending-moment diagrams. Draw the corresponding diagrams when the load is not concentrated at C and D but is uniformly distributed over the beam. (L.U.)

3. Draw the bending-moment diagram for two uniform beams, AB, BDC, hinged at B and resting in a horizontal position on three supports at A, D and C, where $AB = BD = \dfrac{DC}{2}$, and find the maximum bending moment. (L.U., Pt. I)

4. A bridge AB of 15 ft. span is end-supported. The portion CD, where $AC = 3$ ft., $CD = 10$ ft., is covered with a uniformly distributed load

of 2 cwt./ft. Find the magnitude and position of the maximum bending moment.

5. A uniform beam AC, 32 ft. long, whose middle point is B, rests on supports at the same level at its ends and carries a load of 4 tons uniformly distributed along the half AB. Assuming that the weight of the beam itself is negligible, draw a bending-moment diagram indicating the magnitude of the maximum bending moment and the point at which it acts. (L.U.)

6. A train of weight W and length l is in the centre of a bridge of twice its own length. Assuming that the weight of the train is uniformly distributed throughout its length, calculate the bending moment at the centre of the bridge, and compare it with the value when one end of the train just reaches one of the piers of the bridge. (L.U.)

7. A beam AB, 12 ft. long, is simply supported at its ends, so as to be horizontal. It carries a load which varies uniformly from zero at A to 1 ton/ft. at B. Determine formulae for the bending moment and shearing force at x ft. from A, plot the S.F. and B.M. diagrams, and calculate the position and amount of the maximum bending moment. (L.U., Pt. I)

8. A light beam ABC carrying a uniformly distributed load over its whole length is supported at A and maintained horizontally by a strut BD pivoted at B and to a point D vertically below A. $AB = 12$ ft., $AC = BD = 15$ ft., and the load is 2 tons/ft. Find the thrust in DB, and the tension in AB and the position and amount of the greatest bending moment; draw carefully to scale the bending-moment diagram for the beam. (L.U., Pt. I)

9. A light girder is supported at its ends in a horizontal position. A trolly is suspended from two wheels at distance a apart which run along the girder; the load is equally divided between the wheels. Draw the bending-moment diagram for a given position of the trolley, and prove that the bending moment is greatest under the wheel which is nearest the centre of the girder. Determine also the position of the trolley when this greatest bending moment is as great as possible. (L.U., Pt. II)

10. A rod of length l, whose weight is negligible, rests horizontally with its ends supported and carries a movable weight w; the rod will break if the bending moment at any point is L. Prove that the least value of w which can break the rod is $4L/l$. (L.U.)

11. A uniform beam of weight W and length l rests on two supports each distant a from the middle section of the beam. Find the bending moment at the middle and at one support, and find a in terms of l if these two bending moments are equal in magnitude, but opposite in sign. (L.U.)

12. A beam of length l carries a uniformly distributed load and is propped horizontally on two supports, one of which is at one end of the beam. Find the position of the second support which makes the maximum bending moment for the beam least.

11.5 Simple Bending

In calculations relating to the bending of beams certain simplifying assumptions are made to establish what is called the *theory of simple bending.*

It is assumed in the first place that the beam is perfectly elastic so that the strain in any fibre of the beam is proportional to the stress in it. That is,

$$\frac{\text{force in any fibre}}{\text{cross-sectional area of fibre}} = E \times \frac{\text{extension}}{\text{original length}},$$

E being Young's Modulus for the material.

Secondly, it is assumed that a transverse section of the beam which was plane before bending remains plane.

Thirdly, it is assumed, that adjacent layers of the beam can extend or contract independently.

Fourthly, it is assumed that the bending is in one plane and so slight that a simplified expression may be used for the radius of curvature of the beam at any point. Thus if x be the distance of a point along the beam and y its displacement from its original position due to bending, the radius of curvature R at the point is given by the formula

$$R = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}}.$$

It is assumed that $\frac{dy}{dx}$ is small and R is given by

$$R = \frac{1}{\frac{d^2y}{dx^2}}.$$

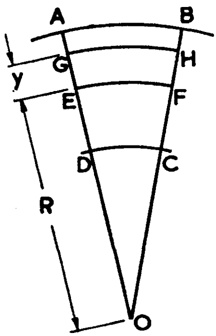

Fig. 282

11.6 The Moment of Resistance

The moment of resistance at any section of a loaded beam is the moment of the couple formed by the tensions and compressions in the fibres or layers of the beam caused by the bending moment at the section. Thus, if the beam does not break the moment of resistance is equal to the bending moment.

Consider a small portion $ABCD$ of a loaded beam (Fig. 282) such that AD and BC were originally parallel, but due to bending the upper layer AB is stretched and the lower layer DC is compressed and BC makes an angle $\delta\theta$ with AD. In

between there will be some layer EF which is neither stretched nor compressed and this is called the *neutral layer.* The intersection of the neutral layer with any section of the beam is called the *neutral axis* of the section.

The radius of curvature of the beam at any point is taken as that of the neutral layer. Let R be the radius of curvature of EF; then EF is approximately an arc of a circle of radius R and a layer GH which is at a height y above EF is an arc of a circle of radius $R + y$.

Then
$$EF = R\delta\theta,$$
$$GH = (R + y)\delta\theta,$$
and the extension of GH is $y\delta\theta$. Its original length was $R\delta\theta$, so that the strain is $\dfrac{y}{R}$.

If E be Young's modulus for the material the stress in GH is $E \times \dfrac{y}{R}$. Thus the stress in any layer is proportional to its distance from the neutral layer.

Let $ALMD$ (Fig. 283) be a transverse section of the beam through A and EN its neutral axis. The stress in an element of area δA distant y from the neutral axis is $\dfrac{Ey}{R}$, and the force in the element is $\dfrac{Ey}{R}\delta A$, perpendicular to the section.

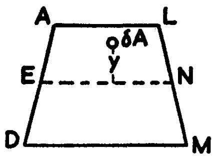

Fig. 283

The total force on the section is therefore
$$\frac{E}{R}\Sigma y\delta A,$$
the summation being over the whole of the section. The tensile force on the area $ALNE$ above the neutral axis must balance the compressive force on the area $ENMD$, so that the total force must be zero, and we have
$$\Sigma y\delta A = 0.$$
It follows that the neutral axis must pass through the centroid of the section.

The sum of the moments of the forces in the elements about the neutral axis is the moment of resistance M_R, and we have
$$M_R = \frac{E}{R}\Sigma y^2\delta A,$$
$$= \frac{EI}{R},$$
where I is the second moment of area of the section about the neutral axis. The quantity EI is called the *flexural rigidity* of the beam. Since the bending moment at any point is equal to the moment of resistance

this is the fundamental equation connecting the curvature of a beam at any point with the bending moment, and hence with the loading.

11.7 Maximum Stress Due to Bending

Let Y be the greatest distance of a layer of the beam from the neutral axis of a section and f the stress in this layer. Then f will be the greatest stress at the section.

We have
$$f = \frac{EY}{R},$$
and
$$\frac{E}{R} = \frac{f}{Y}.$$

We have, therefore, a second formula for the moment of resistance, namely,
$$M_R = \frac{I}{Y}f,$$

I being the second moment of area about the neutral axis, Y the greatest distance of material of the beam from the neutral axis and f the greatest stress at the section due to bending.

The maximum stress due to bending at any section of a loaded beam is found by equating the bending moment M at the section to the expression for the moment of resistance, so that
$$f = \frac{MY}{I}.$$

Alternatively, knowing the greatest safe stress for the material the greatest bending moment which can be tolerated at the section is found and the safe loading for the beam is deduced.

The quantity I/Y is called the *section modulus* and is easily found for beams of regular shape.

Thus for a rectangular beam b inches wide and d inches deep (Fig. 284) the neutral axis is at $\frac{1}{2}d$ from the outer layer and the second moment about it is $\frac{1}{12}bd^3$.

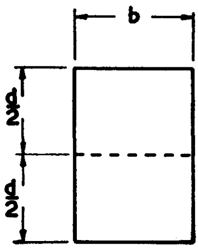

Fig. 284

Then
$$\frac{I}{Y} = \frac{1}{6}bd^2 \text{ in.}^3,$$
and if f be the greatest stress in lb. per sq. in.,
$$M = \frac{1}{6}bd^2f \text{ lb.in.}$$

For a solid circular beam of diameter d

$$I = \frac{\pi d^4}{64},$$

$$\frac{I}{Y} = \frac{\pi}{32} d^3.$$

For a tubular beam of external and internal diameters d_1 and d_2 respectively,

$$I = \frac{\pi}{64}(d_1^4 - d_2^4),$$

$$\frac{I}{Y} = \frac{\pi}{32}\frac{d_1^4 - d_2^4}{d_1}.$$

Steel girders are rolled or built up in the form of I-sections, L-sections, etc., and their second moments of area and section moduli are tabulated for bending about neutral axes parallel and perpendicular to the flanges.

Example 3. *A solid rectangular beam* 10 *ft. long is* 4 *in. wide and* 6 *in. deep. It is supported at its ends and carries a uniformly distributed load of* 1 *ton. Find the greatest stress due to bending.*

The maximum bending moment is

$$M = \frac{2240 \times 10}{8} \text{ ft.lb.}$$

$$= 33{,}600 \text{ in.lb.}$$

The section modulus

$$= \frac{1}{6} \times 4 \times 6^2$$

$$= 24 \text{ in.}^3.$$

The greatest stress

$$= \frac{33{,}600}{24}$$

$$= 1400 \text{ lb./in.}^2$$

11.8 Distribution of Shear Stress

The average sheer stress over the cross-section of a beam is the shearing force at the section divided by the area of the section. The greatest sheer stress in the section may be considerably greater than this.

The shearing force at a vertical section causes a tendency to shear not only along the vertical section but also between horizontal layers of the beam. This may be seen by considering a small prism of square cross-

section running through the beam (Fig. 285). If the force tending to cause shear is q upwards on one vertical side, it will be, to this order of magnitude, q downwards on the other side. These two forces q will tend to make the prism rotate so that an equal and opposite couple is induced

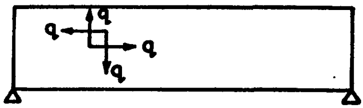

Fig. 285

consisting of forces q on the two horizontal faces. It is the horizontal shear stress which will often cause failure of a timber beam since there is less resistance to shear parallel to the fibres.

We can find a formula for the intensity of shear stress at any level of a loaded beam. Let $ABCD$ be a small portion of a loaded beam (Fig. 286) bounded by vertical sections AD and BC, δx apart, and let $ALMD$

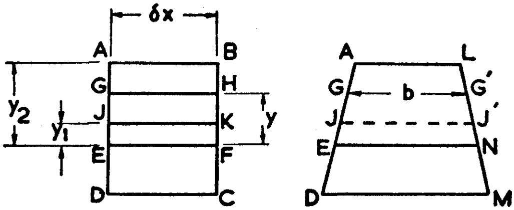

Fig. 286

be the section through AD. Let the bending moment be M at A and $M + \delta M$ at B.

Let EF be the neutral layer and let JK be a horizontal layer at height y_1 above EF and let AB be at height y_2 above EF.

Consider the force due to bending on the area $ALJ'J$ of the vertical section at A. At height y above the neutral axis the tensile stress is $= \dfrac{Ey}{R}$, where E is Young's modulus and R is the radius of curvature.

But $$\frac{EI}{R} = M,$$

therefore $$f = \frac{My}{I},$$

I being the second moment of area of the section about the neutral axis.

Let the section at height y above EF have breadth b. Then the *force* on an element of width δy at this height is

$$b\delta y . f = \frac{M}{I} by\delta y.$$

Therefore the total force on the area $ALJ'J$ is

$$\frac{M}{I}\int_{y_1}^{y_2} by\,dy,$$

$$=\frac{MA_1\bar{y}_1}{I},$$

where A_1 is the area of $ALJ'J$ and $\bar{y}_1$ is the height of its centroid above the neutral axis. Similarly the total force, in the opposite direction, on the portion of the section BC of which BK is one side

$$=\frac{(M+\delta M)A_1\bar{y}_1}{I}.$$

Thus there is a force on the portion $ABKJ$ of the beam tending to move it to the right, and this force is

$$\frac{\delta M}{I}A_1\bar{y}_1.$$

This causes a shear stress over the horizontal face through JK. If $JJ' = b_0$ this area is $b_0\delta x$, and the horizontal shear stress is

$$s=\frac{\delta M A_1\bar{y}_1}{\delta x b_0 I}.$$

In the limit this becomes

$$s=\frac{dM}{dx}\cdot\frac{A_1\bar{y}_1}{b_0I}$$

$$=S\frac{A_1\bar{y}_1}{b_0I},$$

where S is the shearing force at A.

Example 4. *A beam has shearing force S at a rectangular section of width b and depth d. Find the distribution of shear stress over the section.*

Let s_1 be the shear stress at height y_1 above the neutral axis (Fig. 287). The neutral axis passes through the centroid of the section and the area above height y_1 is

$$A_1 = b\left(\frac{1}{2}d - y_1\right).$$

The distance of the centroid of this area from the neutral axis is

$$\bar{y}_1 = \frac{1}{2}y_1 + \frac{1}{4}d.$$

Also $$I = \frac{1}{12}bd^3,$$

therefore $$s_1 = S\frac{\frac{1}{2}b(\frac{1}{2}d+y_1)(\frac{1}{2}d-y_1)}{\frac{1}{12}b^2d^3}$$

$$=\frac{S}{bd}\times\frac{3}{2}\left(1-\frac{4y_1^2}{d^2}\right)$$

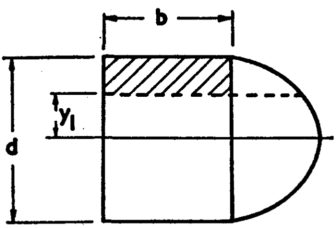

Fig. 287

The average shear stress over the section is S/bd, and it is seen that the stress is greatest for $y_1 = 0$ and is then 1·5 times the average. The intensity of stress at different levels is shown graphically (Fig. 287) and the curve of variation is a parabola.

Example 5. *A T-section has the dimensions shown* (Fig. 288). *Draw a graph showing the distribution of shear stress over the section.*

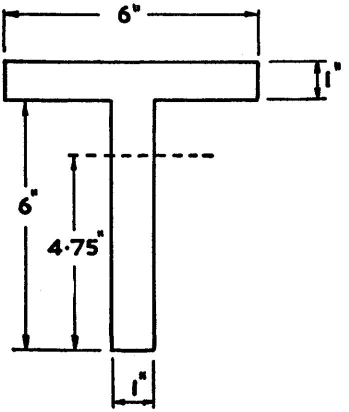

Fig. 288

The neutral axis is 4·75 in. from the bottom of the section and the second moment of area about it is

$$I = 55{\cdot}25 \text{ in.}^4.$$

The first moment of area about the neutral axis of the flange above a distance y_1 from the neutral axis

$$= 6(2{\cdot}25 - y_1)\frac{1}{2}(2{\cdot}25 + y_1)$$

$$= \frac{3}{16}(81 - 16y_1^2).$$

Hence if S be the shearing force the shear stress at this level is

$$s = S\frac{3(81 - 16y_1^2)}{16 \times 6 \times 55{\cdot}25}$$

$$= S\frac{81 - 16y_1^2}{1768}.$$

In the web at distance y_2 *below* the neutral axis the first moment of area is

$$\frac{1}{2}(4{\cdot}75 - y_2)(4{\cdot}75 + y_2),$$

$$= \frac{1}{32}(361 - 16y_2^2).$$

The stress at this level is

$$s = S\frac{361 - 16y_2^2}{32 \times 55{\cdot}25}$$

$$= S\frac{361 - 16y_2^2}{1768}.$$

The greatest stress in the web is at the neutral axis and is

$$s = \frac{361}{1768}S$$

$$= 2{\cdot}45\left(\frac{S}{12}\right),$$

that is 2·45 times the average shear stress over the section.

At the top of the web $y_2 = -1{\cdot}25$ and we have

$$s = \frac{336}{1768}S$$

$$= 2{\cdot}28\left(\frac{S}{12}\right).$$

At the bottom of the flange $y_1 = 1{\cdot}25$ and we have

$$s = \frac{56S}{1768}$$
$$= 0{\cdot}38\left(\frac{S}{12}\right).$$

There is thus a large difference of stress between the bottom of the flange and the top of the web.

The graph of stress distribution is shown in Fig. 289.

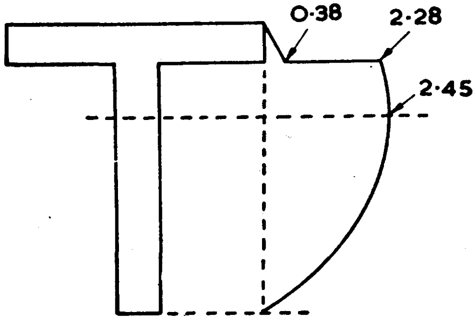

Fig. 289

EXERCISES 11 (*b*)

1. A bridge spans a gap 14 ft. wide and is supported on five road bearers of circular section of timber in which the stress must not exceed 1200 lb./in.². The total uniform load of 9 cwt. per foot run is assumed to be equally shared by the road bearers. Find what their diameter should be.

2. A spar 3 in. in diameter and 10 ft. long is used as a lever with the fulcrum 18 in. from one end. The safe stress in the material is 1600 lb./in.². Find the greatest load that can be lifted without exceeding this stress and the force which must be exerted at the other end of the lever.

3. A timber beam 20 ft. long, 6 in. wide and 9 in. deep weighs 240 lb. It is supported horizontally at points 3 ft. and 15 ft. from one end. The safe stress for the timber is 1400 lb.in.². Find the greatest weight which can safely be suspended from the mid-point of the beam.

4. A rolled-steel joist which is an I-section weighs 32 lb. per foot run and its section modulus with its flanges horizontal is 36·36 in.³. The joist is built into a wall and projects 10 ft. horizontally. Find the greatest weight which can be suspended from its free end if the stress due to bending must not exceed 8 tons/in.².

5. A floor has to carry a load of 3 cwt. per square foot. The floor joists are 12 in. deep and 4·5 in. thick and have a span of 14 ft. Determine the distance apart from centre to centre at which these joists must be spaced if the maximum stress is not to exceed 1000 lb./in.².

6. Show that the stress due to a shearing force S on a circular beam of radius a at distance y_1 from the neutral axis is

$$\frac{S}{\pi a^2}\cdot\frac{4}{3}\left(1-\frac{y_1^2}{a^2}\right)$$

A circular beam of radius 3 in. is 10 ft. long and its ends are supported at the same level. It carries a uniform load of 200 lb. per foot run. Find (i) the greatest tensile stress due to bending, (ii) the greatest shear stress.

7. The rungs of a ladder are 10 in. long and 1 in. in diameter. The safe bending stress and shear stress for the timber are 1800 lb./in.^{2} and 150 lb./in.^{2} respectively. What concentrated load at the centre of a rung will make it unsafe (*a*) by bending, (*b*) by shear? The ends of the rungs may be taken as encastré.

8. A rectangular beam of length l and depth d is end-supported. The safe bending stress is r and the safe shear stress is s. Show that the safe bending stress will be exceeded before the safe shear under an increasing uniformly distributed load if $l/d > r/s$.

9. A rolled-steel joist is an I-section 5 in. wide and 12 in. deep overall. The thickness of the flanges is $\frac{1}{2}$ in. and that of the web $\frac{3}{8}$ in. Find the shear stress at the extremities of the web and on the neutral axis due to a shearing force of 16 tons.

11.9 Deflexions of Beams

Let y be the displacement of a point of the neutral layer of a loaded beam, distant x from one end, from its original position due to bending. We shall measure y vertically downwards, that is in the same direction as that in which the load acts (Fig. 290). For the moment of resistance of a loaded beam we have

$$M_R = \frac{EI}{R},$$

and we shall take the radius of curvature as being positive when the centre of curvature is below the beam. In this case $\frac{dy}{dx}$ is increasing (Fig. 291) and therefore $\frac{d^2y}{dx^2}$ is positive, so that since $\frac{1}{R} = \frac{d^2y}{dx^2}$,

$$M_R = EI\frac{d^2y}{dx^2}.$$

Fig. 290

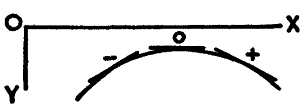

Fig. 291

This corresponds to a positive or hogging bending moment so that if M_x be the bending moment at distance x from one end of the beam,

$$M_x = EI\frac{d^2y}{dx^2}.$$

Let w be the loading at this point and S_x the shearing force.

Then
$$\begin{aligned} w &= \frac{d}{dx}S_x \\ &= \frac{d^2}{dx^2}M_x. \end{aligned}$$

Hence we have
$$\frac{d^2}{dx^2}\,EI\,\frac{d^2y}{dx^2} = w.$$

If EI be constant along the beam we have

$$EI\frac{d^4y}{dx^4} = w.$$

The deflexion at any point is found by integrating this equation. Four arbitrary constants appear in the solution and these must be determined by the boundary conditions, usually the values of y and its derivatives at the ends of the beam.

If a beam of length l is freely supported at both ends the deflexions and the bending moments at both ends are zero and the constants are determined by the conditions $y = \dfrac{d^2y}{dx^2} = 0$, for $x = 0$, and $x = l$.

If the beam is encastré, that is built into a wall or clamped so that the ends are horizontal, we have the conditions $y = \dfrac{dy}{dx} = 0$ for $x = 0$ and $x = l$.

A cantilever with one end free has $y = \dfrac{dy}{dx} = 0$ for $x = 0$, and since the shearing force and bending moment are both zero at the free end $\dfrac{d^2y}{dx^2} = \dfrac{d^3y}{dx^3} = 0$ for $x = l$.

11.10 Standard Cases

(i) *Cantilever with Uniform Load*

Let l be the length and w the loading (Fig. 292).

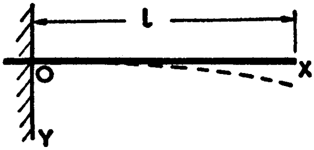

Fig. 292

We have $EI\dfrac{d^4y}{dx^4} = w,$

$$EI\frac{d^3y}{dx^3} = wx + A,$$

$$EI\frac{d^2y}{dx^2} = \frac{1}{2}wx^2 + Ax + B,$$

$$EI\frac{dy}{dx} = \frac{1}{6}wx^3 + \frac{1}{2}Ax^2 + Bx + C,$$

$$EIy = \frac{1}{24}wx^4 + \frac{1}{6}Ax^3 + \frac{1}{2}Bx^2 + Cx + D.$$

Since $y = \dfrac{dy}{dx} = 0$ for $x = 0$, we have $C = D = 0$.

Also, $\dfrac{d^2y}{dx^2} = \dfrac{d^3y}{dx^3}$ for $x = l$, therefore

$$A = -wl,$$

$$B = \frac{1}{2}wl^2,$$

and we have
$$EIy = \frac{1}{24}wx^4 - \frac{1}{6}wlx^3 + \frac{1}{4}wl^2x^2,$$

$$y = \frac{w}{24EI}x^2(x^2 - 4lx + 6l^2).$$

The maximum deflexion is at $x = l$, and this is

$$y = \frac{wl^4}{8EI}.$$

(ii) *Cantilever with Single Load*

Let the load be W at distance l from the fixed end (Fig. 293) and assume the weight of the beam is negligible. We have for $x < l$,

$$EI\frac{d^4y}{dx^4} = 0,$$

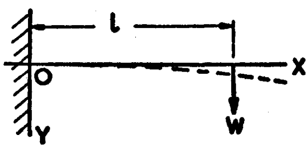

Fig. 293

$$EI\frac{d^3y}{dx^3} = A,$$

$$EI\frac{d^2y}{dx^2} = Ax + B,$$

$$EI\frac{dy}{dx} = \frac{1}{2}Ax^2 + Bx + C,$$

$$EIy = \frac{1}{6}Ax^3 + \frac{1}{2}Bx^2 + Cx + D.$$

Since $y = \frac{dy}{dx} = 0$ when $x = 0$, $C = D = 0$. Also the shearing force is $-W$ and the bending moment Wl when $x = 0$, therefore $A = -W$, $B = Wl$.

Therefore, $$EIy = -\frac{1}{6}Wx^3 + \frac{1}{2}Wlx^2.$$

The maximum deflexion at $x = l$ is $\frac{Wl^3}{3EI}$.

(iii) *End-supported Beam with Uniform Load*

Let l be the length of the beam and w the loading (Fig. 294). We have

$$EI\frac{d^4y}{dx^4} = w,$$

$$EI\frac{d^3y}{dx^3} = wx + A,$$

$$EI\frac{d^2y}{dx^2} = \frac{1}{2}wx^2 + Ax + B,$$

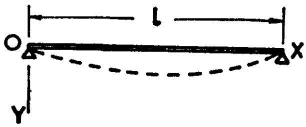

Fig. 294

$$EI\frac{dy}{dx} = \frac{1}{6}wx^3 + \frac{1}{2}Ax^2 + Bx + C,$$

$$EIy = \frac{1}{24}wx^4 + \frac{1}{6}Ax^3 + \frac{1}{2}Bx^2 + Cx + D.$$

The bending moment is zero at both ends, therefore $\frac{d^2y}{dx^2} = 0$, for $x = 0$ and $x = l$, therefore $B = 0$, and $A = -\frac{1}{2}wl$.

Also $y = 0$ for $x = 0$ and $x = l$, therefore $D = 0$ and

$$0 = \frac{1}{24}wl^4 - \frac{1}{12}wl^4 + Cl,$$

$$C = \frac{1}{24}wl^3.$$

Therefore $$EIy = \frac{w}{24}(x^4 - 2x^3l + xl^3)$$

$$= \frac{w}{24}x(x - l)(x^2 - xl - l^2).$$

When $x = \frac{1}{2}l$, $\frac{dy}{dx} = 0$ and we have the maximum deflexion

$$y = \frac{5wl^4}{384EI}.$$

(iv) *End-supported Beam with Load at Centre*

If W be the load we have (Fig. 295)

$$EI\frac{d^4y}{dx^4} = 0,$$

$$EI\frac{d^3y}{dx^3} = A + \Big[W\Big]_{x>\frac{l}{2}},$$

$$EI\frac{d^2y}{dx^2} = Ax + B + \Big[W\Big(x - \frac{l}{2}\Big)\Big]_{x>\frac{l}{2}},$$

$$EI\frac{dy}{dx} = \frac{1}{2}Ax^2 + Bx + C + \Big[\frac{1}{2}W\Big(x - \frac{l}{2}\Big)^2\Big]_{x>\frac{l}{2}},$$

$$EIy = \frac{1}{6}Ax^3 + \frac{1}{2}Bx^2 + Cx + D + \Big[\frac{1}{6}W\Big(x - \frac{l}{2}\Big)^3\Big]_{x>\frac{l}{2}}.$$

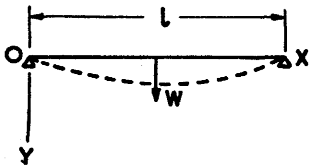

Fig. 295

When $x = 0$, $y = \dfrac{d^2y}{dx^2} = 0$, therefore $D = B = 0$.

When $x = l$, $y = \dfrac{d^2y}{dx^2} = 0$, therefore

$$Al + \frac{1}{2}Wl = 0,$$

$$\frac{1}{6}Al^3 + Cl + \frac{1}{48}Wl^3 = 0,$$

$$A = -\frac{1}{2}W,$$

$$C = \frac{1}{16}Wl^2.$$

Therefore $$EIy = \frac{Wx}{48}(3l^2 - 4x^2) + \Big[\frac{W}{48}(2x - l)^3\Big]_{x>\frac{l}{2}}.$$

The maximum deflexion at $x = \frac{1}{2}l$ is

$$y = \frac{Wl^3}{48EI}.$$

(v) *Encastré Beam with Uniform Load*

Let l be the length and w the loading (Fig. 296).

$$EI\frac{d^4y}{dx^4} = w,$$

$$EI\frac{d^3y}{dx^3} = wx + A,$$

$$EI\frac{d^2y}{dx^2} = \frac{1}{2}wx^2 + Ax + B,$$

$$EI\frac{dy}{dx} = \frac{1}{6}wx^3 + \frac{1}{2}Ax^2 + Bx + C,$$

$$EIy = \frac{1}{24}wx^4 + \frac{1}{6}Ax^3 + \frac{1}{2}Bx^2 + Cx + D.$$

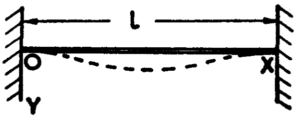

Fig. 296

The boundary condition $y = \dfrac{dy}{dx} = 0$ when $x = 0$ gives $C = D = 0$.

Also, since $y = \dfrac{dy}{dx} = 0$ when $x = l$,

$$\frac{1}{6}wl^3 + \frac{1}{2}Al^2 + Bl = 0,$$

$$\frac{1}{24}wl^4 + \frac{1}{6}Al^3 + \frac{1}{2}Bl^2 = 0,$$

$$A = -\frac{1}{2}wl, \quad B = \frac{1}{12}wl^2.$$

Therefore $$EIy = \frac{w}{24}x^2(l - x)^2.$$

The maximum deflexion at $x = \frac{1}{2}l$ is

$$y = \frac{wl^4}{384EI}.$$

B is the bending moment when $x = 0$ and this is the fixing couple making the beam horizontal at the wall. The bending moment when $x = \frac{1}{2}l$ is $-\frac{1}{24}wl^2$.

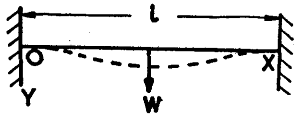

Fig. 297

vi) *Encastré Beam with Load at Centre*

Let W be the load (Fig. 297).

$$EI\frac{d^4y}{dx^4} = 0,$$

$$EI\frac{d^3y}{dx^3} = A + \Big[W\Big]_{x>\frac{l}{2}},$$

$$EI\frac{d^2y}{dx^2} = Ax + B + \left[W\left(x - \frac{1}{2}l\right)\right]_{x>\frac{l}{2}},$$

$$EI\frac{dy}{dx} = \frac{1}{2}Ax^2 + Bx + C + \left[\frac{1}{2}W\left(x - \frac{1}{2}l\right)^2\right]_{x>\frac{l}{2}},$$

$$EIy = \frac{1}{6}Ax^3 + \frac{1}{2}Bx^2 + Cx + D + \left[\frac{1}{6}W\left(x - \frac{1}{2}l\right)^3\right]_{x>\frac{l}{2}}.$$

Since $y = \dfrac{dy}{dx} = 0$ when $x = 0$, $C = D = 0$.

Since $y = \dfrac{dy}{dx} = 0$ when $x = l$,

$$\frac{1}{2}Al^2 + Bl + \frac{1}{8}Wl^2 = 0,$$

$$\frac{1}{6}Al^3 + \frac{1}{2}Bl^2 + \frac{1}{48}Wl^3 = 0,$$

$$A = -\frac{1}{2}W, \quad B = \frac{1}{8}Wl.$$

Therefore $$EIy = \frac{W}{48}x^2(3l - 4x) + \left[\frac{W}{48}(2x - l)^3\right]_{x>\frac{l}{2}}.$$

The greatest deflexion is at $x = \frac{1}{2}l$ and is

$$y = \frac{Wl^3}{192EI}.$$

$B = \frac{1}{8}Wl$ is the fixing couple at the wall, and the bending moment at the centre is $-\frac{1}{8}Wl$.

Example 6.—*A cantilever of uniform stiffness EI projects horizontally from a wall and its free end is propped level with the wall end. It carries a uniform load w over its length a. Find the fixing couple at the wall and the upward thrust of the support. Find also the greatest deflexion of the cantilever and the greatest bending moment.*

Let P be the upward thrust at the wall and G the fixing couple. At x from the wall (Fig. 298),

$$EI\frac{d^4y}{dx^4} = w,$$

$$EI\frac{d^3y}{dx^3} = wx - P,$$

$$EI\frac{d^2y}{dx^2} = \tfrac{1}{2}wx^2 - Px + G.$$

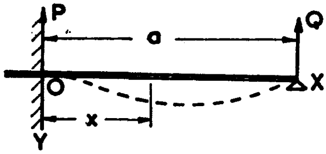

Fig. 298

Since the bending moment is zero at the support,

we have $$0 = \frac{1}{2}wa^2 - Pa + G. \tag{1}$$

$$EI\frac{dy}{dx} = \frac{1}{6}wx^3 - \frac{1}{2}Px^2 + Gx + C,$$

$$EIy = \frac{1}{24}wx^4 - \frac{1}{6}Px^3 + \frac{1}{2}Gx^2 + Cx + D.$$

Slope and deflexion being zero at the wall, $C = D = 0$. Also, the deflexion is zero at the support, so that

$$0 = \frac{1}{24}wa^4 - \frac{1}{6}Pa^3 + \frac{1}{2}Ga^2. \tag{2}$$

From (1) and (2) $$P = \frac{5}{8}wa,$$

$$G = \frac{1}{8}wa^2,$$

and the upward thrust at the support is $wa - P$, that is $\frac{3}{8}wa$. Substituting for P and G we have

$$EI\frac{dy}{dx} = \frac{w}{48}x(8x^2 - 15ax + 6a^2),$$

$$EIy = \frac{w}{48}x^2(x - a)(2x - 3a).$$

The greatest deflexion occurs when $\frac{dy}{dx} = 0$, that is,

when $$8x^2 - 15ax + 6a^2 = 0,$$

$$x = 0{\cdot}5785a.$$

With this value of x we have

$$y = \frac{wa^4}{185EI}, \text{ approximately.}$$

The maximum bending moment occurs at $x = \frac{5}{8}a$ and is $-\frac{9}{128}wa^2$. This is less than the bending moment $\frac{1}{8}wa^2$ at the wall.

Example 7. *A uniform beam of length 2a is freely supported with its ends level. It carries a load w per unit length over its whole length and in addition a load w per unit length over one-half of the beam between one support and the centre. Find the greatest deflexion of the beam in terms of its rigidity EI.*

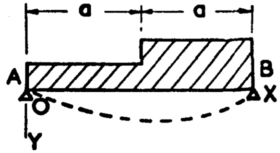

Fig. 299

If x is measured from A (Fig. 299) we have

$$EI\frac{d^4y}{dx^4} = w + \Big[w\Big]_{x>a},$$

$$EI\frac{d^3y}{dx^3} = wx + A + \Big[w(x-a)\Big]_{x>a},$$

$$EI\frac{d^2y}{dx^2} = \frac{1}{2}wx^2 + Ax + B + \left[\frac{1}{2}w(x-a)^2\right]_{x>a}.$$

The bending moment is zero when $x = 0$ and when $x = 2a$, therefore $B = 0$ and

$$2wa^2 + 2Aa + \frac{1}{2}wa^2 = 0,$$

$$A = -\frac{5}{4}wa.$$

$$EI\frac{dy}{dx} = \frac{1}{6}wx^3 - \frac{5}{8}wax^2 + C + \left[\frac{1}{6}w(x-a)^3\right]_{x>a},$$

$$EIy = \frac{1}{24}wx^4 - \frac{5}{24}wax^3 + Cx + D + \left[\frac{1}{24}w(x-a)^4\right]_{x>a}.$$

Since $y = 0$ when $x = 0$ and when $x = 2a$ we have $D = 0$, and

$$\frac{16}{24}wa^4 - \frac{40}{24}wa^4 + 2Ca + \frac{1}{24}wa^4 = 0,$$

$$C = \frac{23}{48}wa^3.$$

The slope is zero when

$$\frac{1}{6}wx^3 - \frac{5}{8}wax^2 + \frac{23}{48}wa^3 + \left[\frac{1}{6}w(x-a)^3\right]_{x>a} = 0.$$

For x less than a this gives

$$8x^3 - 30ax^2 + 23a^3 = 0.$$

This quantity has the same sign at $x = 0$ and $x = a$ and has no real root between $x = 0$ and $x = a$. Hence we assume $x > a$ and the equation becomes

$$16x^3 - 54ax^2 + 24a^2x + 15a^3 = 0.$$

A solution is $x = 1{\cdot}03a$, approximately.

$$y = \frac{w}{48EI}\{2x^4 - 10ax^3 + 23a^3x + 2(x-a)^4\}$$

$$= \frac{5wa^4}{16EI}, \text{ approximately.}$$

11.11 The Theorem of Three Moments

A beam supported at more than two points is called a continuous beam. Where such a beam passes over a support there is a positive or hogging bending moment. The theorem of three moments gives a relation between the bending moments at three successive supports for a uniform beam carrying a uniformly distributed load.

Let A, B, C (Fig. 300) be three successive supports of a uniform

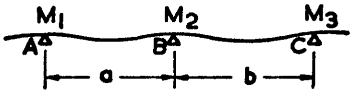

Fig. 300

beam carrying a load of constant intensity w. Let $AB = a$, $BC = b$ and let the bending moments at A, B, C be M_1, M_2, M_3 respectively.

If x be the distance of a point of the beam between A and B from A, we have

$$EI\frac{d^4y}{dx^4} = w,$$

$$EI\frac{d^3y}{dx^3} = wx + A,$$

$$EI\frac{d^2y}{dx^2} = \frac{1}{2}wx^2 + Ax + B,$$

$$EI\frac{dy}{dx} = \frac{1}{6}wx^3 + \frac{1}{2}Ax^2 + Bx + C,$$

$$EIy = \frac{1}{24}wx^4 + \frac{1}{6}Ax^3 + \frac{1}{2}Bx^2 + Cx + D.$$

Since $EI\dfrac{d^2y}{dx^2}$ is the bending moment, $B = M_1$, the bending moment at $x = 0$, and putting $x = a$,

$$M_2 - M_1 = \frac{1}{2}wa^2 + Aa.$$

Since $y = 0$ when $x = 0$, $D = 0$, and since $y = 0$ when $x = a$,

$$\frac{1}{24}wa^4 + \frac{1}{6}Aa^3 + \frac{1}{2}M_1a^2 + Ca = 0,$$

$$C = -\frac{1}{24}wa^3 - \frac{1}{2}M_1a - \frac{1}{6}a\left(M_2 - M_1 - \frac{1}{2}wa^2\right)$$

$$= \frac{1}{24}wa^3 - \frac{1}{3}M_1a - \frac{1}{6}M_2a.$$

Then when $x = a$,

$$EI\left(\frac{dy}{dx}\right)_{x=a} = \frac{1}{6}wa^3 + \frac{1}{2}a\left(M_2 - M_1 - \frac{1}{2}wa^2\right) + M_1a + C$$

$$= -\frac{1}{24}wa^3 + \frac{1}{3}M_2a + \frac{1}{6}M_1a.$$

Similarly, for the length BC, measuring x from C we find for the value of $\frac{dy}{dx}$ at B,

$$EI\left(\frac{dy}{dx}\right)_{x=b} = -\frac{1}{24}wb^3 + \frac{1}{3}M_2b + \frac{1}{6}M_3b.$$

Either expression gives the slope at B, but with opposite signs since x is measured in opposite directions, therefore

$$-\frac{1}{24}wa^3 + \frac{1}{3}M_2a + \frac{1}{6}M_1a = -\left\{-\frac{1}{24}wb^3 + \frac{1}{3}M_2b + \frac{1}{6}M_3b\right\},$$

that is $$M_1a + 2M_2(a + b) + M_3b = \frac{1}{4}w(a^3 + b^3).$$

Example 8. *A uniform beam of length 2a is simply supported at its ends and at its centre and carries a uniform load w. Find the bending moment and the shearing force at the central support* (Fig. 301).

Here $M_1 = M_3 = 0$, therefore

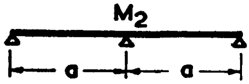

Fig. 301

$$4M_2a = \frac{1}{2}wa^3,$$

$$M_2 = \frac{1}{8}wa^2.$$

Also for either span, $$EI\frac{d^4y}{dx^4} = w,$$

$$EI\frac{d^3y}{dx^3} = wx + A,$$

$$EI\frac{d^2y}{dx^2} = \frac{1}{2}wx^2 + Ax,$$

and when $x = a$, $$\frac{1}{8}wa^2 = \frac{1}{2}wa^2 + Aa,$$

$$A = -\frac{3}{8}wa.$$

The shearing force at the central support is the value of $EI\frac{d^3y}{dx^3}$ at $x = a$,

that is $$wa - \frac{3}{8}wa,$$

$$= \frac{5}{8}wa.$$

This is the shearing force at one side of the central support. At the other side it is the same but of opposite sign and the upthrust at the support is therefore $\frac{5}{4}wa$.

Example 9. *A beam of length 3a is supported at the same level at its ends and at its points of trisection and carries a uniform load w. Find the bending moment at the central supports and the greatest deflexion of the central span.*

By symmetry the bending moments at the inner supports are equal and from the theorem of three moments are given by

$$4aM + aM = \frac{1}{2}wa^3,$$

$$M = \frac{1}{10}wa^2.$$

For the central span (Fig. 302),

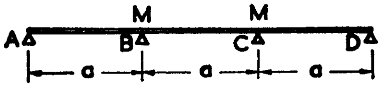

Fig. 302

$$EI\frac{d^2y}{dx^2} = \frac{1}{2}wx^2 + Ax + \frac{1}{10}wa^2,$$

and since the bending moment is the same at B and C,

$$\frac{1}{2}wa^2 + Aa = 0,$$

and
$$EI\frac{d^2y}{dx^2} = \frac{1}{2}wx^2 - \frac{1}{2}wax + \frac{1}{10}wa^2,$$

$$EI\frac{dy}{dx} = \frac{1}{6}wx^3 - \frac{1}{4}wax^2 + \frac{1}{10}wa^2x + C,$$

$$EIy = \frac{1}{24}wx^4 - \frac{1}{12}wax^3 + \frac{1}{20}wa^2x^2 + Cx + D.$$

Since $y = 0$ when $x = 0$ and when $x = a$, $D = 0$ and

$$\left(\frac{1}{24} - \frac{1}{12} + \frac{1}{20}\right)wa^4 + Ca = 0,$$

$$C = -\frac{1}{120}wa^3.$$

The deflexion when $x = \frac{1}{2}a$ is then

$$y = \frac{wa^4}{1920EI}.$$

11.12 Supports at Different Levels

Let the three supports A, B, C of the continuous beam be at depths $\delta_1, \delta_2, \delta_3$ respectively below some standard level (Fig. 303).

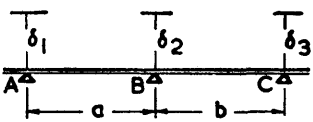

Fig. 303

With the notation of § 11.11 we have for the deflexion in AB,

$$EIy = \frac{1}{24}wx^4 + \frac{1}{6}Ax^3 + \frac{1}{2}M_1x^2 + Cx + D,$$

and hence

$$EI(\delta_2 - \delta_1) = \frac{1}{24}wa^4 + \frac{1}{6}Aa^3 + \frac{1}{2}M_1a^2 + Ca.$$

A has the same value as before and hence the value of C is increased by the amount $EI(\delta_2 - \delta_1)/a$.

We have, therefore,

$$EI\left(\frac{dy}{dx}\right)_{x=a} = -\frac{1}{24}wa^3 + \frac{1}{3}M_2a + \frac{1}{6}M_1a + \frac{EI}{a}(\delta_2 - \delta_1),$$

and for BC,

$$EI\left(\frac{dy}{dx}\right)_{x=b} = -\frac{1}{24}wb^3 + \frac{1}{3}M_2b + \frac{1}{6}M_3b + \frac{EI}{b}(\delta_3 - \delta_2).$$

These quantities are equal but of opposite signs and hence we have

$$M_1a + 2M_2(a + b) + M_3b = \frac{1}{4}w(a^3 + b^3) + 6EI\left(\frac{\delta_1}{a} - \frac{\delta_2}{a} - \frac{\delta_2}{b} + \frac{\delta_3}{b}\right).$$

11.13 Beam on Elastic Foundation

A beam may be supported along its length by an elastic foundation, such as the ground, so that at any point there is an upthrust proportional to the deflexion. Then if y be the deflexion there is an additional load $-ky$ per unit length and k is called the modulus of the foundation.

The differential equation for the deflexion is then

$$EI\frac{d^4y}{dx^4} = -ky + w.$$

Writing $k = 4EI\beta^4$, this becomes

$$\frac{d^4y}{dx^4} + 4\beta^4y = \frac{w}{EI}.$$

The complementary function in the solution is easily obtained. The auxiliary equation is

$$\lambda^4 + 4\beta^4 = 0,$$

and we have $$\lambda = \beta(\pm 1 \pm i).$$

The complementary function is

$$\begin{aligned} y &= Pe^{\beta x+\beta ix} + Qe^{\beta x-\beta ix} + Re^{-\beta x+\beta ix} + Se^{-\beta x-\beta ix}, \\ &= e^{\beta x}(A\cos\beta x + B\sin\beta x) + e^{-\beta x}(C\cos\beta x + D\sin\beta x), \end{aligned}$$

where P, Q, R, S and A, B, C, D are constants.

The particular integral is

$$\frac{1}{D^4 + 4\beta^4}\frac{w}{EI}$$

$$= \frac{w}{k}, \text{ if } w \text{ is constant.}$$

The complete solution is the sum of the complementary function and the particular integral and the constants are determined by the end conditions.

For example let a uniform beam of length l carry a uniform load of intensity w and rest on a foundation of modulus k, the ends being pinned at the same level (Fig. 304). If the origin is taken at the mid-point of the undeflected beam the equation

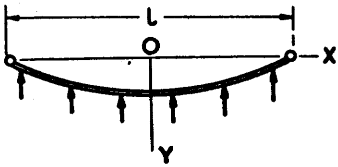

Fig. 304

$$\frac{d^4y}{dx^4} + 4\beta^4 y = \frac{w}{EI},$$

where $k = 4EI\beta^4$, must have a solution which is an even function of x since the deflexion is the same on either side of the centre.

Using hyperbolic functions the complete solution may be written as

$$y = \frac{w}{k} + A \cosh \beta x \cos \beta x + B \sinh \beta x \sin \beta x + C \cosh \beta x \sin \beta x + D \sinh \beta x \cos \beta x,$$

and since y is even $C = D = 0$.

Also, $y = 0$ when $x = \frac{1}{2}l$, therefore

$$A \cosh \frac{1}{2}\beta l \cos \frac{1}{2}\beta l + B \sinh \frac{1}{2}\beta l \sin \frac{1}{2}\beta l = -\frac{w}{k}. \tag{1}$$

Since the bending moment is zero at the ends, $\frac{d^2y}{dx^2}$ must be zero.

$$\frac{dy}{dx} = \beta(B - A) \cosh \beta x \sin \beta x + \beta(A + B) \sinh \beta x \cos \beta x,$$

$$\frac{d^2y}{dx^2} = 2\beta^2 B \cosh \beta x \cos \beta x - 2\beta^2 A \sinh \beta x \sin \beta x.$$

Therefore $$B \cosh \frac{1}{2}\beta l \cos \frac{1}{2}\beta l - A \sinh \frac{1}{2}\beta l \sin \frac{1}{2}\beta l = 0. \tag{2}$$

From (1) and (2),

$$A = -\frac{2w}{k}\frac{\cos \frac{1}{2}\beta l \cosh \frac{1}{2}\beta l}{\cos \beta l + \cosh \beta l},$$

$$B = -\frac{2w}{k}\frac{\sin \frac{1}{2}\beta l \sinh \frac{1}{2}\beta l}{\cos \beta l + \cosh \beta l}.$$

The deflexion at the centre $= \frac{w}{k} + A.$

The bending moment at the centre

$$= 2EI\beta^2 B.$$

EXERCISES 11 (c)

1. A girder AB of length l is simply supported at both ends and carries a load varying uniformly from 0 at A to w at B. Find the formula for the deflexion of the beam at a distance x from A. (L.U., Pt. II)

2. A beam of length $2a$ is clamped horizontally at both ends and is supported at the same level at its mid-point. It carries a uniformly distributed load of intensity w over its whole length. Express the bending moment at a distance x from one end ($x < a$) in terms of the end bending moment M and the support reaction R. By obtaining the equation for the transverse deflexion of the beam determine the values of M and B. (L.U., Pt. II)

3. A uniform beam is supported at its ends and carries a uniformly distributed load along the middle half. Show that the additional deflexion due to the load is 57/64 times the additional deflexion had the load been concentrated at the mid-point. (L.U., Pt. II)

4. A light uniform beam of length l is clamped horizontally at one end and freely supported to the same level at the other. It carries a load of total amount W which varies uniformly from zero at the free support. Determine formulae for the bending moment and shear force at x ft. from the clamp and obtain the reactions at the ends and the deflexion at the mid-point. (L.U., Pt. II)

5. A uniform beam of length $3a$ and weight W is clamped horizontally at one end and supported at a point distant $2a$ from the clamped end at the same level. Show that the reaction at the support is $17W/24$ and draw a rough sketch of the form of the deflected beam. (L.U., Pt. II)

6. A uniform beam is clamped horizontally at one end and carries a load uniformly distributed between the mid-point and the free end. If this load has the same weight per unit length as the beam, prove that the deflexion at the free end is 89/31 times that at the mid-point of the beam. (L.U., Pt. II)

7. A naturally straight uniform elastic rod of length $2l$ is laid across a rigid horizontal table of breadth $2a$ with equal lengths $(l - a)$ projecting beyond the table. If a straight portion AB of the rod is in contact with the table, show that the bending moments at A and B are zero, and that $l/a < 1 + \frac{1}{2}\sqrt{2}$. (L.U., Pt. II)

8. A uniform beam of length $2l$ and weight W is clamped horizontally at one end and supported at its mid-point so that the free end is at the same level as the clamp. Find the reactions at the supports and the deflexion and slope at the mid-point. (L.U., Pt. II)

9. A uniform rod of weight W and flexural rigidity EI has its ends constrained by smooth horizontal guides which are in the same horizontal line and at a distance $2l$ apart. If a weight W' is supported at the middle point, prove that the bending couples at the end are of magnitude $Wl/6 + W'l/4$ and calculate the deflexion at the middle point. (L.U., Pt. II)

10. A uniform beam of length $2l$ and constant flexural rigidity EI is clamped horizontally at one end and pinned to the same level at the other. The load intensity at any point is proportional to the product of the distances of the point from the ends. Show that the reaction at the clamp is 13/20 of the total load and find the deflexion at the mid-point. (L.U., Pt. II)

11. A beam AB of constant flexural rigidity EI and length $2a$, clamped horizontally at both ends, has a concentrated load $4wa$ at its mid-point C. The load intensity is w per unit length over AC and $2w$ over CB. Show that the bending moment at A is $\frac{23}{16}wa^2$ and find the deflexion at C. (L.U., Pt. II)

12. A uniform beam of weight W and length $6a$ is supported symmetrically at its centre C and at two points B, D at the same level, each at a distance $2a$ from C. Express the bending moment at any point P in BC, distant x from the end A nearer to B, in terms of x, a, W and R, the reaction at B. By integration show that this reaction is $17W/48$ and find the deflexion of A below B. (L.U., Pt. II)

13. A light beam of length $(a + b)$ is clamped horizontally at one end and its other end is propped at the same level. It carries a concentrated load W at distance a from the clamped end. Find the reaction at the prop and the couple exerted by the clamp. If $a = b$, find the greatest deflexion, the flexural rigidity being EI.

14. Prove the theorem of three moments in the form

$$L_1a + 2L_2(a + b) + L_3b = \frac{w}{4}(a^3 + b^3).$$

A uniform beam of weight w per unit length and of length $a + b$ is clamped horizontally at each end and is propped, level with the ends, at a point distant a from one end. Show that the bending moment at the prop is

$$\frac{w}{12}(a^2 + b^2 - ab).$$

Find also the thrust on the prop. (L.U., Pt. II)

15. A uniform beam of length $4a$ rests on five supports at the same level, so that there are four bays each of length a. It carries over its whole length a uniformly distributed load of intensity w. Find the reactions at the supports and the greatest bending moment.

11.14 Columns and Struts

A column or strut is a member of a structure which is in compression. A short strut may fail if the compressive stress along its length is too great. A longer strut may also fail by bending. If the member has or is given a small deflexion, the force along its length causes a bending moment which may be sufficient to hold or increase this deflexion. Thus there may be two kinds of compressive stress, one due to bending

and one due to direct compression, and failure may occur if their total exceeds the safe stress for the material. We shall assume that the struts considered bend in one plane; the condition for this is that the plane of bending contains one of the principal axes of the cross-section.

The ends of a strut may be position-fixed or they may be fixed both in position and direction, or one end may be completely free. These differing end conditions determine the constants in the solution of the differential equation of the strut.

11.15 Strut with Position-fixed Ends

Consider a strut of negligible weight, of uniform cross-section and of length l, supporting a load P (Fig. 305). Let OX be the vertical through O and OY perpendicular to it. If the strut is slightly deflected in the plane OXY, the bending moment at any point (x, y) caused by the thrust P is $-Py$ (the sagging side of the strut being that which is farther along OY).

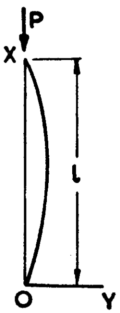

Fig. 305

We have then $$EI\frac{d^2y}{dx^2} = -Py.$$

Writing $P = EIn^2$, this equation becomes

$$\frac{d^2y}{dx^2} + n^2y = 0.$$

The complete solution is

$$y = A\cos nx + B\sin nx.$$

Since $y = 0$ when $x = 0$, $A = 0$, and we have

$$y = B\sin nx.$$

But $y = 0$ when $x = l$, therefore

$$0 = B\sin nl.$$

Hence, if there is any deflexion we must have

$$\sin nl = 0,$$

$$nl = \pi,$$

$$n^2 = \frac{\pi^2}{l^2},$$

$$P = \frac{\pi^2 EI}{l^2}.$$

If A be the area of the cross-section and $I = Ak^2$, we have

$$\frac{P}{A} = \frac{E\pi^2}{(l/k)^2}.$$

The value found for P is called Euler's crippling load for the strut. P is the load which is sufficient to hold the strut in the deflected position,

and if the load is greater than P the deflexion will increase rapidly. It is assumed in the calculation that the compressive stress P/A is less than the safe compressive stress f, that is,

$$\frac{E\pi^2}{(l/k)^2} < f,$$

that is
$$\left(\frac{l}{k}\right)^2 > \frac{E\pi^2}{f}.$$

The quantity l/k, where k is the least radius of gyration of the cross-section about an axis through the centroid, is called the *slenderness ratio* of the strut. Thus Euler's formula applies only to struts with a high slenderness ratio.

11.16 Strut with Direction-fixed Ends

In this case there must be a couple G at either end (Fig. 306) to hold the direction and the bending moment at (x, y) is $-Py + G$.

Then
$$EI\frac{d^2y}{dx^2} = -Py + G,$$

$$\frac{d^2y}{dx^2} + n^2y = \frac{G}{P}n^2,$$

$$y = A\cos nx + B\sin nx + \frac{G}{P}.$$

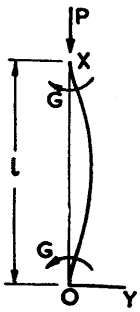

Fig. 306

Since $y = \frac{dy}{dx} = 0$ for $x = 0$, $B = 0$ and $A = -\frac{G}{P}$, and

$$y = \frac{G}{P}(1 - \cos nx).$$

Also, $y = 0$ when $x = l$, therefore $\cos nl = 1$,

$$nl = 2\pi,$$

$$P = \frac{4\pi^2EI}{l^2},$$

$$\frac{P}{A} = \frac{4\pi^2E}{(l/k)^2}.$$

Thus the crippling load is increased fourfold, and the effective safe length for a given load is doubled. The least slenderness ratio for which the formula holds is also doubled.

11.17 Strut with Upper End Free

In this case the lower end must be direction-fixed and the fixing couple G is Pa, where a is the deflexion of the free end (Fig. 307). We have

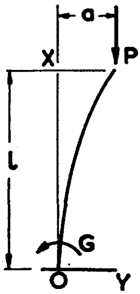

Fig. 307

$$EI\frac{d^2y}{dx^2} + Py = Pa,$$

$$y = a + A\cos nx + B\sin nx$$
$$= a(1 - \cos nx),$$

since $y = \dfrac{dy}{dx} = 0$ for $x = 0$.

Since $y = a$ when $x = l$,

$$\cos nl = 0,$$

$$nl = \frac{1}{2}\pi,$$

$$P = \frac{\frac{1}{4}\pi^2 EI}{l^2},$$

$$\frac{P}{A} = \frac{\frac{1}{4}\pi^2 E}{(l/k)^2}.$$

Thus the crippling load is $\frac{1}{4}$ of that for position-fixed ends and the effective safe length for a given load is halved. The least slenderness ratio for which the formula holds is also halved.

11.18 Strut with One End Direction-fixed

In this case there must be a force R at the position-fixed end (Fig. 308) to balance the fixing couple G and $Rl = G$. The bending moment at (x, y) is

$$-Py + R(l - x),$$

and we have

$$EI\frac{d^2y}{dx^2} + Py = R(l - x),$$

$$\frac{d^2y}{dx^2} + n^2y = \frac{R}{P}n^2(l - x),$$

$$y = \frac{R}{P}(l - x) + A\cos nx + B\sin nx.$$

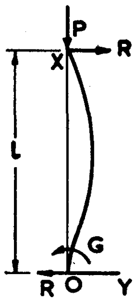

Fig. 308

Since $y = 0$ when $x = 0$, $A = -Rl/P$, and since $\dfrac{dy}{dx} = 0$ for $x = 0$, $B = R/(nP)$, and we have

$$y = \frac{R}{P}(l - x) - \frac{R}{P}l\cos nx + \frac{R}{nP}\sin nx.$$

Since $y = 0$ when $x = l$, we have

$$nl = \tan nl.$$

The least positive solution of this equation is $nl = 4{\cdot}493$ and hence

$$P = \frac{(4{\cdot}493)^2 EI}{l^2}$$

$$= \frac{2{\cdot}05\pi^2 EI}{l^2},$$

$$\frac{P}{A} = \frac{2{\cdot}05\pi^2 E}{(l/k)^2}.$$

Thus the crippling load is approximately doubled by fixing the direction of one end. The effective safe length for a given load is increased by about 40 per cent and the least slenderness ratio by the same percentage.

It will be seen that the deflexion curves obtained in all the above four cases are sine curves, but in each case a different length of the curve must be chosen to fit the end conditions. The lengths in which the sine curve varies through half of a complete wave are in the four cases l, $\frac{1}{2}l$, $2l$, $0{\cdot}7l$ respectively, and these lengths are called the free lengths of the struts. The crippling load is $\dfrac{\pi^2 EI}{(l')^2}$ in each case, where l' is the free length.

11.19 Maximum Stress in a Strut

If a compressive force P acts along the axis of a strut and A is the area of the cross-section the compressive stress at every point of the strut, which we suppose to be homogeneous and of uniform cross-section, is P/A. If, in addition, the force P sets up a bending moment in the strut and M_0 is the maximum bending moment there will be a compressive stress due to M_0 at the section where it acts. The greatest compressive stress due to bending will then be

$$\frac{M_0 Y}{I},$$

where I is the second moment of area of the section about the neutral axis and Y the greatest distance of a layer which is in compression from the neutral axis.

Then the greatest compressive stress in the strut is

$$\frac{P}{A} + \frac{M_0 Y}{I}.$$

In tie-bars P is a tensile force and the greatest tensile stress will be given by a similar formula.

We shall find the maximum bending moment in struts and tie-bars for various conditions of loading by finding first the deflexion curve of the member.

11.20 Strut with Initial Curvature—Direction-fixed Ends

Some small initial curvature will usually be found in a strut before it takes any load and we shall suppose that the curve is a sine curve. Let OA (Fig. 309) be a strut of length l which initially has the shape

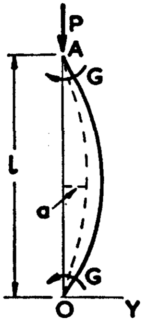

Fig. 309

$$y_1 = a \sin \frac{\pi x}{l},$$

so that a is its greatest deflexion from the line joining its ends.

Let y be the deflexion of a point (x, y) from the line OA when a load P is applied. Since the ends are direction-fixed there must be a fixing couple G and the bending moment at (x, y) is $-Py + G$.

We have then,

$$EI\frac{d^2}{dx^2}(y - y_1) = -Py + G.$$

Writing $P = EIn^2$ and substituting for $\dfrac{d^2y_1}{dx^2}$ we have

$$\frac{d^2y}{dx^2} + n^2y = n^2\frac{G}{P} - \frac{a\pi^2}{l^2}\sin\frac{\pi x}{l}.$$

A particular integral is $\dfrac{G}{P} + \dfrac{a\pi^2}{\pi^2 - n^2l^2}\sin\dfrac{\pi x}{l}$, and the complete solution is

$$y = \frac{G}{P} + \frac{a\pi^2}{\pi^2 - n^2l^2}\sin\frac{\pi x}{l} + A\cos nx + B\sin nx.$$

If $y = 0$ for $x = 0$ and $x = l$ we have $A = -G/P$ and

$$B = -\frac{G}{P}\cdot\frac{1 - \cos nl}{\sin nl}$$
$$= -\frac{G}{P}\tan\frac{1}{2}nl.$$

If $\dfrac{dy}{dx} = 0$ for $x = 0$, that is

$$0 = \frac{\pi}{l}\cdot\frac{a\pi^2}{\pi^2 - n^2l^2} + nB,$$

then
$$\frac{G}{P} = \frac{\pi}{nl}\cdot\frac{a\pi^2}{\pi^2 - n^2l^2}\cot\frac{1}{2}nl.$$

The greatest deflexion is at $x = \frac{1}{2}l$ and at this point we have

$$y_0 = \frac{G}{P} + \frac{a\pi^2}{\pi^2 - n^2l^2} + A\cos\frac{1}{2}nl + B\sin\frac{1}{2}nl,$$

$$= \frac{G}{P} + \frac{a\pi^2}{\pi^2 - n^2l^2} - \frac{G}{P}\sec\frac{1}{2}nl,$$

$$= \frac{G}{P} + \frac{a\pi^2}{\pi^2 - n^2l^2}\left(1 - \frac{\pi}{nl}\operatorname{cosec}\frac{1}{2}nl\right).$$

The greatest bending moment M_0 is

$$M_0 = -Py_0 + G$$

$$= -\frac{Pa\pi^2}{\pi^2 - n^2l^2}\left(1 - \frac{\pi}{nl}\operatorname{cosec}\frac{1}{2}nl\right).$$

11.21 Strut with Initial Curvature and Eccentric Load

Let a strut whose initial curve is given by $y_1 = a\sin\frac{\pi x}{l}$ carry a load P which acts at distance e from the centroid of the end section (Fig. 310). We may suppose the ends to be position-fixed but direction free. If the ends were direction-fixed the eccentricity of loading would have no effect.

The bending moment at (x, y) is $P(y + e)$ and we have

$$EI\frac{d^2}{dx^2}(y - y_1) = -P(y + e),$$

$$\frac{d^2y}{dx^2} + n^2y = -n^2e - a\frac{\pi^2}{l^2}\sin\frac{\pi x}{l},$$

where $EIn^2 = P$.

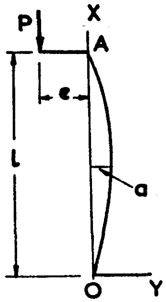

Fig. 310

The particular integral is $-e + \frac{a\pi^2}{\pi^2 - n^2l^2}\sin\frac{\pi x}{l}$,

and the complete solution is

$$y = -e + \frac{a\pi^2}{\pi^2 - n^2l^2}\sin\frac{\pi x}{l} + A\cos nx + B\sin nx.$$

Since $y = 0$ for $x = 0$ and $x = l$, $A = e$, and

$$B = e\frac{1 - \cos nl}{\sin nl} = e\tan\frac{1}{2}nl.$$

The greatest deflexion at $x = \frac{1}{2}l$ is

$$y_0 = -e + \frac{a\pi^2}{\pi^2 - n^2l^2} + e\cos\frac{1}{2}nl + e\tan\frac{1}{2}nl\sin\frac{1}{2}nl$$

$$= -e + \frac{a\pi^2}{\pi^2 - n^2l^2} + e\sec\frac{1}{2}nl.$$

The greatest bending moment M_0 is

$$M_0 = -P(y_0 + e)$$

$$= -\frac{a\pi^2P}{\pi^2 - n^2l^2} - Pe\sec\frac{1}{2}nl.$$

It is clear that M_0 is the sum of the bending moments due to initial curvature and to eccentricity of loading.

11.22 Strut with Uniform Lateral Load

Consider a strut of length l under the action of a compressive force P at its ends and carrying a uniform lateral load of intensity w (Fig. 311). Let the axis OX be along the undeflected line of the strut

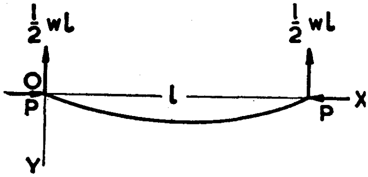

Fig. 311

and OY perpendicular in the direction of deflexion. We shall suppose the ends to be position-fixed but direction free.

Taking into account the end reaction $\frac{1}{2}wl$ and the distributed load we have for the bending moment at (x, y),

$$M = -Py - \frac{1}{2}wlx + \frac{1}{2}wx^2.$$

Then
$$EI\frac{d^2y}{dx^2} + Py = -\frac{1}{2}wlx + \frac{1}{2}wx^2,$$

$$\frac{d^2y}{dx^2} + n^2y = \frac{w}{2EI}(x^2 - lx).$$

A particular integral is

$$\frac{w}{2n^2EI}\left(x^2 - lx - \frac{2}{n^2}\right)$$
$$= \frac{w}{P}\left(\frac{1}{2}x^2 - \frac{1}{2}lx - \frac{1}{n^2}\right).$$

The complete solution is

$$y = \frac{w}{P}\left(\frac{1}{2}x^2 - \frac{1}{2}lx - \frac{1}{n^2}\right) + A\cos nx + B\sin nx.$$

Since $y = 0$ for $x = 0$ and $x = l$ we have

$$A = \frac{w}{n^2P}, \quad B = \frac{w}{n^2P}\tan\frac{1}{2}nl,$$

and
$$y = \frac{w}{n^2P}\left\{\frac{1}{2}n^2x(x - l) - 1 + \cos nx + \tan\frac{1}{2}nl\sin nx\right\}.$$

The greatest deflexion at $x = \frac{1}{2}l$ is

$$y_0 = \frac{w}{n^2P}\left(-\frac{n^2l^2}{8} - 1 + \sec\frac{1}{2}nl\right).$$

The greatest bending moment is

$$M_0 = -Py_0 - \frac{1}{8}wl^2$$

$$= -\frac{w}{n^2}\left(\sec\frac{1}{2}nl - 1\right).$$

11.23 Strut with Central Lateral Load

Let a strut of length l under the action of a compressive force P at its ends carry a concentrated lateral load W at its centre (Fig. 312), and let the ends be position-fixed but direction free.

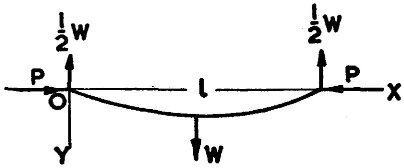

Fig. 312

With the axes as shown, the bending moment at (x, y) is

$$-Py - \frac{1}{2}Wx, \quad \text{for } x < \frac{1}{2}l.$$

Then $$EI\frac{d^2y}{dx^2} = -Py - \frac{1}{2}Wx,$$

$$\frac{d^2y}{dx^2} + n^2y = -\frac{1}{2}\frac{W}{EI}x,$$

$$y = -\frac{1}{2}\frac{W}{P}x + A\cos nx + B\sin nx.$$

Using the conditions $y = 0$ when $x = 0$ and $\frac{dy}{dx} = 0$ when $x = \frac{1}{2}l$, we have $A = 0$, $B = \frac{W}{2Pn}\sec\frac{1}{2}nl$, and

$$y = \frac{1}{2}\frac{W}{P}\left(\frac{1}{n}\sec\frac{1}{2}nl\sin nx - x\right).$$

This is the deflexion curve for $0 < x < \frac{1}{2}l$ and gives the greatest deflexion,

$$y_0 = \frac{1}{2}\frac{W}{P}\left(\frac{1}{n}\tan\frac{1}{2}nl - \frac{1}{2}l\right).$$

The maximum bending moment is

$$M_0 = -Py_0 - \frac{1}{4}Wl$$

$$= -\frac{W}{2n}\tan\frac{1}{2}nl.$$

11.24 Tie Bar with Eccentric Load

The maximum bending moment in a tie bar is found in the same way as for a strut. The essential difference is that since the load P acts in the opposite direction the differential equation obtained is of the form

$$EI\frac{d^2y}{dx^2} - Py = f(x),$$

that is

$$\frac{d^2y}{dx^2} - n^2y = \frac{1}{EI}f(x).$$

The complete solution, therefore, involves hyperbolic functions instead of sines and cosines of nx and the bending moments can for the most part be deduced from the corresponding expressions for struts by substituting $n\sqrt{(-1)}$ for n.

As an example we shall consider the case of a tie bar with an eccentric load (Fig. 313). Let the tensile force P act at distance e from the centroid of the end sections. We take the axis of x along the undeflected line of the tie bar and the axis of y in the opposite direction to the eccentricity and suppose the ends to be direction free.

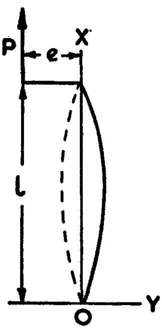

Fig. 313

The bending moment at (x, y) is $P(y + e)$ and we have

$$EI\frac{d^2y}{dx^2} = P(y + e),$$

$$\frac{d^2y}{dx^2} - n^2y = n^2e,$$

where $P = EIn^2$.

Then $$y = -e + A\cosh nx + B\sinh nx.$$

Since $y = 0$ for $x = 0$ and $x = l$, $A = e$ and $0 = -e + e\cosh nl + B\sinh nl$,

$$B = -e\,\frac{\cosh nl - 1}{\sinh nl}$$

$$= -e\tanh\frac{1}{2}nl.$$

The greatest deflexion at $x = \frac{1}{2}l$ is

$$y_0 = -e + e\cosh\frac{1}{2}nl - e\tanh\frac{1}{2}nl\sinh\frac{1}{2}nl$$
$$= -e\left(1 - \operatorname{sech}\frac{1}{2}nl\right).$$

Since $\operatorname{sech}\frac{1}{2}nl < 1$, this deflexion is negative and $P(y_0 + e)$ is less than Pe the bending moment at the ends. Thus the maximum bending moment is Pe.

Example 10. *The lower end of a uniform light cantilever of length l and flexural rigidity EI is clamped at an angle α to the vertical. A vertical load W is applied to the upper end. Obtain the differential equation for y, the deflexion (from the undeflected position) of a point on the strut distant x from the lower end in terms of the end deflexion a. Show that, at the free end, the value of dy/dx is $\tan\alpha(\sec nl - 1)$ where $EIn^2 = W\cos\alpha$.* (L.U., Pt. II)

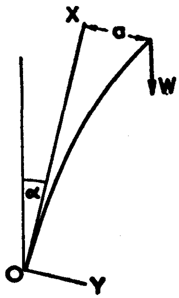

Fig. 314

Let the x-axis be along the undeflected line of the cantilever and the y-axis perpendicular (Fig. 314). The bending moment at a point of the beam whose coordinates are (x, y) is

$$W\sin\alpha(l - x) + W\cos\alpha(a - y),$$

and this is a positive or hogging moment.
We have therefore,

$$EI\frac{d^2y}{dx^2} = W\sin\alpha(l - x) + W\cos\alpha(a - y),$$

that is $\dfrac{d^2y}{dx^2} + n^2y = n^2a + n^2\tan\alpha(l - x).$

A particular integral of this equation is $a + \tan\alpha(l - x)$, and the complete solution is

$$y = a + \tan\alpha(l - x) + A\cos nx + B\sin nx.$$

Now $y = 0$ when $x = 0$, therefore $A = -(a + l\tan\alpha)$. Also $\dfrac{dy}{dx} = 0$ when $x = 0$, therefore $-\tan\alpha + Bn = 0$.
Therefore

$$y = a + \tan\alpha(l - x) - (a + l\tan\alpha)\cos nx + \frac{1}{n}\tan\alpha\sin nx.$$

We have also that $y = a$ when $x = l$,

therefore $$a = a - (a + l\tan\alpha)\cos nl + \frac{1}{n}\tan\alpha\sin nl,$$

$$a = -l\tan\alpha + \frac{1}{n}\tan\alpha\tan nl.$$

$$\left(\frac{dy}{dx}\right)_l = -\tan\alpha + n(a + l\tan\alpha)\sin nl + \tan\alpha\cos nl$$
$$= -\tan\alpha + \tan\alpha\sec nl.$$

EXERCISES 11 (*d*)

1. A uniform strut AB, whose length l is great compared with the dimensions of its cross-section, is set up vertically with A and B fixed so that the tangents to the shape of the strut at A and B are vertical. The strut supports a load W at B its upper end, W being large compared with the weight of the strut. If W is just great enough to cause the strut to bend, prove that, if y, supposed small, is the displacement of any point of the strut from the vertical line AB, y satisfies the equation

$$EI\frac{d^2y}{dx^2} = -Wy + M$$

where M is the fixing couple at B, and that $W = 4\pi^2EI/l^2$.
(L.U., Pt. II)

2. A wooden beam 3 in. by 6 in. in cross-section and 12 ft. long is used as a strut. Find its slenderness ratio. Find the load which it can support with its lower end fixed in position and direction (*a*) if its upper end is free, (*b*) if its upper end is fixed in position only. E for the wood is $1{\cdot}2 \times 10^6$ lb./in.^{2}.

3. A strut of length l and modulus of rigidity EI has an initial curvature so that its shape is approximately that of a parabola with its centre at a distance a from the line joining its ends. If the strut is subjected to a compressive force P at its ends, which are fixed in position only, show that its deflexion is given by the equation

$$\frac{d^2y}{dx^2} + n^2y = -\frac{8a}{l^2},$$

where $EIn^2 = P$. Hence show that the greatest bending moment in the strut is $8aP(\sec \frac{1}{2}nl - 1)/(n^2l^2)$.

4. A strut of length l and modulus of rigidity EI has an initial curvature so that its shape is approximately that of a parabola with its centre at a distance a from the line joining its ends. The ends are fixed in position only and the strut is subjected to a compressive force P which acts at distance e from the centroid of its ends. Show that its deflexion is given by the equation

$$\frac{d^2y}{dx^2} + n^2y = -n^2e - \frac{8a}{l^2},$$

where $EIn^2 = P$. Hence show that the greatest bending moment in the strut is $Pe \sec \frac{1}{2}nl + 8aP(\sec \frac{1}{2}nl - 1)/n^2l^2$.

5. A uniform tie bar of length l and flexural rigidity EI is subjected to a tensile force P at its ends and carries a uniform lateral load of intensity w. Show that the deflexion of the tie bar is given by the equation

$$\frac{d^2y}{dx^2} - n^2y = \frac{wn^2}{2P}(x^2 - lx),$$

where $EIn^2 = P$. Hence show that the greatest bending moment in the bar is $-w(1 - \text{sech}\,\frac{1}{2}nl)/n^2$.

6. A uniform tie bar of length l and flexural rigidity EI is subjected to a tensile force P at its ends and carries a concentrated lateral load W at its mid-point. Show that the deflexion of the tie bar is given by the equation

$$\frac{d^2y}{dx^2} - n^2y = -\frac{Wn^2}{2P}x,$$

where $EIn^2 = P$. Hence show that the greatest bending moment in the bar is $-(W/2n)\tanh \tfrac{1}{2}nl$.

7. A uniform strut of length l and flexural rigidity EI has an initial curvature so that its shape is that of a sine curve with its centre at distance a from the line joining its ends. The strut is subjected to a compressive force P at its ends, which are fixed in position only, and carries a uniform lateral load of intensity w. Show that its deflection is given by the equation

$$\frac{d^2y}{dx^2} + n^2y = -a\frac{\pi^2}{l^2}\sin\frac{\pi x}{l} + \frac{wn^2}{P}(x^2 - lx),$$

where $EIn^2 = P$. Show that the greatest bending moment is

$$-Pa\pi^2/(\pi^2 - n^2l^2) - w(\sec \tfrac{1}{2}nl - 1)/n^2.$$

8. A uniform strut of length l and flexural rigidity EI is subjected to a compressive force P which acts at a distance e from the centroid of its end. The strut carries a uniform lateral load of intensity w. Show that its deflexion is given by the differential equation

$$\frac{d^2y}{dx^2} + n^2y = -n^2e + \frac{wn^2}{2P}(x^2 - lx),$$

where $EIn^2 = P$. Show that the greatest bending moment is

$$-Pe\sec\tfrac{1}{2}nl - w(\sec\tfrac{1}{2}nl - 1)/n^2.$$

9. A light uniform pole of length l and constant flexural rigidity EI is fixed vertically in the ground at its lower end A, and its upper end B is acted upon by a force T which makes an angle α with the downward vertical. The consequent small horizontal deflexion of B is a. Taking the origin at A, measuring x vertically and y horizontally, state the bending moment at any point $P(x, y)$ of the pole, and show that

$$(D^2 + n^2)y = n^2a + n^2(l - x)\tan\alpha,$$

where $D = \dfrac{d}{dx}$ and $EIn^2 = T\cos\alpha$. Solve this differential equation and show that $na = \tan\alpha(\tan nl - nl)$. (L.U., Pt. II)

10. A uniform thin lath of length l and constant flexural rigidity EI is clamped vertically at its lower end and at its upper end carries a small light bracket of length a fixed perpendicularly to the lath. When a load W is hung from the bracket it deflects a small horizontal distance b and negligible vertical distance.

State the bending moment at a point on the lath distant x vertically and y horizontally from the clamped end. Find b and the bending moment at the clamp in terms of the other quantities given. Evaluate W when $b = a$. (L.U., Pt. II)

CHAPTER 12

MOTION OF A FLUID

12.1 Stability of Floating Bodies

If a body is floating in a liquid its weight W acting vertically downwards through the centre of gravity G must be balanced by the force of buoyancy acting vertically upwards through the centre of buoyancy H (Fig. 315), and hence GH must be a vertical line. A vertical displacement downwards increases the buoyancy and an upward displacement decreases it, so that in either case the body tends to return to its original position. Thus the equilibrium of a floating body is stable for vertical displacement, and it is clearly neutral for lateral displacements.

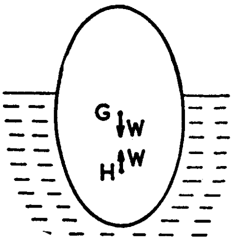

Fig. 315

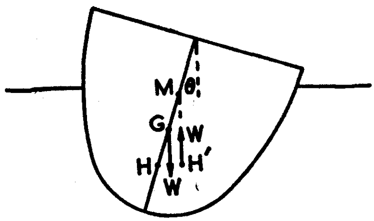

Fig. 316

We shall consider the stability of a floating body when given a small rotational displacement, which is such that the volume of liquid displaced by the body is unaltered by the rotation.

Let H be the centre of buoyancy in the undisturbed position and H' the centre of buoyancy when the body has turned through a small angle θ (Fig. 316), the force of buoyancy being W in each case. Let the vertical through H' meet the line HG in a point M.

Then if M is above G there is a restoring couple

$$W \times GM \sin \theta,$$
$$= W \times GM \times \theta, \text{ approximately,}$$

tending to diminish θ and restore the body to its equilibrium position. If M is below G the couple tends to increase θ, and thus the equilibrium is stable if M is above G and unstable if it is below G. The point M is called the ***metacentre*** and its height above G is called the ***metacentric height***.

We shall see how the position of the metacentre can be determined both theoretically and experimentally, and that M is in fact the centre

of curvature of the locus of H as the body rotates. We shall first find the position of the centre of buoyancy in the undisturbed and disturbed positions.

12.2 Coordinates of the Centre of Buoyancy

Let ABC (Fig. 317 (i)) be the section of the body by the free surface of the liquid in the equilibrium position. This section is called the *plane of flotation.*

Let OY be the axis about which the plane of flotation turns, OX and OZ being perpendicular axes *fixed in the body* with OZ vertical before rotation. Let an element of area δA of the plane of flotation be distant x from OY and let z be the height of a prism with base δA, parallel to OZ and terminated by the surface of the body.

Then the volume immersed is

$$V = \int z dA,$$

the integration being over the plane of flotation.

When the plane of flotation turns through a small angle θ about OY, the height of the prism increases by $x \tan \theta$ (Fig. 317 (ii)) $= x\theta$, approximately, and the volume immersed is

$$V' = \int (z + x\theta) dA.$$

Since $V = V'$, we have

$$0 = \theta \int x dA,$$

that is, the first moment of area of the plane of flotation about OY is zero. Therefore the centroid of the plane of flotation must lie on OY. Thus if the body turns without altering the volume immersed the axis of rotation must pass through the centroid of the plane of flotation.

With reference to the axes OX, OY, OZ (Fig. 317), let $(\bar{x}, \bar{y}, \bar{z})$ be the

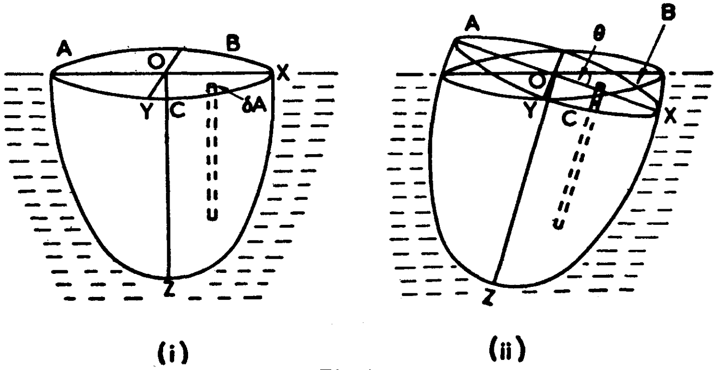

Fig. 317

coordinates of the centre of buoyancy H in the undisturbed position and $(\bar{x}', \bar{y}', \bar{z}')$ its coordinates in the disturbed position. Since H is the centre of gravity of the volume (of liquid) cut off we have

$$V\bar{x} = \int xz dA, \quad V\bar{y} = \int yz dA, \quad V\bar{z} = \frac{1}{2}\int z^2 dA,$$

the integral in each case being taken over the plane of flotation.

In the disturbed position z is increased by $x\theta$ and the centre of gravity of the element $(z + x\theta)dA$ is at a depth $\frac{1}{2}(z + x\theta)$ below its highest point, that is $\frac{1}{2}(z - x\theta)$ below the plane OXY.

Then

$$V\bar{x}' = \int x(z + x\theta)dA, \quad V\bar{y}' = \int y(z + x\theta)dA, \quad V\bar{z}' = \frac{1}{2}\int (z^2 - x^2\theta^2)dA.$$

$$V(\bar{x}' - \bar{x}) = \theta\int x^2 dA,$$

$$V(\bar{y}' - \bar{y}) = \theta\int xy dA,$$

$$V(\bar{z}' - \bar{z}) = -\frac{1}{2}\theta^2\int x^2 dA.$$

Since θ is a small quantity and $\bar{z}' - z$ is proportional to θ^2 it is zero to the first order. $\int xy dA$ is the product of inertia (of area) of the plane of flotation about the axes OX and OY. Hence, if OX and OY are principal axes of the plane of flotation we have $\bar{y}' = \bar{y}$.

$\int x^2 dA$ is the second moment of area of the plane of flotation about OY. Thus if A be the area of the plane of flotation and k its radius of gyration about OY, we have

$$\bar{x}' - \bar{x} = \frac{Ak^2\theta}{V}.$$

We shall assume then that the rotation is about a principal axis of the plane of flotation and hence that the only first-order displacement of the centre of buoyancy is parallel to OX.

12.3 Metacentric Height

For a small rotation θ we have found that the displacement HH' of the centre of buoyancy is horizontal and equal to $Ak^2\theta/V$.

Therefore, since (Fig. 318)

$$HH' = HM \tan\theta,$$
$$= HM.\theta, \text{ approximately},$$

we have
$$HM = \frac{Ak^2}{V}.$$

The metacentric height is then

$$GM = \left(\frac{Ak^2}{V} - GH\right),$$

and the condition for stability is

$$\frac{Ak^2}{V} - GH > 0.$$

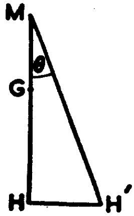

Fig. 318

The restoring couple is $\quad W\left(\frac{Ak^2}{V} - GH\right)\theta$

$$= V\varrho\left(\frac{Ak^2}{V} - GH\right)\theta$$

$$= \varrho\theta(Ak^2 - GH.V).$$

In these formulae,

A is the area of the plane of flotation,
k is its radius of gyration about the axis of rotation, which is a principal axis,
V is the volume immersed,
ϱ is the density of the liquid.

Since a plane section has two principal axes through its centroid, it is the height of the metacentre for rotation about the principal axis with the lesser radius of gyration that will determine the stability.

If I be the moment of inertia of the body about an axis through the centre of gravity parallel to the axis of rotation, we have

$$\frac{I}{g}\ddot{\theta} = -W\left(\frac{Ak^2}{V} - GH\right)\theta,$$

and hence the period of the roll is

$$2\pi\left\{\frac{I}{g} \div W\left(\frac{Ak^2}{V} - GH\right)\right\}^{1/2}.$$

Example 1. *A uniform circular cylinder of specific gravity s, length h and radius r floats in water with its axis vertical. Find the condition that it should be stable for small rotational displacements.*

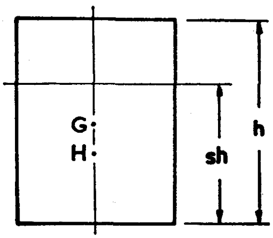

Fig. 319

The plane of flotation is a circle of radius a (Fig. 319) and the second moment about a diameter is

$$Ak^2 = \frac{1}{4}\pi r^4.$$

The volume immersed is

$$V = \pi r^2 sh,$$

$$\frac{Ak^2}{V} = \frac{r^2}{4sh}.$$

The centre of gravity is at height $\frac{1}{2}h$, and the centre of buoyancy at height $\frac{1}{2}sh$ above the base, therefore

$$HG = \frac{1}{2}h(1 - s).$$

Therefore, for stability, we must have

$$\frac{r^2}{4sh} > \frac{1}{2}h(1 - s),$$

$$\frac{r^2}{h^2} > 2s(1 - s).$$

Example 2. *A homogeneous solid whose specific gravity with respect to a liquid is s floats in stable equilibrium. Prove that if the body is inverted and placed in a second liquid with respect to which its specific gravity is* $(1 - s)$, *it will float in stable equilibrium with the same plane of flotation.*

Let V be the volume of the body and XX' the free surface when it floats in the first liquid (Fig. 320). Let H be the centre of buoyancy and G the

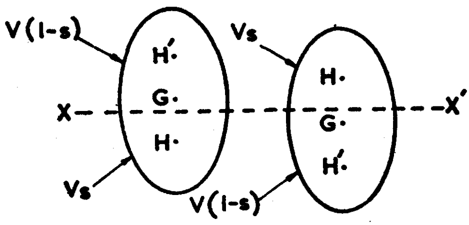

Fig. 320

centre of gravity. Then if H' be the centre of gravity of the volume $V(1 - s)$ which is out of the liquid, HGH' is a straight line and

$$Vs \times GH = V(1 - s) \times GH'.$$

Hence since G is vertically above H the body can float in the second liquid with G vertically above H' with the same plane of flotation.
Let Ak^2 be the second moment of area of the plane of flotation about a principal axis.
Since the equilibrium is stable we have

$$\frac{Ak^2}{Vs} - HG > 0,$$

that is

$$\frac{Ak^2}{V(1 - s)} - \frac{s}{(1 - s)}HG > 0,$$

$$\frac{Ak^2}{V(1 - s)} - H'G > 0.$$

This is the condition for stability of flotation in the second liquid.

Example 3. *A log of length l and specific gravity s whose cross-section is an isosceles triangle of base a and height h, floats in water with the surface containing the base horizontal. Find the condition that the equilibrium be stable.*

Let k be the height of the vertex above the free surface and x the width of the plane of flotation (Fig. 321).

Then
$$\frac{1}{2}kxl = (1-s)\frac{1}{2}ahl,$$
$$kx = (1-s)ah.$$

Also
$$\frac{x}{k} = \frac{a}{h},$$

therefore
$$k^2 = (1-s)h^2.$$

The depth of the centre of buoyancy H below the vertex is

$$\frac{\frac{1}{2}ah \times \frac{2}{3}h - \frac{1}{2}xk \times \frac{2}{3}k}{\frac{1}{2}ah - \frac{1}{2}kx},$$
$$= \frac{2}{3}\cdot\frac{h^3-k^3}{h^2-k^2}.$$

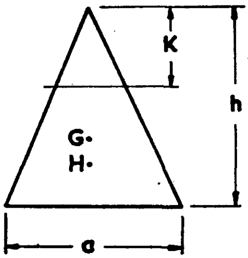

Fig. 321

Therefore
$$GH = \frac{2}{3}\left\{\frac{h^3-k^3}{h^2-k^2} - h\right\},$$
$$= \frac{2}{3}\,\frac{k^2(h-k)}{h^2-k^2}.$$

Also
$$\frac{I}{V} = \frac{\frac{1}{12}lx^3}{\frac{1}{2}l(ah-xk)},$$
$$= \frac{1}{6}\,\frac{a^2}{h^2}\cdot\frac{k^3}{h^2-k^2}.$$

Therefore, for stability in the plane of the cross-section we must have

$$\frac{1}{6}\,\frac{a^2}{h^2}\cdot\frac{k^3}{h^2-k^2} > \frac{2}{3}\,\frac{k^2(h-k)}{h^2-k^2},$$

that is
$$\frac{a^2}{4h^2} > \frac{h}{k} - 1,$$
$$\left(1 + \frac{a^2}{4h^2}\right)^2 > \frac{1}{1-s},$$
$$s < \frac{a^2(a^2+8h^2)}{(a^2+4h^2)^2}.$$

For longitudinal stability we must have

$$\frac{I_1}{V} = \frac{\frac{1}{12}xl^3}{\frac{1}{2}l(ah-kx)} = \frac{1}{6}\,\frac{l^2k}{h^2-k^2} > \frac{2}{3}\,\frac{k^2(h-k)}{h^2-k^2},$$

that is
$$\frac{l^2}{4h^2} > \sqrt{(1-s)} - (1-s).$$

12.4 Floating Body containing Liquid

Let G be the centre of gravity and M the metacentre for rolling displacements of a body of weight W (Fig. 322). The force of buoyancy is

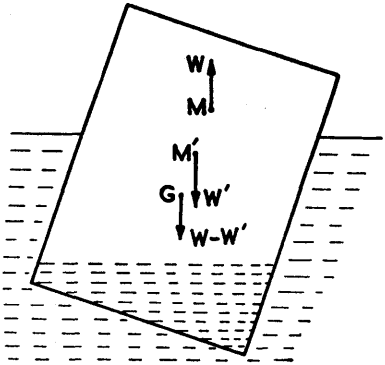

Fig. 322

W and the restoring couple when the body tilts through a small angle θ is

$$W \times GM \times \theta.$$

If the body contains liquid of weight W' with a free surface in the body we have the weight $W - W'$ of the body acting at the centre of gravity of the body and W' acting at the centre of gravity of the internal liquid. The line of action of W' is found in the same way as if the body were displacing this liquid W'. When the body tilts through a small angle θ the line of action passes through a point M' on GM; the point M' would be the metacentre if the body were floating so as to displace the internal liquid.

Then the restoring couple is seen to be

$$(W \times GM - W' \times GM')\theta,$$

and this quantity must be positive for stability.

Example 4. *A uniform thin metal cylinder which has a bottom but no top floats in water with its axis vertical. The cylinder is* 1 *ft. high,* 8 *in. radius and would contain three times its weight of water. Show that if it floats with its axis vertical the equilibrium is stable. Show also that if water of weight equal to half that of the cylinder is put into it the equilibrium is unstable.*

The centre of gravity of the cylinder (Fig. 323) is at height h in. above the base where

$$h = \frac{2\pi \times 8 \times 12 \times 6}{2\pi \times 8 \times 12 + \pi \times 8^2}$$
$$= 4{\cdot}5 \text{ in.}$$

In the first case 4 in. of the side is immersed and

$$GM = \frac{\pi \times 8^4}{\pi \times 8^2 \times 4 \times 4} - (4{\cdot}5 - 2)$$
$$= 4 - 2{\cdot}5 = 1{\cdot}5 \text{ in.}$$

Thus the equilibrium is stable.

Let W be the weight of the empty cylinder. When the water is put into the cylinder it sinks 2 in. and we have

$$GM = \frac{\pi \times 8^4}{\pi \times 8^2 \times 6 \times 4} - (4{\cdot}5 - 3$$
$$= \frac{7}{6} \text{ in.}$$

$$\frac{3}{2}W \times GM = \frac{7}{4}W.$$

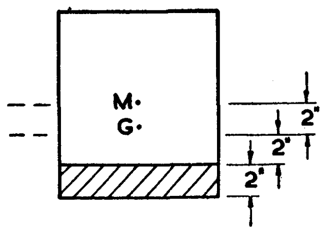

Fig. 323

Also
$$GM' = \frac{\pi \times 8^4}{\pi \times 8^2 \times 2 \times 4} - (4\cdot5 - 1)$$
$$= 4\cdot5 \text{ in.}$$
$$\frac{1}{2}W \times GM' = \frac{9}{4}W.$$

Then $\frac{3}{2}W \times GM - \frac{1}{2}W \times GM' = -\frac{1}{2}W$, and the equilibrium is unstable.

12.5 Experimental Determination of the Metacentric Height

The position of the metacentre of a ship may be found by moving a weight across the deck. If the weight is w and is displaced a distance l (Fig. 324) a couple wl is applied and if the ship thereupon lists through a small angle θ, the couple wl balances the restoring couple

$$W \times GM \times \theta.$$

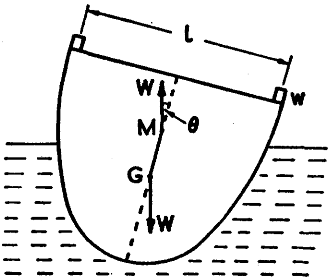

Fig. 324

Hence
$$GM = \frac{wl}{W\theta}.$$

Example 5. *A ship whose displacement is* 5000 *tons lists* 1 *degree when a load of* 10 *tons is moved* 35 *ft. across the deck. Find the metacentric height.*

$$\theta = 1 \text{ degree}$$
$$= 0\cdot0175 \text{ radians.}$$
$$GM = \frac{10 \times 35}{5000 \times 0\cdot0175}$$
$$= 4 \text{ ft.}$$

EXERCISES 12 (*a*)

1. A uniform solid cylinder has height h, radius r, and specific gravity s. Find the limits between which h/r must lie in order that the cylinder may float in water with its axis vertical. (L.U., Pt. II)

2. A uniform circular cylinder whose radius is two-thirds of its height floats in water with its axis vertical. Show that if its specific gravity lies between $\frac{1}{4}$ and $\frac{3}{4}$ its equilibrium cannot be stable. (L.U., Pt. II)

3. A uniform thin-walled tube, open at both ends, of length h and radius r floats in water with its axis vertical. If its specific gravity is 0·64, show that this position is stable if $12h < 25r$. (L.U., Pt. II)

4. A cylindrical block of wood of radius r, length $2r$ and weight W floats in water in unstable equilibrium with its axis vertical and one-quarter of its volume immersed. Show that the equilibrium may be rendered just stable by fastening to the base of the cylinder a metal plate of approximate weight $0\cdot41W$. (The volume of the plate may be neglected.) (L.U., Pt. I)

5. A uniform *thin* hollow circular cylinder without ends, made of material of specific gravity s floats in water with its axis vertical. Prove that the equilibrium will be stable if the ratio of the height to the radius of the base is less than $\{s(1 - s)\}^{1/2}$. (L.U., Pt. II)

6. A cube of side 2 ft. has a cylindrical hole of radius 6 in. drilled centrally through it at right-angles to one face. If it floats in water with the axis of the hole vertical, find the values between which its specific gravity must lie for stability. (L.U., Pt. II)

7. A uniform right circular solid cylinder of radius r, length $2l$ and specific gravity $\frac{1}{2}$ floats in water with its axis horizontal. Find approximately the work done in rotating the cylinder through a small angle θ about a horizontal axis through its centre perpendicular to the axis of symmetry. Deduce that the original position is stable for this type of displacement if $l > r$. (L.U., Pt. II)

8. A uniform wooden beam of square section floats in water with one face horizontal. Show that the equilibrium will be stable only if the specific gravity of the wood is either between 0·79 and 1·0 or less than 0·21 approximately. (L.U., Pt. II)

9. A uniform wooden rod whose cross-section is a right-angled isosceles triangle floats on water with the face containing the hypotenuse of each section horizontal and immersed. Show that, for the equilibrium to be stable, the specific gravity of the wood must be less than $\frac{8}{9}$. (L.U., Pt. I)

10 Determine the condition of stability of equilibrium for a solid of revolution floating with its axis vertical. A cylindrical hole of radius r is drilled centrally through a cube at right-angles to a pair of faces. If the cube floats in a liquid of double its density with the axis of the hole vertical, prove that the equilibrium is stable provided the radius of the hole lies between the positive roots of the equation $6\pi r^4 - 3\pi a^2 r^2 + a^4 = 0$, where a is the length of an edge of the cube. (L.U., Pt. II)

11. A homogeneous solid floating in equilibrium is removed from the liquid and placed in the inverted position in a second liquid of such density that the plane of flotation is the same as before. Show that the two positions are both stable or unstable.

A uniform lamina in the form of a regular pentagon floats symmetrically with its plane vertical, with three edges completely immersed. Prove that the equilibrium is stable for displacements in the vertical plane of the lamina if not more than one-quarter of each of the remaining edges is immersed. (L.U., Pt. II)

12. If a body be floating in equilibrium in a liquid of weight w per unit volume prove that the couple required to give it a small displacement θ about a horizontal axis is $w\theta(I - V.HG)$, where I is the second moment of the section of flotation about the axis, V the volume of liquid displaced, H the centre of buoyancy and G the centre of gravity of the body.

A rectangular block measuring 3 ft. $\times$ 2 ft. $\times$ 1 ft. floats in water

with its shortest edges vertical and one-third of its volume immersed. If the transference of a 1 lb. weight from one side to the other of the top face parallel to the 2 ft. edge tilts the block through $2\frac{1}{2}°$, find the height of the centre of gravity of the block above its base. (L.U., Pt.II)

13. A uniform thin rod, of length a, cross-section α and specific gravity s (< 1), has one end freely hinged at a point at a height na above the surface of the water in a large vessel, and is partially immersed making an angle θ with the vertical. Find the moment about the hinge which must be applied in order to maintain the rod in this position. Hence, show that, if V is the total potential energy of the system,

$$\frac{dV}{d\theta} = \frac{1}{2}a^2\alpha g \sin\theta(s - 1 + n^2 \sec^2\theta).$$

Prove that the vertical position is stable if $n^2 > (1 - s)$, and find the stable positions if $n^2 < (1 - s)$. (L.U., Pt. II)

14. A uniform solid cylinder, whose length is equal to 3/4 of its diameter and whose specific gravity is 1/2, floats in water. Show that there is a position of equilibrium in which the axis of the cylinder is inclined to the vertical, and that the equilibrium in this position is stable. (L.U., Pt. II)

15. A cylindrical can, open at one end, is made of uniform thin metal. Its height is equal to its diameter, its weight is W, and it would hold a weight $2W$ of water when full. Show that if floated empty in water, with its axis vertical and the open end uppermost, the equilibrium would be unstable, and find the least weight of water which must be put into the can to make it stable. (L.U., Pt. II)

12.6 The Quantity Equation

We now consider the motion of a perfect fluid without viscosity, and we consider, in the first place, the motion of incompressible fluids. When a liquid, such as water, is flowing through a channel there is evidence of viscosity in a tendency of particles of the liquid to adhere to the sides of the channel and to each other, so that the rate of flow may be greater in the centre than at the sides. We can treat such motion as if it were uniform with the mean velocity of the stream allowing, if necessary, for loss of velocity due to friction.

A *streamline* in a moving fluid is a line drawn in the fluid such that its direction at any point is the direction of motion of the particle at the point. If the motion of the fluid is steady, so that the velocity at any point is independent of the time, the streamlines are the paths of particles of the fluid and can be seen by introducing colouring matter in the fluid. If water is flowing along a channel at low velocity the streamlines will generally be parallel to the sides of the passage. If the velocity is large there may be cross-currents and eddies and the streamlines curved in various ways; this is called turbulent flow.

A *stream filament* is a tube bounded by streamlines and we shall

generally assume the cross-sectional area of a stream filament to be infinitesimal. No liquid can cross the streamline boundaries of a filament since the direction of motion is always tangential to the streamlines. Thus a stream filament resembles a material tube in that what goes in at one end must come out at the other end. It is assumed that there is no forming of cavities in the liquid such as may occur in water when the pressure (including atmospheric pressure) is below about 8 lb./in.^{2}.

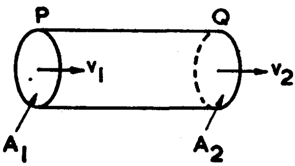

Fig. 325

Thus if A_1 and A_2 be the cross-sectional areas at two points P and Q on a stream filament, and v_1 and v_2 the velocities of the fluid at P and Q respectively (Fig. 325), the quantity of fluid passing the point P in unit time is A_1v_1, while the quantity passing Q in the same time is A_2v_2.

Thus $$A_1v_1 = A_2v_2.$$

This is known as the *quantity equation* and applies not only to motion along a stream filament but to steady motion along a pipe or channel, A_1 and A_2 being the cross-sectional area of the liquid at two points and v_1 and v_2 the corresponding velocities. The quantity of liquid Q which passes a point in unit time is therefore constant and the quantity equation may be written

$$Av = Q \text{ (a constant).}$$

Q is usually measured in cubic feet per second and is called the *discharge.* If ϱ be the density of the liquid, we have

$$\varrho Av = \varrho Q \text{ (a constant),}$$

and ϱQ is the *mass discharged,* usually in pounds per second. In this form the equation applies to the steady motion of a compressible fluid along a stream filament.

Example 6. *An open horizontal channel of rectangular cross-section varies gradually in width, being* 8 *ft. wide at a point A and* 10 *ft. wide at a point B. If the depth of the water is* 4 *ft. at A and* 2·2 *ft. at B, and the velocity at A is* 10 *ft./sec., find the velocity at B.*

The quantity passing the point A in 1 second is

$$8 \times 4 \times 10 = 320 \text{ cu. ft.}$$

If v ft./sec. be the velocity at B, the quantity passing B in 1 second is

$$10 \times 2{\cdot}2 \times v = 22v \text{ cu. ft.}$$

Therefore $$v = \frac{320}{22} = 14{\cdot}5 \text{ ft./sec.}$$

12.7 The Head of a Liquid

Consider a small mass w lb. of liquid which is under a pressure of p lb./sq. ft., which is at a height h ft. above some standard position and which is moving with a velocity of v ft./sec.

(i) *Pressure Head*

If ϱ lb./ft.^{3} be the density of the liquid, the pressure p could be due to the depth h_1 ft. of the mass below the free surface of the liquid, and $p = \varrho h_1$. If the liquid is under pressure without having a free surface we may still think of the pressure as due to a head h_1 of liquid, where $h_1 = p/\varrho$.

The quantity p/ϱ ft. is called the ***pressure head*** of the mass of liquid. wp/ϱ ft.lb. is called its ***pressure energy.***

(ii) *Potential Head*

The potential energy of the mass of liquid referred to the standard position is wh ft.lb. The height h ft. is called the ***potential head*** of the mass of liquid.

(iii) *Velocity Head*

If the velocity v were due to falling freely under gravity through a height h_2 ft. we would have $v^2 = 2gh_2$, and $h_2 = v^2/2g$ is called the ***velocity head*** of the mass of liquid. The kinetic energy is $wh_2 = wv^2/2g$ ft.lb.

A mass of liquid has thus three kinds of head and its total head H is

$$H = \frac{p}{\varrho} + \frac{v^2}{2g} + h \text{ ft.}$$

Bernoulli's theorem states that for a liquid of uniform density in steady motion the total head H is constant throughout the liquid. Alternatively, we may take Bernoulli's theorem as stating that the total energy, that is the sum of the kinetic, potential and pressure energy, is constant throughout the liquid.

12.8 Proof of Bernoulli's Theorem

Consider a stream filament AC in fluid of density ϱ between the cross-sections of the filament at A and C (Fig. 326). Let a_1 and a_2 be the areas of the cross-section at A and C respectively and let p_1, h_1, v_1 be the pressure, the height and the velocity at A and p_2, h_2, v_2 the corresponding quantities at C.

After small time δt let the liquid in the filament have taken up the position between A' and C'.

Then $AA' = v_1\delta t$ and $CC' = v_2\delta t$.

In this displacement of the fluid the thrusts on the end-sections are p_1a_1 and p_2a_2 respectively and the work done by these forces is

$$p_1a_1v_1\delta t - p_2a_2v_2\delta t.$$

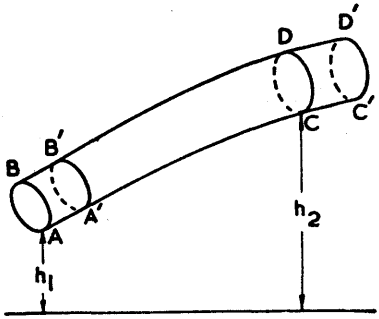

Fig. 326

The liquid has gained the potential energy of the mass be-

tween C and C' and lost the potential energy of the mass between A and A'. Hence the increase of potential energy is

$$\varrho a_2 v_2 \delta t \times h_2 - \varrho a_1 v_1 \delta t \times h_1.$$

It has also gained the kinetic energy of the mass between C and C' and lost that of the mass between A and A'. Hence the increase of kinetic energy is

$$\frac{1}{2g}(\varrho a_2 v_2 \delta t)v_2^2 - \frac{1}{2g}(\varrho a_1 v_1 \delta t)v_1^2.$$

Hence since the increase of energy is equal to the work done, we have

$$p_1 a_1 v_1 \delta t - p_2 a_2 v_2 \delta t$$
$$= \varrho a_2 v_2 \delta t\left(h_2 + \frac{v_2^2}{2g}\right) - \varrho a_1 v_1 \delta t\left(h_1 + \frac{v_1^2}{2g}\right).$$

Now from the quantity equation we have

$$a_1 v_1 = a_2 v_2,$$

therefore $$p_1 - p_2 = \varrho\left(h_2 + \frac{v_2^2}{2g}\right) - \varrho\left(h_1 + \frac{v_1^2}{2g}\right),$$

that is $$\frac{p_1}{\varrho} + \frac{v_1^2}{2g} + h_1 = \frac{p_2}{\varrho} + \frac{v_2^2}{2g} + h_2.$$

Hence the total head $\dfrac{p}{\varrho} + \dfrac{v^2}{2g} + h$ is constant along the stream filament.

If the liquid is in steady motion in a channel or pipe so that the velocity is uniform over a cross-section the velocity head is constant at all points of the cross-section. The total of the pressure head and potential head is also constant since the difference in potential head of two points of the cross-section is equal to their difference of pressure head. Thus the total head

$$H = \frac{p}{\varrho} + \frac{v^2}{2g} + h$$

is constant throughout the liquid.

Example 7. (*Toricelli's theorem.*) *An aperture in the side of a tank is at a depth h below the free surface of the liquid in the tank. Show that the velocity of the liquid at the aperture is* $\sqrt{(2gh)}$.

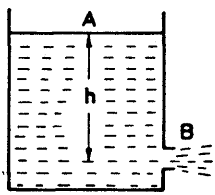

Fig. 327

Let A be a point on the free surface of the liquid and B the aperture (Fig. 327). We suppose that at A and B the atmospheric pressure is p_a and that the level of liquid in the tank is maintained constant.

At A the velocity head is zero, the potential head is h with respect to the level at B and the pressure head is p_a/ϱ.

Thus $$H = \frac{p_a}{\varrho} + h.$$

At B, let v be the velocity, the potential head is zero and the pressure head is p_a/ϱ, therefore
$$H = \frac{v^2}{2g} + \frac{p_a}{\varrho}.$$
Equating the two values of H we have
$$v^2 = 2gh.$$

Example 8. *Water is flowing along a pipe* 100 *ft. long. The pipe has a diameter of* 8 *in. at one end and tapers gradually to a diameter of* 4 *in. at the other. The end with the larger diameter is* 10 *ft. higher than the other. If the pipe discharges* 1000 *gallons per minute, find the difference of pressure between the ends of the pipe.*

The quantity discharged per second
$$= \frac{1000 \times 10}{62{\cdot}5 \times 60} \text{ cu. ft.}$$
$$= \frac{8}{3} \text{ cu. ft.}$$
Hence the velocities at the lower and upper ends are, respectively,
$$v_1 = \frac{8}{3} \times \frac{36}{\pi} = 30{\cdot}56 \text{ ft./sec.},$$
$$v_2 = \frac{8}{3} \times \frac{9}{\pi} = 7{\cdot}46 \text{ ft./sec.}$$
$$\frac{v_1^2}{2g} = \frac{(30{\cdot}56)^2}{64} = 14{\cdot}59 \text{ ft.},$$
$$\frac{v_2^2}{2g} = \frac{(7{\cdot}46)^2}{64} = 0{\cdot}91 \text{ ft.}$$
Taking the potential heads at the two ends as 0 and 10 ft. respectively and the pressures as p_1 and p_2 lb./ft.^{2}, we have, from Bernoulli's equation,
$$\frac{p_1}{\varrho} + 0 + 14{\cdot}59 = \frac{p_2}{\varrho} + 10 + 0{\cdot}91,$$
$$\frac{p_2 - p_1}{\varrho} = 3{\cdot}68 \text{ ft.},$$
$$p_2 - p_1 = 3{\cdot}68 \times 62{\cdot}5 \text{ lb./ft.}^2$$
$$= 1{\cdot}60 \text{ lb./in.}^2.$$

12.9 The Venturi Meter

The Venturi meter for measuring the flow of a liquid consists essentially of a length of pipe tapering to a narrow throat (Fig. 328). The difference in the pressure at the throat and at another point is measured by tubes inserted in the pipe at these points.

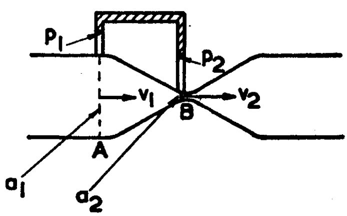

Fig. 328

Only the difference in pressure is required so that a U-tube containing mercury may be used.

Let a_1 and a_2 be the area of the cross-sections at the two points A and B and let v_1 and v_2 be the velocities at the points. Let the difference in the pressure heads at A and B be h. Then, assuming the pipe to be horizontal, we have from Bernoulli's equation.

$$\frac{p_1}{\varrho} + \frac{v_1^2}{2g} = \frac{p_2}{\varrho} + \frac{v_2^2}{2g},$$

$$\frac{p_1}{\varrho} - \frac{p_2}{\varrho} = h = \frac{v_2^2 - v_1^2}{2g}.$$

From the quantity equation we have

$$a_1 v_1 = a_2 v_2 = Q,$$

therefore
$$2gh = v_1^2\left(\frac{a_1^2}{a_2^2} - 1\right),$$

$$v_1 = \frac{a_2}{\sqrt{(a_1^2 - a_2^2)}}\sqrt{(2gh)},$$

$$Q = \frac{a_1 a_2}{\sqrt{(a_1^2 - a_2^2)}}\sqrt{(2gh)},$$
$$= ch^{1/2},$$

where c is a constant for the meter.

Due to frictional losses the total head at the throat is reduced and the observed value of h is too great. This is allowed for by introducing a coefficient k and taking the flow as $Q = kch^{1/2}$.

Values of k of about 0·97 are common.

Example 9. *A Venturi meter placed in a horizontal water pipe of* 4 *in. diameter reduces to* 1 *in. diameter and the pressure difference recorded at the two diameters is* 1·3 *in. of mercury* (*S.G.* 13·6). *Find the flow, assuming that the constant of the meter is* 0·97.

Here
$$h = \frac{1{\cdot}3 \times 13{\cdot}6}{12}$$
$$= 1{\cdot}473 \text{ ft. of water.}$$

$$c = \frac{a_1 a_2 \sqrt{(2g)}}{\sqrt{(a_1^2 - a_2^2)}},$$

$$a_1 = \frac{4\pi}{144} \text{ sq. ft.,}$$

$$a_2 = \frac{\pi}{4 \times 144} \text{ sq. ft.,}$$

$$c = \frac{4\pi \times 8}{144\sqrt{(255)}} = 0{\cdot}0437.$$

$$0{\cdot}97c = 0{\cdot}042,$$
$$h^{1/2} = 1{\cdot}213,$$
$$Q = 0{\cdot}051 \text{ cu. ft./sec.}$$
$$= 3{\cdot}1 \text{ cu. ft./min.}$$

12.10 Flow in a Rectangular Channel

If a liquid of density ϱ flows in a channel with a rectangular cross-section and a horizontal bottom (Fig. 329), let b be the width of the channel, h the depth of the liquid and u the velocity assumed uniform over a cross-section. The pressure on the free surface being atmospheric and equal to p_a, we have, from Bernoulli's equation,

$$\frac{p_a}{\varrho} + \frac{u^2}{2g} + h = \text{constant}.$$

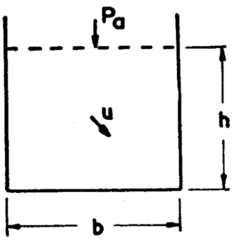

Fig. 329

The variation of atmospheric pressure being negligible as the depth varies we have

$$u^2 + 2gh = \text{constant}.$$

If the width b varies gradually, the quantity equation gives

$$ubh = \text{constant}.$$

If we consider small variations of u, b and h we have

$$u\delta u + g\delta h = 0,$$

$$\frac{\delta u}{u} + \frac{\delta b}{b} + \frac{\delta h}{h} = 0.$$

Eliminating δu we have $\quad \dfrac{\delta h}{\delta b} = \dfrac{hu^2}{b(gh - u^2)}.$

Thus if $u^2 < gh$, h and b increase together and u decreases as b increases.

To find the velocity u_1 at a point where $b = b_1$, we have

$$u^2 + 2gh = u_1^2 + 2gh_1,$$

$$ubh = u_1 b_1 h_1,$$

hence
$$u_1^3 - (u^2 + 2gh)u_1 + 2g\frac{hub}{b_1} = 0.$$

Example 10. *Water is flowing uniformly in a channel of rectangular cross-section with a horizontal bottom. If the channel is 8 ft. wide and the water 4 ft. deep at a point A, and the velocity at A is 10 ft./sec., find the velocity at a point B where the channel has increased in width to 10 ft.*

Let h ft. be the depth of water at B and v ft./sec. the velocity there. If the total head is the same at A and B, omitting the head due to atmospheric pressure we have

$$\frac{v^2}{2g} + h = \frac{10^2}{2g} + 4.$$

From the quantity equation we have

$$10 \times h \times v = 8 \times 4 \times 10.$$

Eliminating h we have $$\frac{v^2}{64} + \frac{32}{v} = \frac{100}{64} + 4,$$

$$v^3 - 356v + 2048 = 0,$$

whence $$v = 6{\cdot}54 \text{ ft./sec.}$$

$$h = 4{\cdot}90 \text{ ft.}$$

12.11 The Pitot Tube

The Pitot tube is an instrument by which the velocity of a liquid may be measured. It consists essentially of a tube containing mercury with one limb open at the end and placed parallel to the streamlines of the liquid (Fig. 330). The other limb, also parallel to the streamlines, is sealed at the end but has an opening in the side at B.

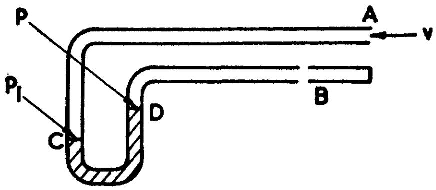

Fig. 330

Let p be the pressure of the liquid at A and v the velocity, and let p_1 be the pressure on the surface of the mercury at C. Then assuming Bernoulli's equation to hold inside the tube and the potential head to be constant we have, since the velocity is zero at C,

$$\frac{p}{\varrho} + \frac{v^2}{2g} = \frac{p_1}{\varrho}.$$

Since the limb at B is parallel to the stream lines no liquid enters the opening and the pressure just inside the tube at B is equal to that outside which is p. Thus the pressure on the mercury at D is the pressure p and the difference in the level of the mercury at D and C measures the difference $p_1 - p$.

Thus if h be the difference of the levels at D and C, and s the specific gravity of the mercury, we have

$$p_1 - p = s\varrho h,$$

and hence $$v^2 = 2gsh.$$

A simple form of Pitot tube consists of a glass tube with its lower end bent through a right-angle and open at both ends. The tube is placed in a stream with the bent open end facing the direction of flow (Fig. 331). The water enters the tube and rises above the free surface inside the tube. Thus the

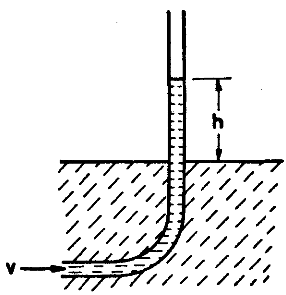

Fig. 331

velocity head of the flowing liquid is converted into a potential head. If v be the velocity and h the height of liquid in the tube above the free surface we have

$$\frac{v^2}{2g} = h,$$

and thus the velocity is determined.

12.12 Free Surface of a Rotating Liquid

Let a circular cylinder containing liquid rotate about its axis, which is vertical, with angular velocity ω (Fig. 332). Let an element of the liquid of mass w be distant r from the axis of rotation on the free surface at P and at a height z above the lowest point of the free surface O. The centrifugal force on the mass due to rotation is $\frac{w}{g}\omega^2 r$ acting horizontally and its weight w acts vertically downwards. The components of these two forces along the tangent to the free surface must balance, therefore, if ψ be the inclination to the horizontal of the tangent at P, we have

$$\frac{w}{g}\omega^2 r \cos\psi = w \sin\psi,$$

$$\tan\psi = \frac{\omega^2}{g} r,$$

that is

$$\frac{dz}{dr} = \frac{\omega^2}{g} r,$$

$$z = \frac{\omega^2}{2g} r^2 + \text{constant.}$$

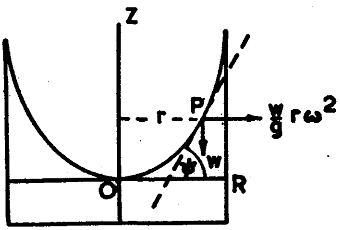

Fig. 332

Since $z = 0$ when $r = 0$ the constant is zero and we have the equation of a parabola

$$z = \frac{\omega^2}{2g} r^2.$$

For the surface as a whole, writing $r^2 = x^2 + y^2$, we have the equation of a paraboloid.

$$2gz = \omega^2(x^2 + y^2).$$

If a be the radius of the cylinder, the height h of the paraboloid is

$$h = \frac{\omega^2}{2g} a^2$$

$$= \frac{v^2}{2g},$$

where v is the greatest fluid velocity.

The centrifugal thrust on a small radial element inside the liquid of length δr and volume $A\delta r$ distant r from the axis is

$$\frac{\varrho A\delta r}{g}\omega^2 r.$$

Thus the centrifugal pressure per unit area is $\frac{\varrho\omega^2}{g}r\delta r$.

Integrating this quantity from r to a, we have the centrifugal intensity of pressure at the boundary

$$\begin{aligned} p_1 &= \int_r^a \frac{\varrho\omega^2}{g} r\,dr \\ &= \frac{\varrho\omega^2}{2g}(a^2 - r^2), \end{aligned}$$

r being the radius to the inner surface.

Thus the centrifugal head at the boundary is

$$\begin{aligned} \frac{p_1}{\varrho} &= \frac{\omega^2}{2g}(a^2 - r^2), \\ &= \frac{v_2^2}{2g} - \frac{v_1^2}{2g}, \end{aligned}$$

where v_1 and v_2 are the velocities of the fluid at the inner and outer surfaces.

12.13 Fluid-filled Rotating Cylinder

Let a closed cylinder of radius R full of liquid of density ϱ be turning about its axis, which is vertical, with angular velocity ω.

The centrifugal intensity of pressure at distance r from the centre is, from § 12.12,

$$\frac{\varrho\omega^2}{2g}r^2.$$

This pressure acts outwards from the centre, but as the pressure is the same in all directions, it acts on the top of the cylinder.

The thrust on a small ring of the top of radius r and width δr is therefore

$$2\pi r\delta r . \frac{\varrho\omega^2}{2g}r^2.$$

The total upward thrust on the top of the cylinder is

$$\begin{aligned} &\frac{\pi\varrho\omega^2}{g}\int_0^R r^3 dr \\ &= \frac{\pi\varrho\omega^2 R^4}{4g}. \end{aligned}$$

Example 11. *A closed cylinder of* 6 *in. internal diameter is filled with mercury and rotates about its axis, which is vertical, at* 900 *r.p.m. Find the liquid thrust on the top of the cylinder.*

Here $$\omega = 30\pi \text{ rad./sec.},$$

and the thrust is
$$\frac{\pi \times 62{\cdot}5 \times 13{\cdot}6 \times (30\pi)^2 \times (1/4)^4}{4 \times 32}$$
$$= 724 \text{ lb.}$$

12.14 Time to Empty a Tank

When a liquid flows through an orifice in the base or side of a tank there is a convergence of streamlines at the orifice with the result that the jet which issues has a cross-section appreciably less than that of the orifice. The section of the jet xx' at which the streamlines first become parallel (Fig. 333) is called the *vena contracta* and this is the effective cross-section of the orifice at which the discharge is to be calculated.

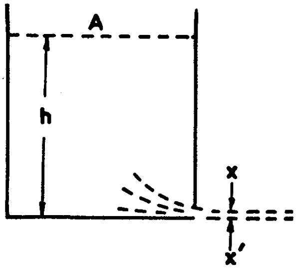

Fig. 333

Let A be the uniform cross-sectional area of a tank, h the height of the liquid in the tank at any instant, a the effective area of an orifice in the base and v the velocity of the emerging liquid.

Since the potential head h is being converted to a velocity head $v^2/(2g)$, we have
$$v = \sqrt{(2gh)}.$$
The quantity emerging in time δt is
$$av\delta t = a(\sqrt{2gh})\delta t.$$
If the free surface falls through $-\delta h$ in this time the quantity emerging is $-A\delta h$.

Therefore
$$-A\delta h = a\sqrt{(2gh)}\delta t,$$
$$\frac{dh}{dt} = -\frac{a\sqrt{(2gh)}}{A}.$$
If T be the time for the level to fall from H to 0, we have
$$T = -\frac{A}{a\sqrt{(2g)}}\int_H^0 h^{-1/2}dh$$
$$= -\frac{2A}{a\sqrt{(2g)}}\Big[h^{1/2}\Big]_H^0$$
$$= \frac{2A}{a\sqrt{(2g)}}.H^{1/2}$$
$$= \frac{AH}{\frac{1}{2}a\sqrt{(2gH)}}.$$

This is the volume of the liquid divided by one-half of the maximum rate of discharge.

If the tank is discharging into another tank of uniform cross-sectional area A_1 let h_1 be the height of the free surface in the second tank, and let $h - h_1 = H$ initially (Fig. 334).

Then $\quad v^2 = 2g(h - h_1)$, and $-A\delta h = A_1\delta h_1$.

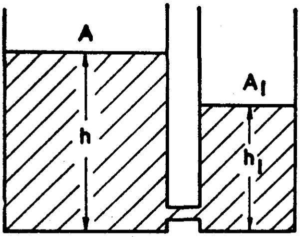

Fig. 334

Therefore $a\{2g(h - h_1)\}^{1/2}\delta t = -A\delta h$.

Writing $x = h - h_1$,

$$\delta x = \delta h - \delta h_1$$

$$= \delta h\left(1 + \frac{A}{A_1}\right),$$

$$\delta h = \frac{A_1}{A + A_1}\delta x.$$

Then
$$\delta t = -\frac{AA_1}{A + A_1}\cdot\frac{1}{a\sqrt{(2g)}}x^{-1/2}\delta x,$$

$$T = \frac{AA_1}{A + A_1}\cdot\frac{1}{a\sqrt{(2g)}}\int_0^H x^{-1/2}dx$$

$$= \frac{AA_1H}{A + A_1}\cdot\frac{1}{\frac{1}{2}a\sqrt{(2gH)}}.$$

It is easily seen that $AA_1H/(A + A_1)$ is the volume of liquid eventually transferred to the second tank, and the time is this volume divided by one-half of the maximum rate of discharge.

EXERCISES 12 (*b*)

1. Water at a pressure of 4 lb./in.^{2} is pumped into a tank through an orifice 10 ft. below the free surface. Neglecting friction, find the velocity of flow through the orifice and the rate at which work is being done.

2. Water flows through an orifice in the vertical side of a tank the diameter of the jet being 2 in. The jet moves 4 ft. horizontally while falling 2 ft. Find the rate of discharge of water in gallons per minute and the height of the free surface above the orifice.

3. A water channel has a rectangular section and a horizontal bottom. The channel is 12 ft. wide and the water is 4 ft. deep, the flow being 384 cu. ft./sec. The channel gradually widens to a point at which the velocity of the water is 6 ft./sec. Find the width of the section and the depth of the water at this point.

4. A water channel has a rectangular section and a horizontal bottom. At a point where the velocity is 6 ft./sec., the channel is 8 ft. wide and

the water is 3 ft. deep. The channel gradually widens to 9 ft. Find the new velocity and depth of the water.

5. A channel has a horizontal bottom and vertical sides. At first the breadth is uniform and equal to b, then the bottom widens gradually and symmetrically until it is again of uniform breadth now equal to b_1. Water flows steadily along the channel. If u and h are the speed and depth where the breadth is b, and if h_1 is the depth where the breadth is b_1, prove that $h_1 < h$ if $u^2 > 2gh_1^2/(h_1 + h)$.

(L.U., Pt. II)

6. A water tap of a diameter 1/4 in. is 60 ft. below the level of the reservoir which supplies water to a town. Find the maximum amount of water which would be delivered by the tap in gallons per hour.

(L.U., Pt. II)

7. A vertical water pipe reduces gradually from 9 in. diameter at a point A to 8 in. diameter at a point B, 20 ft. below A. The flow is 5 cu. ft./sec. and the pressure at A is 10 lb./in.^{2}. Find the pressure at B.

8. Petrol is led steadily through a pipeline which passes over a hill into a valley. If the crest of the hill is at a height h above the level of the pipe in the valley show that by properly adjusting the ratio of the cross-sections of the pipe at the crest and in the valley the pressure may be equalised at these two places. (L.U., Pt. II)

9. A horizontal pipe running full of water tapers from 8 in. diameter at a point A to $1\frac{1}{2}$ in. diameter at a point B. The difference of the pressures at A and B is that due to a head of 10 ft. of water. Find the velocity of the water at the larger section and the flow in gallons per minute.

10. Liquid of density ϱ is flowing along a horizontal pipe of variable cross-section, and is connected with a differential pressure gauge at two points A and B. Show that, if $p_1 - p_2$ is the pressure indicated by the gauge, the mass of liquid flowing through the pipe per second is

$$\{2\sigma_1^2\sigma_2^2\varrho(p_1 - p_2)/(\sigma_1^2 - \sigma_2^2)\}^{1/2}$$

where σ_1, σ_2 are the areas of the cross-sections at A and B respectively.

(L.U., Pt. II)

11. Water flows steadily through a horizontal pipe at the rate of 2 cubic feet per second. At a point A the sectional area of the pipe is 0·25 sq. ft. and the pressure at the centre of this section is 3000 lb. per sq. ft. At a point B, down-stream from A, the sectional area is 0·125 sq. ft. and the pressure at the centre of this section is 1200 lb. per sq. ft. Find the loss of energy per cubic foot of water between A and B.

(L.U., Pt. II)

12. A vertical water pipe is 2 in. in diameter at a point A and 1 in. in diameter at a point B, 6 ft. below A. Pressure gauges at A and B record a pressure difference of 4·5 lb./in.^{2}. Neglecting frictional losses, find the quantity of water flowing through the pipe.

13. A Venturi meter tapers from 12 in. diameter to 4 in. diameter at the throat and the recorded pressure difference at the two points is 2 in. of mercury (S.G. 13·6). If the coefficient for the meter is 0·97, calculate the discharge in gallons per minute.

14. A Pitot tube placed in a water pipe 9 in. in diameter parallel to the flow records a difference of pressure of 3 in. of water at its two orifices. Find the discharge through the pipe.

15. Water is flowing from a tank A in which the water level is 20 ft. above the orifice into a tank B in which the water level is 10 ft. above the orifice. Find the velocity of the water. If the tanks are of the same uniform cross-section of 8 sq. ft. and the effective cross-section of the orifice is 1 in.^{2}, find how long it will take before the levels in the two tanks are equal.

16. A hemispherical bowl with its axis vertical is filled with water and discharges through an orifice at its lowest point. Show that time of discharge is the volume of water divided by $\frac{5}{7}$ths of the maximum rate of discharge.

17. A boiler is a cylinder of 4 ft. diameter with its axis horizontal and 10 ft. long. If it is half full, find the time to empty the boiler through an orifice of effective diameter 2 in.

18. A cylinder 8 in. in diameter containing liquid is placed centrally on a circular disc rotating in a horizontal plane. The paraboloid formed by the rotating liquid is 8 in. in height. Find the rate at which the disc is rotating.

19. A closed cylinder of 4 in. radius is filled with water and placed centrally on a disc rotating in a horizontal plane. When the thrust of the liquid on the top of the cylinder is 50 lb., find the speed of rotation of the disc.

20. Water flows up a vertical pipe and then flows radially between two horizontal circular plates of radius R a small distance d apart. Show that if p_0 is the pressure, v_0 the velocity in the pipe and r its radius, the pressure on the plates at distance x from the centre is

$$p_0 + \frac{\varrho v_0^2}{2g}\left(1 - \frac{r^4}{4d^2x^2}\right).$$

Hence, show that the total pressure on one of the plates is

$$\varrho\pi(R^2 - r^2)\left(\frac{p_0}{\varrho} + \frac{v_0^2}{2g}\right) - \frac{\pi\varrho r^2 v_0^2}{4gd^2}\log_e\frac{R}{r}.$$

12.15 Stream Functions

Two-dimensional flow of a fluid is completely defined by the flow in one plane, the flow in any parallel plane being exactly the same. Thus streamlines drawn in the plane represent the flow in all parallel planes. An obstacle bounded by surfaces perpendicular to the plane will be represented by a curve in the plane, and the streamlines will give the pattern of the flow round the obstacle. If the flow is steady the condition of continuity must hold everywhere in the plane and we may apply the continuity equation to a unit depth of fluid moving parallel to the plane.

Consider a point P in the plane whose coordinates with reference to a fixed origin O are (x, y) and let OCP be any curve in the plane joining the points O and P (Fig. 335). The *flux* ψ across the line OCP is defined as the volume of fluid per unit depth crossing the boundary OCP in unit time, the flux being positive if to an observer at O looking along OP the flow is from left to right.

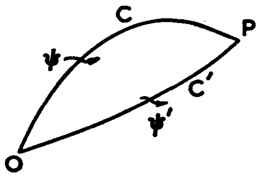

Fig. 335

Let ψ' be the flux across any other curve $OC'P$ in the plane. From the condition of continuity we have $\psi' = \psi$, and hence the flux across any curve joining O to P depends only on the coordinates of P and we may write it as $\psi(x, y)$.

The function of position $\psi(x, y)$ is called the *stream function* of the flow. The stream function is defined with reference to a particular origin O, but a change of origin to O' merely adds to the flux across OP the constant flux across $O'O$ and increases the stream function by a constant. The stream function may also be written as $\psi(r, \theta)$ in terms of polar coordinates r and θ.

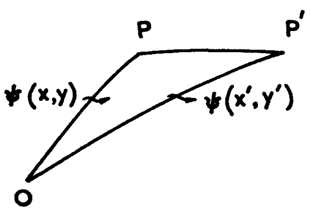

Fig. 336

It is evident that if (x', y') be the coordinates of another point P' in the plane (Fig. 336) the flux across PP' is

$$\psi(x', y') - \psi(x, y).$$

If P and P' be two points on a streamline there is no flux across PP' and hence

$$\psi(x', y') = \psi(x, y).$$

Thus the stream function has a constant value along a streamline and the equations of the streamlines are therefore

$$\psi(x, y) = C_1,$$
$$\psi(x, y) = C_2, \text{ etc.}$$

where C_1, C_2, etc., are constants. The flux across any line joining the streamlines $\psi = C_1$ and $\psi = C_2$ is $C_1 - C_2$, and streamlines are usually drawn with equal differences $C_1 - C_2$, $C_2 - C_3$, etc., so that equal quantities of fluid pass between the streamlines.

12.16 Differentiation of a Stream Function

Let $\psi(x, y)$ be the stream function at a point P whose coordinates are (x, y) in a steady two-dimensional flow, and let u and v be the components of the velocity of flow at P parallel to the axes of x and y respectively. Let P' be the point $(x + \delta x, y + \delta y)$ and $P'NP$ a right-angled triangle with sides parallel to the axes (Fig. 337).

The flow across NP' is $u\delta y$, and across PN it is $-v\delta x$. Hence, from the condition of continuity, the flow across PP' is $-v\delta x + u\delta y$.

Now the flow across PP' is the increase of the stream function ψ, and, δx and δy being infinitesmal, we have

$$\delta\psi = \frac{\partial\psi}{\partial x}\delta x + \frac{\partial\psi}{\partial y}\delta y.$$

Therefore
$$u = \frac{\partial\psi}{\partial y},$$
$$v = -\frac{\partial\psi}{\partial x}.$$

If the components of velocity are known at any point in the flow, the stream function may be found by integrating the equation

$$\delta\psi = -v\delta x + u\delta y.$$

Thus if the flow has everywhere constant velocity u_0 parallel to the x-axis we have

$$\frac{\partial\psi}{\partial y} = u_0,\ \frac{\partial\psi}{\partial x} = 0,$$
$$\psi = u_0 y + \text{constant},$$

and the streamlines are parallel to the x-axis.

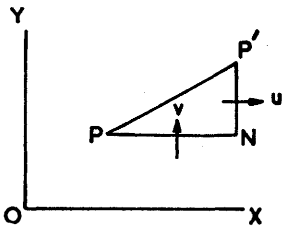

Fig. 337

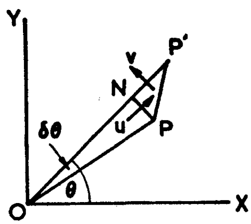

Fig. 338

If the flow has everywhere a velocity with components u_0 and v_0 parallel to the x- and y-axes respectively, we have

$$\frac{\partial\psi}{\partial y} = u_0,\ \frac{\partial\psi}{\partial x} = -v_0,$$
$$\psi = u_0 y - v_0 x + \text{constant},$$

and the streamlines are parallel to the straight line $u_0 y - v_0 x = 0$.

In polar coordinates, let the coordinates of the points P and P' be (r, θ) and $(r + \delta r, \theta + \delta\theta)$ respectively, and let the velocity of flow at P have components u along the radius vector and v perpendicular to it in the direction of θ increasing (Fig. 338). Considering the flow across the sides of the triangle PNP' whose lengths are $r\delta\theta$ and δr, we find that the flow across PN and NP' is

$$ur\delta\theta - v\delta r.$$

The flow across PP' is $\delta\psi = \frac{\partial\psi}{\partial r}\delta r + \frac{\partial\psi}{\partial\theta}\delta\theta.$

Hence
$$u = \frac{1}{r}\frac{\partial\psi}{\partial\theta},$$
$$v = -\frac{\partial\psi}{\partial r}.$$

12.17 Sources and Sinks

A *source* is a point in the plane of a two-dimensional flow at which liquid appears, and the strength of the source is the quantity of liquid (per unit depth) which appears in unit time. If m be the strength of the source a quantity m must cross the circumference of a circle of radius r about the source in unit time, and hence the velocity of flow at a distance r from the source is $\frac{m}{2\pi r}$ along the radius from the source.

Thus in polar coordinates, the origin being at the source, we have
$$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{m}{2\pi r}, \quad \frac{\partial\psi}{\partial r} = 0,$$
and hence
$$\psi = \frac{m}{2\pi}\theta + \text{constant}.$$
Thus the streamlines are straight lines radiating from the source. The quantity ψ is rendered single valued and the source itself is excluded by erecting an imaginary barrier along the positive x-axis. Thus we write
$$\psi = \frac{m}{2\pi}\theta, \text{ for } 0 \leqslant \theta < 2\pi.$$

In Cartesian coordinates we have
$$\psi = \frac{m}{2\pi}\tan^{-1}\frac{y}{x}, \text{ for } 0 \leqslant \tan^{-1}\frac{y}{x} < 2\pi.$$

If the source is at a point (a, b) we have
$$\psi = \frac{m}{2\pi}\tan^{-1}\frac{y-b}{x-a}.$$

A *sink* is a point at which liquid disappears, that is a negative source, and for a sink of strength m at the point (a, b) the stream function is
$$\psi = -\frac{m}{2\pi}\tan^{-1}\frac{y-b}{x-a}.$$

12.18 Superposition of Stream Functions

Stream functions for more complicated patterns of flow may be obtained by adding the stream functions for linear flow and those for sources and sinks.

Example 12. *The stream function for a source and an equal sink.*

Let the source and the sink each be of strength m and lie on the x-axis at distances c and $-c$ respectively from the origin.

The stream function due to the source is $\frac{m}{2\pi}\tan^{-1}\frac{y}{x-c}$, and that due to the sink is $-\frac{m}{2\pi}\tan^{-1}\frac{y}{x+c}$.

Hence, for the source and sink together we have

$$\begin{aligned}\psi(x, y) &= \frac{m}{2\pi}\tan^{-1}\frac{y}{x-c} - \frac{m}{2\pi}\tan^{-1}\frac{y}{x+c}\\ &= \frac{m}{2\pi}\tan^{-1}\frac{2cy}{x^2+y^2-c^2}.\end{aligned}$$

The streamlines are given by $\psi(x, y) = \text{constant}$, that is

$$\frac{x^2+y^2-c^2}{2cy} = b,$$

where b is a constant,

that is $\quad x^2 + (y - bc)^2 = c^2(1 + b^2).$

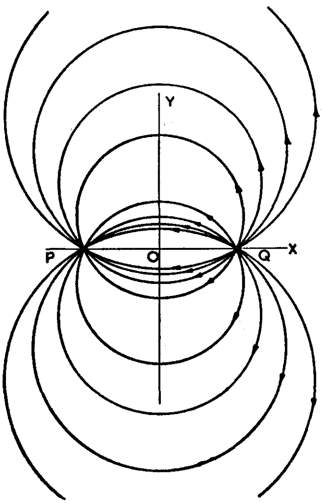

Fig. 339

Thus the streamlines are circles with centres at $(0, bc)$ on the y-axis and passing through both the source and the sink (Fig. 339). Each circle is divided into two parts by the line PQ joining the source to the sink and each arc is a different streamline. This is evident since the flow pattern must be the same above and below the x-axis. The discontinuity on the line PQ arises from the fact that $\tan^{-1}(1/b)$ may have one of two values which differ by π, and thus there are two values of the stream function for any value of b. The line PQ therefore forms an imaginary barrier and the flow pattern is as shown in Fig. 339.

Example 13. *The stream function for a doublet.*

If the source and the sink of Example 12 approach each other until they coincide, and at the same time their strengths increase so that cm remains constant, we obtain an imaginary flow pattern known as *doublet* flow.

The expression obtained for the stream function in Example 12 was

$$\psi = \frac{m}{2\pi}\tan^{-1}\frac{2cy}{x^2+y^2-c^2}.$$

In this expression let c approach zero, while $2cm = \mu$, a constant.

Then
$$\psi = \lim_{c\to 0}\frac{\mu}{4\pi c}\tan^{-1}\frac{2cy}{x^2+y^2-c^2}$$
$$= \lim_{c\to 0}\frac{\mu}{4\pi c}.\frac{2cy}{x^2+y^2-c^2}$$
$$= \frac{\mu}{2\pi}.\frac{y}{x^2+y^2}.$$

Thus the streamlines are given by
$$\frac{x^2+y^2}{2y} = b, \text{ a constant,}$$
$$x^2+(y-b)^2 = b^2.$$

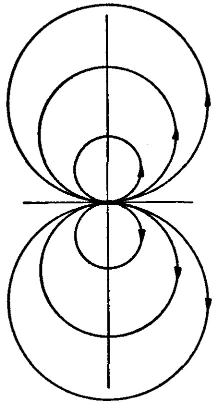

Fig. 340

The streamlines are therefore circles with centres at $(0, b)$ and radius b (Fig. 340), μ is called the *moment of the doublet* and the line joining the original source and sink is called the *axis of the doublet*. The direction of flow along the axis of the doublet is in the positive direction of the x-axis, and this flow would be reversed by changing the sign of μ.

Example 14. *The stream function for a source in an infinite stream.*

Let the stream have velocity U in the direction of the negative x-axis and let the source of strength m be at the origin.

Then
$$\psi(x, y) = -Uy + \frac{m}{2\pi}\tan^{-1}\frac{y}{x},$$

and we may take $-\pi < \tan^{-1} y/x \leqslant \pi$, which is equivalent to taking a barrier along the negative x-axis.

The streamline $\psi = 0$ is
$$-Uy + \frac{m}{2\pi}\tan^{-1}\frac{y}{x} = 0,$$
$$y = x\tan\frac{2\pi Uy}{m}.$$

This is satisfied by the straight line $y = 0$, and by the curve $y = x\tan\dfrac{2\pi Uy}{m}$ for $\left|\dfrac{2\pi Uy}{m}\right| \leqslant \pi$, that is $|y| < \dfrac{m}{2U}$.

If y is small we have approximately
$$y = x.\frac{2\pi Uy}{m},$$

and hence the curve crosses the x-axis at $\left(\dfrac{m}{2\pi U}, 0\right)$. The shape of this curve is seen in Fig. 341.

For other values of ψ lying between $-\frac{1}{2}m$ and $\frac{1}{2}m$ there are two branches of each streamline, one lying inside the curve $\psi = 0$ and the other outside in the opposite half plane. The flow pattern shows the flow due to the source contained within the streamline $\psi = 0$ and the uniform stream divided by this curve and passing outside it.

If we consider the streamline $\psi = 0$ as the boundary of a solid placed in the

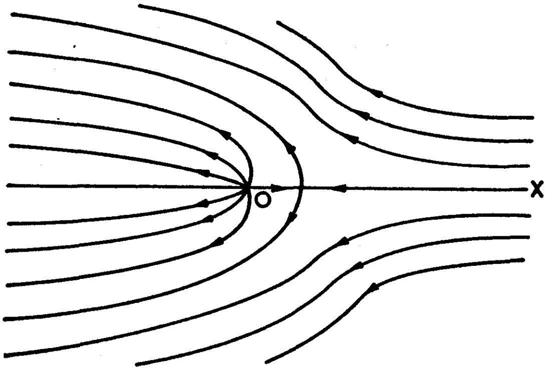

Fig. 341

infinite stream the outside pattern of flow is unchanged and we thus have the flow around a body of this shape placed in the infinite stream.

For the velocity of flow in directions parallel to the axes we have

$$u = \frac{\partial\psi}{\partial y} = -U + \frac{m}{2\pi}.\frac{x}{x^2+y^2},$$

$$v = -\frac{\partial\psi}{\partial x} = \frac{m}{2\pi}.\frac{y}{x^2+y^2}.$$

Thus at the point $\left(\frac{m}{2\pi U}, 0\right)$ we have $u = v = 0$. This point is called the *stagnation point* for the flow round the body.

Example 15. *A doublet in an infinite stream.*

Let the stream have velocity U in the direction of the negative x-axis and let the x-axis be the axis of a doublet at the origin and μ its moment.

Then $$\psi(x, y) = -Uy + \frac{\mu}{2\pi}.\frac{y}{x^2+y^2}.$$

The streamline $\psi = 0$ gives, $\quad y = 0,$

or $$x^2 + y^2 = \frac{\mu}{2\pi U}.$$

Thus we have the x-axis and a circle of radius $\{\mu/(2\pi U)\}^{1/2}$ about the origin. Denoting this radius by a we have

$$\psi(x, y) = -Uy\left(1 - \frac{a^2}{x^2+y^2}\right).$$

Since there is no flow across the streamline $\psi = 0$, ψ is the stream function for the flow round a circular cylinder of radius a placed in the stream and the flow pattern is as shown in Fig. 342.

In polar coordinates we have

$$\psi = -Ur \sin\theta\left(1 - \frac{a^2}{r^2}\right),$$

$$-\frac{\partial\psi}{\partial r} = U \sin\theta\left(1 + \frac{a^2}{r^2}\right),$$

$$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = -Ur \cos\theta\left(1 - \frac{a^2}{r^2}\right).$$

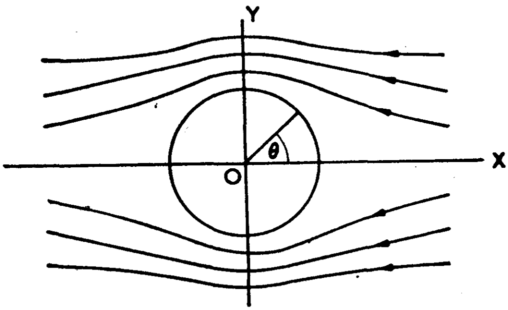

Fig. 342

Thus on the circumference where $r = a$, the radial velocity is zero and the tangential velocity is $2U \sin \theta$.

If p_0 be the pressure in the stream at a great distance from the cylinder where the velocity is U, and p the pressure on the cylinder at the point (a, θ), we have, from Bernoulli's equation,

$$\frac{p}{\varrho} + \frac{4U^2 \sin^2 \theta}{2g} = \frac{p_0}{\varrho} + \frac{U^2}{2g},$$

$$p - p_0 = \frac{\varrho U^2}{2g}(1 - 4 \sin^2 \theta).$$

We thus have an expression for the pressure at any point of a cylinder placed in an infinite stream of an ideal fluid. It will be noticed that the same pressure is obtained for θ and $\pi - \theta$, and hence that the total pressure on the cylinder is zero.

EXERCISES 12 (*c*)

1. Given the stream function for a two-dimensional field of flow, show how to obtain the x and y velocity components at a point in the field.

   If $u = 2x$, $v = -2y$ are respectively the x and y components of a possible fluid motion, determine the stream function and plot the streamlines $\psi = 1$, 2 and 3. (L.U., Pt. I)

2. Explain the meaning of the term "stream function" and state the conditions of flow for which it can be used. Derive from your definition the stream function in polar coordinates, for the following cases:
   (*a*) Uniform velocity in a straight line inclined at an angle a to the x-axis.
   (*b*) Flow from a source.
   (*c*) Flow in a doublet.

   Use a sketch to show in reasonable proportions the result of adding cases (*a*) and (*b*). (L.U., Pt. I)

3. A line source of strength 50π ft./sec. units is situated in a uniform stream flowing at 40 ft./sec. At a point 2 ft. downstream of the source there is an equal sink. Locate the two points of zero velocity in the resultant field of flow and show how to trace the streamlines passing through these two points. (L.U., Pt. I)

4. Considering two-dimensional flow of an incompressible fluid determine the stream function for a doublet of strength 50 ft. sec. units situated in a uniform stream whose undisturbed velocity is 30 ft./sec., in a direction parallel to the axis of the doublet and opposite to the doublet flow along this axis. (L.U., Pt. I)

12.19 Properties of Compressible Fluids

The pressure p, the volume V and the temperature T of a given mass of gas are linked by the characteristic equation

$$p.V = K.T,$$

where K is a constant. The equation is usually applied to a unit mass of gas and the volume of the unit mass is denoted by v. Thus if p lb. per sq. ft. be the pressure, v the volume of 1 lb. of gas in cu. ft., T the absolute temperature in degrees centigrade, we have

$$pv = RT,$$

and the gas constant R is in ft.lb. per degree centigrade. For the air of the atmosphere $R = 96$ approximately.

If ϱ be the density of the gas in lb. per cu. ft. we have $\varrho v = 1$, and the characteristic equation may be written as

$$\frac{p}{\varrho} = RT.$$

When heat is absorbed by a gas it may expand and at the same time its temperature may increase. The work done in expanding may be measured in mechanical units such as foot pounds and the heat may be measured in calories. To measure the effect of heat we equate the heat to its equivalent mechanical energy according to the formula

$$1 \text{ calorie} = 3{\cdot}08 \text{ ft.lb., approximately.}$$

We thus have an equation of conservation of energy for a unit mass of gas.

Let H be the heat absorbed by the gas,
W the work done by the gas,
E_1 the internal energy of the gas, that is the heat that it contains, before the heat is absorbed,
E_2 the internal energy after the heat has been absorbed,

then
$$H = W + (E_2 - E_1).$$

Thus part of the heat absorbed goes to expand the gas and part to warm the gas.

If the temperature of the gas remains constant there is no increase in its internal energy and the heat is entirely absorbed in expansion. The expansion is then called *isothermal* and we have $H = W$.

If the gas is contained in a vessel through which no heat can permeate and allowed to expand, we have $H = 0$,

and
$$W + (E_2 - E_1) = 0.$$

Thus the work done in expanding reduces the internal energy of the gas. In this case the expansion is called *adiabatic.*

The assumption that the expansion is isothermal or adiabatic simplifies the equation of energy and shows exactly how the energy is transferred.

12.20 Specific Heats

The specific heat of a gas is the amount of heat required to raise the temperature of a unit mass by one degree.

The specific heat k_v, at constant volume, relates to a gas in a closed vessel so that its volume cannot change.

The specific heat k_p, at constant pressure, relates to a gas which is open to the air or is otherwise kept at a constant pressure.

Either quantity may be measured in heat units or in energy units per degree.

The ratio of the specific heats is denoted by the symbol γ, and we have

$$\frac{k_p}{k_v} = \gamma.$$

For *air* at atmospheric pressure $\gamma = 1{\cdot}404$ approximately. Although the specific heats vary for different initial temperatures, their difference $k_p - k_v$ remains constant.

If the volume of a unit mass of gas remains constant the change in pressure δp due to a small increase of temperature δT is found from the characteristic equation to be

$$\frac{\delta p}{p} = \frac{\delta T}{T}.$$

The heat required to raise the temperature by δT at constant volume is therefore

$$k_v\delta T = k_v T\frac{\delta p}{p}.$$

If the pressure remains constant the change in volume δv due to a rise δT in temperature is, from the characteristic equation,

$$\frac{\delta v}{v} = \frac{\delta T}{T}.$$

Thus the heat required to raise the temperature by δT at constant pressure is

$$k_p\delta T = k_p T\frac{\delta v}{v}.$$

The heat absorbed when the volume changes by δv and the pressure by δp is therefore

$$T\left(k_p\frac{\delta v}{v} + k_v\frac{\delta p}{p}\right).$$

If the expansion of the gas is *isothermal,* so that its temperature remains constant, we have

$$\frac{\delta p}{p} + \frac{\delta v}{v} = 0,$$

and the heat absorbed is

$$T\left(k_p\frac{\delta v}{v} + k_v\frac{\delta p}{p}\right) = T(k_p - k_v)\frac{\delta v}{v}.$$

The total heat absorbed, that is the work done, in an expansion from volume v_1 to volume v_2 is then

$$W = \int_{v_1}^{v_2} T(k_p - k_v)\frac{dv}{v},$$

$$= T(k_p - k_v)\log\frac{v_2}{v_1}. \tag{1}$$

If the expansion is *adiabatic,* then no heat is absorbed and we have

$$T\left(k_p\frac{\delta v}{v} + k_v\frac{\delta p}{p}\right) = 0,$$

that is

$$\frac{\delta p}{p} + \gamma\frac{\delta v}{v} = 0.$$

Integrating this equation we have

$$pv^\gamma = \text{constant}, \tag{2}$$

and this gives the relation between pressure and volume in adiabatic expansion. Thus if p_1, v_1 and p_2, v_2 be the pressure and volume respectively of a unit mass of gas at different times and no heat is absorbed or given off by the gas meanwhile, we have

$$p_1v_1^\gamma = p_2v_2^\gamma.$$

It is clear that the equation will apply to any mass of gas under these conditions, and if V_1 and V_2 be the volumes of the same mass,

$$p_1V_1^\gamma = p_2V_2^\gamma.$$

12.21 Work Done in Expansion

Let S and S' be the bounding surfaces of a unit mass of gas before and after a small expansion against pressure p which is constant over the surface (Fig. 343).

Let the normal displacement of an element δS of S be δn. The consequent increase of volume is $\delta S \times \delta n$, and the total increase of volume is the sum of this quantity over the surface.

Assuming the expansion to be slow, so that no kinetic energy is developed we have, for the work done,

$$\begin{aligned} W &= \Sigma p\delta S \times \delta n \\ &= p\Sigma(\delta S \times \delta n) \\ &= p \times \text{increase of volume} \\ &= p\delta v. \end{aligned}$$

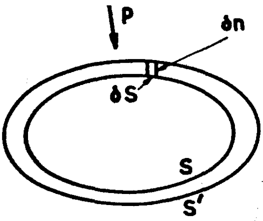

Fig. 343

(i) *Isothermal Expansion*

If a unit mass of gas expands isothermally from volume v_1 to volume v_2 while the pressure changes from p_1 to p_2, we have

$$p_1v_1 = p_2v_2.$$

Then the work done is
$$\begin{aligned} W &= \int_{v_1}^{v_2} p\,dv \\ &= p_1v_1\int_{v_1}^{v_2}\frac{dv}{v} \\ &= p_1v_1 \log\frac{v_2}{v_1} \\ &= RT\log\frac{v_2}{v_1}, \end{aligned}$$

where T is the temperature at which the expansion takes place. We have already found (§ 12.20 (1)) an expression for W, namely

$$W = (k_p - k_v)T\log\frac{v_2}{v_1}.$$

Thus
$$\begin{aligned} R &= k_p - k_v \\ &= k_v(\gamma - 1). \end{aligned}$$

(ii) *Adiabatic Expansion*

If the expansion from volume v_1 to volume v_2 is adiabatic we have (§ 12.20 (2)),

$$\begin{aligned} p_1v_1^\gamma &= p_2v_2^\gamma, \\ W &= \int_{v_1}^{v_2} p\,dv \\ &= p_1v_1^\gamma\int_{v_1}^{v_2} v^{-\gamma}dv \\ &= \frac{p_1v_1^\gamma}{1-\gamma}(v_2^{1-\gamma} - v_1^{1-\gamma}) \\ &= \frac{p_2v_2 - p_1v_1}{1-\gamma}. \end{aligned}$$

If T_1 and T_2 be the temperatures at volumes v_1 and v_2 respectively, $p_1v_1 = RT_1$, $p_2v_2 = RT_2$, and we have,

$$W = \frac{R}{\gamma - 1}(T_1 - T_2)$$
$$= k_v(T_1 - T_2).$$

This work is done at the expense of internal energy, and therefore W represents the loss of internal energy due to the fall in temperature. Thus k_vT represents the internal energy when the temperature is T.

Example 16. *An air compressor takes in 4 cu. ft. of air at each stroke and compresses it so that its pressure is 60 lb./in.². If the process is adiabatic with $\gamma = 1{\cdot}41$ and the air was initially at a pressure of 14·7 lb./in.², find the work done in each stroke.*

We first require the volume of the compressed air from the formula

$$14{\cdot}7 \times 4^\gamma = 60 \times V^\gamma,$$
$$V = 4 \times \left(\frac{14{\cdot}7}{60}\right)^{1/\gamma}$$
$$= 4(0{\cdot}245)^{1/1{\cdot}41}$$
$$= 4(0{\cdot}245)^{0{\cdot}709}$$
$$= 1{\cdot}475 \text{ cu. ft.}$$

The work done in one stroke is

$$W = \frac{1}{\gamma - 1}(p_2V_2 - p_1V_1)$$
$$= \frac{1}{0{\cdot}41}(60 \times 1{\cdot}475 - 4 \times 14{\cdot}7) \times 144 \text{ ft. lb.}$$
$$= 10{,}430 \text{ ft.lb.}$$

12.22 Bulk Modulus of a Fluid

The bulk modulus of a fluid is defined as a quantity K such that

$$K = \frac{\text{increase in pressure}}{\text{volumetric strain}}.$$

If $-\delta v$ be the increase of the volume v of a fluid due to an increase δp in pressure, the volumetric strain

$$= -\frac{\delta v}{v}.$$

Thus $$K = -v\frac{\delta p}{\delta v}$$

$$= -v\frac{dp}{dv}, \text{ as } \delta p \text{ tends to zero.}$$

Since the volume decreases as p increases, $\dfrac{dp}{dv}$ is negative and K is positive.

For water $K = 300{,}000$ lb./in.^{2}, approximately.

If the change of pressure and volume of a gas is isothermal we have

$$pv = \text{constant},$$

$$\frac{dp}{dv} = -\frac{p}{v},$$

$$K = p.$$

If the change of pressure and volume of a gas is adiabatic,

$$pv^{\gamma} = \text{constant},$$

$$\frac{dp}{dv} = -\gamma\frac{p}{v},$$

$$K = \gamma p.$$

12.23 Velocity of Pressure Waves in a Fluid

We shall assume that a pressure wave is travelling through a tube of uniform cross-section of the fluid causing a series of condensations and rarefactions which move through the fluid with a certain velocity u. It is convenient to analyse the problem by imposing a velocity u on the fluid in the opposite direction so that we have in the moving fluid a static pressure wave which causes local variations of velocity (Fig. 344).

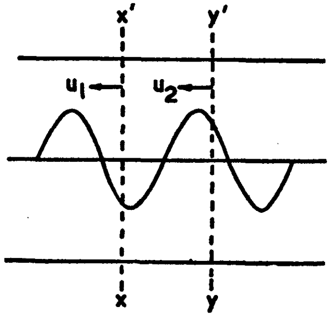

Fig. 344

Let xx' and yy' be two cross-sections of the tube a small distance apart. Let p_1, ϱ_1, u_1 and v_1 be the pressure, density, velocity and volume per unit mass respectively at xx' and p_2, ϱ_2, u_2 and v_2 the corresponding quantities at yy'.

In steady flow the mass of fluid w between the sections is constant and hence if A be the cross-sectional area of the tube we have,

$$w = (u - u_1)\varrho_1 A = (u - u_2)\varrho_2 A. \tag{1}$$

The change in the momentum of the mass w in 1 second is

$$\frac{w}{g}(u - u_1) - \frac{w}{g}(u - u_2), \tag{2}$$

and since this is caused by the differences of the thrusts on the sections we have

$$(p_2 - p_1)A = \frac{w}{g}(u - u_1 - u + u_2),$$

$$= \frac{w}{g}\left\{\frac{w}{A\varrho_1} - \frac{w}{A\varrho_2}\right\}, \text{ from (1)},$$

$$= \frac{w^2}{gA}\left(\frac{1}{\varrho_1} - \frac{1}{\varrho_2}\right),$$

$$= \frac{w^2}{gA}(v_1 - v_2).$$

Therefore
$$\frac{w^2}{A^2} = \frac{g(p_2 - p_1)}{v_1 - v_2}.$$

Now the increase in pressure between xx' and yy' is $p_1 - p_2$ and the volumetric strain is $(v_1 - v_2)/v_1$, hence

$$K = -\frac{(p_1 - p_2)v_1}{v_1 - v_2}$$

$$= \frac{w^2 v_1}{A^2 g}. \tag{3}$$

If there were no wave, the mass passing through the tube per second would be

$$w = \varrho A u = \frac{Au}{v},$$

where ϱ is the density and v the volume per unit mass, and

$$\frac{w}{A} = \frac{u}{v}. \tag{4}$$

Thus assuming that the changes in density due to the wave motion are small we can take $v_1 = v$ and from (3) and (4) we have

$$\left(\frac{u}{v}\right)^2 = \frac{gK}{v},$$

$$u^2 = gKv = \frac{gK}{\varrho},$$

$$u = \sqrt{(gKv)} = \sqrt{\left(\frac{gK}{\varrho}\right)}.$$

Example 17. *Find the velocity of sound in water which has a bulk modulus of* 3×10^5 *lb./in.*^{2} *and density* 62·5 *lb./cu. ft.*

The velocity is
$$u = \sqrt{\left(\frac{gK}{\varrho}\right)}$$
$$= \sqrt{\left(\frac{32 \times 3 \times 10^5 \times 144}{62{\cdot}5}\right)}$$
$$= 1920\sqrt{6}$$
$$= 4702 \text{ ft./sec.}$$

12.24 Velocity of Sound in Air

Sound is propagated in air with the velocity of a longitudinal wave and the process is approximately adiabatic.

Then
$$K = \gamma p,$$
where $\gamma = 1{\cdot}41$ approximately. The speed of sound is
$$c = \sqrt{(\gamma p g v)},$$
and $pv = RT$, where $R = 96$ ft.lb. approximately. At a temperature of 15° Centigrade, $T = 273 + 15 = 288$,
$$c = \sqrt{(1{\cdot}41 \times 96 \times 288 \times 32{\cdot}2)}$$
$$= 1120 \text{ ft./sec.}$$

If the process were isothermal we would have $K = p$ and
$$c = \sqrt{(96 \times 288 \times 32{\cdot}2)}$$
$$= 944 \text{ ft./sec.}$$
which is not in agreement with the observed velocity.

The velocity of sound is thus seen to be proportional to the square root of the absolute temperature of the air, and since the temperature decreases with altitude so does the velocity of sound.

12.25 Bernoulli's Equation for Gas

Bernoulli's equation for the flow of a liquid along a stream filament is proved (§ 12.8) by showing how the difference of the thrusts on two cross-sections of the filament increases the kinetic and potential energy of the liquid between them. When the same argument is applied to the flow of a gas it must be remembered that some of the work done may be taken up in increasing the internal energy of the gas. Thus Bernoulli's equation applied to a unit mass of gas moving along a stream filament takes the form
$$\frac{p_1}{\varrho_1} + h_1 + \frac{u_1^2}{2g} + E_1 = \frac{p_2}{\varrho_2} + h_2 + \frac{u_2^2}{2g} + E_2.$$
Here, p_1, ϱ_1, h_1, u_1, E_1 are the pressure, density, height, velocity and internal energy, respectively, at a point A and p_2, ϱ_2, h_2, u_2, E_2 the

same quantities at a point B. This assumes that no heat is absorbed by the gas between A and B. If a quantity H of heat is absorbed we have

$$H + \frac{p_1}{\varrho_1} + h_1 + \frac{u_1^2}{2g} + E_1 = \frac{p_2}{\varrho_2} + h_2 + \frac{u_2^2}{2g} + E_2.$$

The equation holds for steady flow of gas in a pipe.

If v be the volume of a unit mass at density ϱ we have $\varrho v = 1$, so that

$$\frac{p}{\varrho} = pv.$$

Also,
$$E = k_v T,$$
$$pv = RT$$
$$= k_v(\gamma - 1)T,$$

therefore
$$E = \frac{pv}{\gamma - 1},$$
$$\frac{p}{\varrho} + E = \frac{\gamma}{\gamma - 1} pv$$
$$= \frac{\gamma}{\gamma - 1} \cdot \frac{p}{\varrho}.$$

Thus if no heat is absorbed between A and B, so that the flow is adiabatic, and $h_1 = h_2$ we have $H = 0$ and

$$\frac{\gamma}{\gamma - 1}(p_1 v_1 - p_2 v_2) = \frac{u_2^2}{2g} - \frac{u_1^2}{2g}.$$

We have also
$$p_1 v_1^\gamma = p_2 v_2^\gamma,$$
$$v_2 = v_1 \left(\frac{p_1}{p_2}\right)^{1/\gamma},$$
$$p_2 v_2 = p_1 v_1 \left(\frac{p_2}{p_1}\right)^{1-1/\gamma}.$$

Therefore
$$\frac{u_2^2}{2g} - \frac{u_1^2}{2g} = \frac{\gamma}{\gamma - 1} p_1 v_1 \left\{1 - \left(\frac{p_2}{p_1}\right)^{1-1/\gamma}\right\}.$$

If the flow is isothermal and a quantity of heat H is absorbed between A and B, we have

$$p_1 v_1 = p_2 v_2,$$

and, from § 12.21,
$$H = p_1 v_1 \log \frac{p_1}{p_2}.$$

Thus for level flow $h_1 = h_2$ and we have

$$\frac{u_2^2}{2g} - \frac{u_1^2}{2g} = p_1 v_1 \log \frac{p_1}{p_2}.$$

Example 18. *A stream of air flows past a body. The velocity of the undisturbed air is 250 ft./sec., the pressure is 18 lb./in.² and the density is 0·072 lb. per cu. ft. Find the pressure at a point of the body where the velocity of the air stream is zero. Find also the velocity of flow at a point near the body where the pressure is 17·5 lb./in.². Assume adiabatic flow with* $\gamma = 1{\cdot}4$.

We have in the first case $u_2 = 0$ in Bernoulli's equation and

$$u_1^2 = \frac{2g\gamma}{\gamma - 1}\cdot\frac{p_1}{\varrho_1}\left\{\left(\frac{p_2}{p_1}\right)^{2/7} - 1\right\},$$

$$\frac{p_1}{\varrho_1} = \frac{18 \times 144}{0{\cdot}072} = 36000,$$

$$\left(\frac{p_2}{p_1}\right)^{2/7} = 1 + \frac{(250)^2 \times 0{\cdot}4}{64 \times 1{\cdot}4 \times 36000}$$

$$= 1{\cdot}0078.$$

$$\frac{p_2}{p_1} = (1{\cdot}0078)^{7/2}$$

$$= 1{\cdot}027,$$

$$p_2 = 18{\cdot}49 \text{ lb./in.}^2.$$

In the second case we have

$$\frac{p_2}{p_1} = \frac{17{\cdot}5}{18} = 0{\cdot}9722,$$

$$\left(\frac{p_2}{p_1}\right)^{2/7} = 0{\cdot}9919.$$

$$u_2^2 - u_1^2 = \frac{64 \times 1{\cdot}4}{0{\cdot}4} \times 36000 \times 0{\cdot}0081$$

$$= 65320.$$

$$u_1^2 = 62{,}500,$$

$$u_2^2 = 127{,}800,$$

$$u_2 = 358 \text{ ft./sec.}$$

12.26 Flow in Converging Pipe

Let ϱ be the density and u the velocity of a gas at a point where the cross-section of the pipe is a. If x be the distance measured along the pipe, a varies with x and $\dfrac{da}{dx}$ is negative.

Now from the quantity equation,

$$\varrho a u = \text{constant},$$

therefore, by logarithmic differentiation,

$$\frac{1}{\varrho}\frac{d\varrho}{dx} + \frac{1}{u}\frac{du}{dx} = -\frac{1}{a}\frac{da}{dx}. \tag{1}$$

Assuming that the flow is adiabatic we have from Bernoulli's equation,

$$\frac{u^2}{2g} + \frac{\gamma}{\gamma - 1}\cdot\frac{p}{\varrho} = \text{constant},$$

that is, since $p = C\varrho^\gamma$, where C is constant,

$$\frac{u^2}{2g} + \frac{\gamma}{\gamma - 1}C\varrho^{\gamma-1} = \text{constant.}$$

Differentiating this equation with respect to x we have

$$\frac{1}{g}u\frac{du}{dx} + \gamma C\varrho^{\gamma-2}\frac{d\varrho}{dx} = 0,$$

$$\begin{aligned}\frac{1}{\varrho}\frac{d\varrho}{dx} &= -\frac{u}{g\gamma C\varrho^{\gamma-1}}\frac{du}{dx}\\ &= -\frac{u\varrho}{g\gamma p}\frac{du}{dx}\\ &= -\frac{u}{c^2}\frac{du}{dx},\end{aligned}$$

where c is the velocity of sound.

Therefore from (1) $\dfrac{du}{dx}\left(\dfrac{1}{u} - \dfrac{u}{c^2}\right) = -\dfrac{1}{a}\dfrac{da}{dx}.$

The right-hand side in this equation is positive, therefore $\frac{du}{dx}$ is positive if $\frac{1}{u} > \frac{u}{c^2}$, that is if $u < c$. Thus the velocity increases along the pipe if u is less than the velocity of sound.

12.27 Venturi Meter for Gas

The flow of a gas through a Venturi meter may be taken as adiabatic. Let a_1 be the cross-sectional area of the pipe and a_2 that of the throat. Let ϱ_1, p_1, v_1, u_1, refer to the pipe and ϱ_2, p_2, v_2, u_2, to the throat.

We have the quantity equation,

$$\varrho_1 a_1 u_1 = \varrho_2 a_2 u_2,$$

and Bernoulli's equation gives

$$\frac{u_2^2 - u_1^2}{2g} = \frac{\gamma}{\gamma - 1}p_1 v_1\left\{1 - \left(\frac{p_2}{p_1}\right)^{1-1/\gamma}\right\}.$$

Hence, eliminating u_2, we have

$$\frac{u_1^2}{2g}\left\{\left(\frac{\varrho_1 a_1}{\varrho_2 a_2}\right)^2 - 1\right\} = \frac{\gamma}{\gamma - 1}\cdot\frac{p_1}{\varrho_1}\left\{1 - \left(\frac{p_2}{p_1}\right)^{1-1/\gamma}\right\}.$$

Since

$$\frac{\varrho_1}{\varrho_2} = \frac{v_2}{v_1} = \left(\frac{p_1}{p_2}\right)^{1/\gamma},$$

we have

$$u_1^2 = \frac{\dfrac{2g\gamma}{\gamma - 1}\cdot\dfrac{p_1}{\varrho_1}\left\{1 - \left(\dfrac{p_2}{p_1}\right)^{1-1/\gamma}\right\}}{\left(\dfrac{p_1}{p_2}\right)^{2/\gamma}\cdot\dfrac{a_1^2}{a_2^2} - 1}.$$

The quantity flowing per second through the pipe is then

$$Q = a_1 \varrho_1 u_1.$$

It will be seen that a knowledge of the ratio of the pressures at the inlet and the throat is required to determine the flow as well as a knowledge of the initial state p_1/ϱ_1.

Example 19. *A Venturi meter which is horizontal has diameter* 9 *in. at the inlet and* 3 *in. at the throat. Air flows through the pipe and the pressure at the inlet is* 140 *lb./in.*^{2}, *and* 125 *lb./in.*^{2} *at the throat. If the temperature of the air at the inlet is* 27° *C., find the velocity of the air at the inlet and the mass of air passing through the pipe per second, taking* $\gamma = 1{\cdot}4$ *and* $R = 96$ *ft.lb. per degree C.*

We have $$\left(\frac{p_2}{p_1}\right)^{1-1/\gamma} = \left(\frac{25}{28}\right)^{0\cdot4/1\cdot4} = 0{\cdot}968.$$

$$\left(\frac{p_1}{p_2}\right)^{2/\gamma} = \left(\frac{28}{25}\right)^{10/7} = 1{\cdot}176.$$

$$\frac{a_1^2}{a_2^2} = 81,$$

$$u_1^2 = \frac{\dfrac{64 \times 14}{4} \times 96 \times 300(1 - 0{\cdot}968)}{81 \times 1{\cdot}176 - 1},$$

$$u_1 = 46{\cdot}8 \text{ ft./sec.}$$

Also $$\varrho_1 = \frac{140 \times 144}{96 \times 300} = 0{\cdot}7,$$

$$Q = a_1 \varrho_1 u_1$$

$$= \frac{9\pi}{64} \times 0{\cdot}7 \times 46{\cdot}8$$

$$= 14{\cdot}5 \text{ lb. per sec.}$$

12.28 The Pitot Tube for Gas

We found (§ 12.11) for a Pitot tube in a liquid the formula

$$u^2 = \frac{2g}{\varrho}(p_0 - p),$$

where p_0 is the pressure where the velocity is zero and p is the pressure at velocity u.

For a Pitot tube in a gas, assuming adiabatic expansion, we have from Bernoulli's equation,

$$\frac{u^2}{2g} = \frac{\gamma}{\gamma - 1} p_0 v_0 \left\{1 - \left(\frac{p_1}{p_0}\right)^{1-1/\gamma}\right\}.$$

The velocity of sound at pressure p_0 is c_0 where

$$c_0^2 = \frac{\gamma p_0 g}{\varrho_0} = \gamma p_0 v_0 g,$$

therefore

$$\frac{u^2}{c_0^2} = \frac{2}{\gamma - 1}\left\{1 - \left(\frac{p}{p_0}\right)^{1-1/\gamma}\right\},$$

$$\frac{p}{p_0} = \left(1 - \frac{\gamma - 1}{2}\frac{u^2}{c_0^2}\right)^{\gamma/(\gamma-1)}.$$

Using the binomial expansion we have

$$\frac{p}{p_0} = 1 - \frac{1}{2}\gamma\frac{u^2}{c_0^2} + \frac{1}{8}\gamma\left(\frac{u}{c_0}\right)^4 + \dots.$$

Therefore, if u/c_0 is small we have, approximately,

$$\frac{p}{p_0} = 1 - \frac{1}{2}\gamma\frac{u^2}{c_0^2}$$

$$= 1 - \frac{1}{2}\frac{u^2 \varrho_0}{p_0 g},$$

that is

$$u^2 = \frac{2g}{\varrho_0}(p_0 - p). \tag{1}$$

Thus by neglecting the third and higher terms of the binomial expansion we have the same expression for the velocity from the Pitot tube in air as in liquid. The ratio of the third term to the second is $\frac{1}{4}(u/c_0)^2$, and hence for velocities below about half of the speed of sound the formula (1) gives a good approximation to the velocity.

12.29 Flow through an Orifice

Let gas in a tank be at pressure p_1, density ϱ_1 and at rest. Let it discharge with velocity u through an orifice of effective cross-sectional area a and let p_2 and ϱ_2 be the pressure and density outside the tank (Fig. 345).

Assuming the flow to be adiabatic we have, from Bernoulli's equation,

$$u^2 = \frac{2g\gamma}{\gamma - 1}\frac{p_1}{\varrho_1}\left\{1 - \left(\frac{p_2}{p_1}\right)^{1-1/\gamma}\right\}.$$

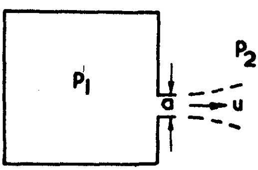

Fig. 345

The mass discharged in unit time is

$$W = \varrho_2 a u$$

$$= a\varrho_2\left[\frac{2g\gamma}{\gamma - 1}\frac{p_1}{\varrho_1}\left\{1 - \left(\frac{p_2}{p_1}\right)^{1-1/\gamma}\right\}\right]^{1/2}.$$

Since $\quad p_1\varrho_1^{-\gamma} = p_2\varrho_2^{-\gamma}$,

$$\varrho_2 = \varrho_1\left(\frac{p_2}{p_1}\right)^{1/\gamma},$$

$$W = a\varrho_1\left[\frac{2g\gamma}{\gamma-1}\frac{p_1}{\varrho_1}\left\{\left(\frac{p_2}{p_1}\right)^{2/\gamma} - \left(\frac{p_2}{p_1}\right)^{1+1/\gamma}\right\}\right]^{1/2}.$$

For a given initial state the mass discharged in unit time is a maximum when

$$\left(\frac{p_2}{p_1}\right)^{2/\gamma} - \left(\frac{p_2}{p_1}\right)^{1+1/\gamma}$$

is a maximum, that is

$$\frac{2}{\gamma}\left(\frac{p_2}{p_1}\right)^{2/\gamma-1} = \frac{\gamma+1}{\gamma}\left(\frac{p_2}{p_1}\right)^{1/\gamma},$$

$$\frac{p_2}{p_1} = \left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)}.$$

Then $\quad \left(\frac{p_2}{p_1}\right)^{2/\gamma} - \left(\frac{p_2}{p_1}\right)^{1+1/\gamma} = \left(\frac{2}{\gamma+1}\right)^{2/(\gamma-1)}\left\{1 - \frac{2}{\gamma+1}\right\}$

$$= \left(\frac{2}{\gamma+1}\right)^{2/(\gamma-1)}\left(\frac{\gamma-1}{\gamma+1}\right),$$

$$W = a\varrho_1\left[\frac{2g\gamma}{\gamma+1}\frac{p_1}{\varrho_1}\left(\frac{2}{\gamma+1}\right)^{2/(\gamma-1)}\right]^{1/2}.$$

If $\gamma = 1{\cdot}4$, the maximum occurs when $p_2/p_1 = 0{\cdot}528$.

12.30 Pressure Altitude Equation for Atmosphere

Consider a vertical column of air of unit cross-sectional area in equilibrium. Let p be the pressure at height h and $p + \delta p$ the pressure at height $h + \delta h$ (Fig. 346). Then p exceeds $p + \delta p$ by the weight of the column of air between the two levels. That is, if ϱ be the density,

$$-\delta p = \varrho\delta h.$$

Therefore, in the limit, $\quad \frac{dh}{dp} = -\frac{1}{\varrho},$

$$= -v,$$

where v is the volume per unit mass. Hence, since $pv = RT$,

$$\frac{dh}{dp} = -\frac{RT}{p}. \qquad (1)$$

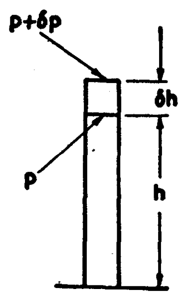

Fig. 346

This equation may be solved to find the values of the pressure and density at different heights, but in

order to obtain a solution some assumption must be made as to the relation between temperature and pressure.

It is found by meteorologists that the temperature of the atmosphere decreases by about 1·98 degrees centigrade for every 1000 ft. above sea level up to a height of about 35,000 ft. Above this height is an isothermal layer of air known as the stratosphere. Thus neither an isothermal nor an adiabatic solution is in agreement with observation, but we shall consider both of these solutions and in the following section we shall show how the *standard atmosphere* is based on a combination of them.

(i) *Isothermal Assumption*

In this case T is constant, and if p_0 be the pressure for $h = 0$ we have

$$\begin{aligned} h &= \int_{p_0}^{p} -\frac{RT}{p}dp \\ &= -RT\log\frac{p}{p_0}, \\ p &= p_0 e^{-h/(RT)}, \\ \varrho &= \varrho_0 e^{-h/(RT)}. \end{aligned}$$

(ii) *Adiabatic Assumption*

In this case

$$\begin{aligned} pv^{\gamma} &= p_0 v_0^{\gamma}, \\ v &= v_0(p_0)^{1/\gamma}(p)^{-(1/\gamma)}, \\ h &= -\int_{p_0}^{p} v\,dp \\ &= -v_0(p_0)^{1/\gamma}\int_{p_0}^{p}(p)^{-(1/\gamma)}dp \\ &= -\frac{\gamma}{\gamma-1}v_0(p_0)^{1/\gamma}\{(p)^{1-1/\gamma}-(p_0)^{1-1/\gamma}\}, \\ (p)^{1-1/\gamma} &= (p_0)^{1-1/\gamma}\left\{1-\frac{\gamma-1}{\gamma}\cdot\frac{h}{p_0v_0}\right\}, \\ p &= p_0\left\{1-\frac{\gamma-1}{\gamma}\cdot\frac{h}{p_0v_0}\right\}^{\gamma/(\gamma-1)}. \\ \varrho &= \varrho_0\left(\frac{p}{p_0}\right)^{1/\gamma} \\ &= \varrho_0\left\{1-\frac{\gamma-1}{\gamma}\frac{h}{p_0v_0}\right\}^{1/(\gamma-1)}. \end{aligned}$$

12.31 The Standard Atmosphere

The standard atmosphere assumes a sea-level temperature of 15° C. falling by 1·98° C. for every 1000 ft. above sea level. This is called the *temperature lapse rate.* The temperature lapse continues up to the stratosphere at a height of 35,300 ft. at which level the temperature has fallen to about −55° C. Above this level the temperature is taken to be constant. The gas constant for the atmosphere is taken as

$$R = 95{\cdot}9 \text{ ft.lb. per degree centigrade.}$$

We have for the temperature at height h ft., for $h < 35{,}300$,

$$T = T_0 - ah,$$

where $a = 0{\cdot}00198$ and $T_0 = 288°$.

From Equation (1) (§ 12.30) we have

$$\frac{dh}{dp} = -\frac{R(T_0 - ah)}{p},$$

$$-\int_0^h \frac{a\,dh}{T_0 - ah} = aR\int_{p_0}^{p} \frac{dp}{p},$$

$$\log \frac{T_0 - ah}{T_0} = aR \log \frac{p}{p_0},$$

$$p = p_0\left(1 - \frac{a}{T_0}h\right)^{1/aR}$$

$$= p_0\left(1 - \frac{aR}{RT_0}h\right)^{1/aR}$$

$$= p_0\left(1 - \frac{aR}{p_0v_0}h\right)^{1/aR}.$$

Also

$$\frac{p}{\varrho} = R(T_0 - ah),$$

$$\varrho = \frac{p}{RT_0}\left(1 - \frac{ah}{T_0}\right)^{-1}$$

$$= \frac{p_0}{RT_0}\left(1 - \frac{ah}{T_0}\right)^{1/aR-1}.$$

It will be seen that this is in agreement with the adiabatic solution of the previous section if

$$aR = \frac{\gamma - 1}{\gamma}.$$

Now

$$aR = 0{\cdot}00198 \times 95{\cdot}9$$

$$= 0{\cdot}190.$$

If
$$\frac{\gamma - 1}{\gamma} = 0{\cdot}190,$$
$$\frac{1}{\gamma} = 0{\cdot}81,$$
$$\gamma = 1{\cdot}235, \text{ approximately.}$$

Thus the pressure and density are as given by the adiabatic formula with $\gamma = 1{\cdot}235$.

In the stratosphere, that is at height $h + 35{,}300$ ft. above sea level, we have the isothermal solution

$$p = p_0 e^{-h/RT},$$

where p_0 and T_0 are the pressure and temperature respectively at 35,300 ft. The figures given in this section relate to the standard atmosphere known as NACA (National Advisory Committee for Aeronautics). The British ICAN (International Commission on Air Navigation) atmosphere is almost identical except that it assumes a temperature of $-56{\cdot}5°$ C. at the stratosphere.

Example 20. *Find the density and pressure in the atmosphere at* 10,000 *ft. above sea level if the temperature and pressure at sea level are* 15° *C. and* 14·7 *lb./in.*^{2} *and* $R = 96$ *ft.lb. per* 1° *C.,*

(*a*) *assuming a temperature lapse rate of* 2° *C. per* 1000 *ft. rise,*

(*b*) *assuming an isothermal atmosphere.*

In case (*a*) we have
$$\alpha = 0{\cdot}002,$$
$$T = 288,$$
$$\frac{\alpha h}{T} = \frac{20}{288},$$
$$\frac{1}{\alpha R} = \frac{1}{0{\cdot}192} = 5{\cdot}208,$$
$$p = 14{\cdot}7\left(1 - \frac{20}{288}\right)^{5\cdot208}$$
$$= 14{\cdot}7 \times 0{\cdot}688$$
$$= 10{\cdot}11 \text{ lb./in.}^2,$$
$$\varrho = \frac{10{\cdot}11 \times 144}{96 \times 268}$$
$$= 0{\cdot}057 \text{ lb. per cu. ft.}$$

In case (*b*) we have
$$\frac{h}{RT} = \frac{10{,}000}{96 \times 288} = 0{\cdot}3616,$$
$$e^{-0\cdot3616} = 0{\cdot}6966,$$
$$p = 14{\cdot}7 \times 0{\cdot}6966$$
$$= 10{\cdot}24 \text{ lb./in.}^2,$$
$$\varrho_0 = \frac{14{\cdot}7 \times 144}{96 \times 288} = 0{\cdot}0767,$$
$$\varrho = 0{\cdot}0767 \times 0{\cdot}6966$$
$$= 0{\cdot}053 \text{ lb. per cu. ft.}$$

EXERCISES 12 (*d*)

1. A mass of 1 lb. of air at a pressure of 30 lb./in.^{2} and temperature 45° C. is allowed to expand isothermally until the pressure is 20 lb./in.^{2}. Find the work done in the expansion.

2. Four cubic feet of gas has absolute temperature 396° C. and is at a pressure of 80 lb./in.^{2}. If the gas is allowed to expand until its temperature falls to 288° C. and its pressure to 20 lb./in.^{2}, find its new volume. If the process is adiabatic, find the work done by the gas in expanding.

3. Show that the work done by a gas in expanding from volume v_1 and pressure p_1 to volume v_2 and pressure p_2 according to the law $pv^{1\cdot 4}$ = constant is $2\cdot 5\,(p_1v_1 - p_2v_2)$. An air compressor takes in 3 cu. ft. of air per stroke at 15 lb.wt./sq. in. and compresses it to 50 lb.wt./sq. in. according to the above law. Find the net h.p. required at 60 strokes per minute. (L.U., Pt. II)

4. The velocity of a pressure wave in a liquid whose specific gravity is 0·8 is 4988 ft./sec. Find the bulk modulus of the liquid.

5. Calculate the velocity of sound in air when the temperature is $-15°$ C., taking $R = 96$ ft.lb. and $\gamma = 1\cdot 4$.

6. A stream of air flowing past a body has a velocity of 200 ft./sec. in the undisturbed portion of the fluid, the pressure there is 15·0 lb./in.^{2} and the temperature 15° C. Assuming adiabatic flow with $\gamma = 1\cdot 4$ and $R = 96$ ft.lb. per degree C., find the velocity at a point near the body at which the pressure is 14·5 lb./in.^{2}.

7. At a point A in a stream of air the velocity is 200 ft./sec., the pressure is 15 lb./in.^{2} and the density is 0·078 lb./cu. ft. At a point B farther along the stream the pressure is 14·5 lb./in.^{2}. Assuming adiabatic flow ($\gamma = 1\cdot 4$), find the velocity at B and express the velocities at A and B as multiples of the speed of sound at the points.

8. In the steady motion of a gas under no external forces, prove that

$$q\frac{\partial q}{\partial s} + \frac{1}{\varrho}\frac{\partial p}{\partial s} = 0,$$

where p, q, ϱ are corresponding values of pressure, speed and density and $\partial/\partial s$ refers to differentiation along a streamline. Deduce Bernoulli's theorem.

If the pressure and density are connected by the adiabatic law, prove that $\dfrac{p}{\varrho}\dfrac{\gamma}{(\gamma-1)} + \dfrac{1}{2}q^2$ is constant along a streamline, where γ is the ratio of the specific heats. (L.U., Pt. II)

9. A gas in which the pressure p and density ϱ are connected by the adiabatic relation $p = k\varrho^\gamma$ flows steadily along a pipe. Prove the velocity q satisfies the relation

$$q^2 + \frac{2\gamma}{\gamma - 1}\frac{p}{\varrho} = \text{constant},$$

external forces being neglected.

If the pipe diverges slightly in the direction of flow, show that the speed of any particle of the gas is increasing if $q > c$ and decreasing if $q < c$, where $c^2 = \gamma p/\varrho$. (L.U., Pt. II)

10. A Venturi meter has a diameter of 6 in. reducing to 2 in. at the throat. At the larger section the pressure of air flowing through the pipe is 150 lb./in.^{2} and its temperature is 15° C. At the throat the pressure is 135 lb./in.^{2}. Find the mass of air per second flowing through the pipe taking $R = 96$ and $\gamma = 1{\cdot}4$.

11. A horizontal Venturi meter has inlet diameter 6 in. and throat diameter 3 in. As air flows through the pipe the ratio of the pressures at the inlet and at the throat is 5/3. If the temperature at the inlet is 15° C., find the flow at the inlet in cu. ft. per sec. ($R = 96$, $\gamma = 1{\cdot}4$).

12. A horizontal Venturi meter has throat diameter 3 in. and inlet diameter 6 in. The pressures at the inlet and at the throat are 90 lb./in.^{2} and 75 lb./in.^{2} respectively, and the temperature of air at the inlet is 20° C. Taking $\gamma = 1{\cdot}4$ and $R = 96$ ft.lb. per degree Centigrade, calculate the mass of air flowing through the pipe per second.

13. In air in which the mercury barometer reads 30 in. and the temperature is 15° C. a Pitot tube on an aircraft records a pressure difference of 0·60 in. of mercury. Find the air speed of the aircraft. ($R = 96$ ft.lb.)

14. A large vessel contains air at a pressure of 60 lb./in.^{2} and temperature 15° C. An orifice in the side of the tank has an effective cross-sectional area of 1 sq. in. Find the mass of air discharged per second into the atmosphere in which the pressure is 15 lb./in.^{2} ($R = 96$, $\gamma = 1{\cdot}4$).

15. Assuming a constant temperature of 15° C. up to a height of 1000 ft., find the pressure and density of the air at this height taking ground-level pressure as 14·7 lb./in.^{2} and $R = 96$ ft.lb. per degree C.

16. In the standard atmosphere find the temperature, pressure and density at a height of 20,000 ft.

17. Find the velocity of sound at a height of 20,000 ft. in the standard atmosphere.

18. Assuming a temperature gradient of 2° C. per 1000 ft., find the density and pressure at 12,000 ft. height when the pressure at the ground level is 14·7 lb./in.^{2} and the temperature is 15° C. R for air $= 96$ ft.lb. per degree C.

19. Find the pressure and density at a height of 35,000 ft., assuming a temperature lapse rate of 2° C. per 1000 ft. from a temperature of 15° C. at the ground. The ground-level pressure is 14·7 lb./in.^{2} and $R = 96$ ft.lb. per degree C. Find the speed of sound at this level.

20. Find the pressure and density of the atmosphere at a height of 40,000 ft., assuming a temperature lapse rate of 2° C. per 1000 ft. up to 35,000 ft. and a constant temperature above this level. The ground level temperature is 15° C. and the pressure 14·7 lb./in.^{2}, $R = 96$ ft.lb. per degree C.

CHAPTER 13

LEGENDRE FUNCTIONS, BESSEL FUNCTIONS, WAVE EQUATION, HEAT-FLOW EQUATION

13.1 Solution of Differential Equations in Series

When it is not possible to obtain a solution of a differential equation in terms of elementary functions, a solution may often be obtained which is an infinite series of powers of x, the independent variable. Certain well-known functions, such as Legendre and Bessel functions, are defined by such series and although the series cannot be summed the values of the functions are computed from the series for different values of x and tabulated in much the same way as sines and cosines.

Consider an ordinary linear differential equation of the second order:

$$P\frac{d^2y}{dx^2} + Q\frac{dy}{dx} + Ry = 0,$$

where P, Q and R are functions of x. We assume a solution of the form

$$y = a_0x^p + a_1x^{p+1} + a_2x^{p+2} + \ldots,$$

which is assumed to be convergent and differentiable term by term if x is sufficiently small.

This series is substituted in the differential equation and the result is itself a series of powers of x. This series must be identically zero and hence the coefficient of each power of x must be zero, and from this the coefficients in the original series may be deduced.

Example 1. ***Find a solution as an infinite series of the differential equation***

$$\frac{dy}{dx} - my = 0.$$

Let us assume in this case that the solution is

$$y = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + \ldots \infty.$$

Then $$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + \ldots + na_nx^{n-1} + \ldots \infty.$$

$$\frac{dy}{dx} - my = (a_1 - ma_0) + (2a_2 - ma_1)x + (3a_3 - ma_2)x^2 + \ldots$$
$$+ \{(n+1)a_{n+1} - ma_n\}x^n + \ldots \infty$$

Since y is a solution this last series is identically zero and we have

$$\begin{aligned} a_1 - ma_0 &= 0, \\ 2a_2 - ma_1 &= 0, \\ 3a_3 - ma_2 &= 0, \\ \cdots & \\ (n+1)a_{n+1} - ma_n &= 0. \end{aligned}$$

Therefore

$$a_1 = ma_0,$$

$$a_2 = \frac{m^2 a_0}{2},$$

$$a_3 = \frac{m^3 a_0}{2.3},$$

$$a_4 = \frac{m^4 a_0}{2.3.4},$$

$$a_n = \frac{m^n a_0}{n!}.$$

Hence
$$y = a_0\left(1 + mx + \frac{m^2x^2}{2!} + \frac{m^3x^3}{3!} + \ldots + \frac{m^nx^n}{n!} + \ldots\right)$$
$$= a_0 e^{mx}.$$

We thus have obtained a solution with an arbitrary constant a_0.

13.2 The Indicial Equation

In the previous example we assumed that the solution started with a constant term a_0. The series solution must in general be taken as starting with some power of x and the first term taken as a_0x^ρ, where ρ may be a positive or negative integer or some fraction. The first step in finding a series solution is then to find possible values of ρ.

Consider the differential equation

$$2x^2(x-1)\frac{d^2y}{dx^2} + 3x(3x-1)\frac{dy}{dx} + (6x+1)y = 0.$$

We may rewrite the equation in two parts as

$$\left\{2x^3\frac{d^2y}{dx^2} + 9x^2\frac{dy}{dx} + 6xy\right\} - \left\{2x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - y\right\} = 0.$$

The terms are so arranged that if $y = x^\rho$ is substituted in the first bracket each term is a multiple of $x^{\rho+1}$, while if it is substituted in the second bracket each term is a multiple of x^ρ. The terms of the equation are thus arranged according to weight in two brackets. It is clear that the lowest power of x in the series obtained by substituting

$$y = a_0x^\rho + a_1x^{\rho+1} + a_2x^{\rho+2} + \ldots$$

in the differential equation will be obtained by substituting a_0x^ρ in the second bracket. Since the coefficient of this power of x must be zero we have,

$$-a_0\{2\rho(\rho-1) + 3\rho - 1\} = 0,$$

and since we suppose a_0 not to be zero, we have,

$$2\rho^2 + \rho - 1 = 0,$$

$$(2\rho - 1)(\rho + 1) = 0.$$

This equation is called the *indicial equation* since it determines the index of the lowest power of x. In this case we have $\varrho = \frac{1}{2}$, or $\varrho = -1$. Thus there are two series solutions possible, one for each value of ϱ, and the complete solution of the differential equation will be the sum of the two solutions. For a linear differential equation of the nth order the indicial equation is, in general, of the nth degree, leading to n separate solutions.

13.3 The Recurrence Relation for Coefficients

The coefficients of the powers of x in the series solution are determined by a recurrence relation as in Example 1. The recurrence relation is obtained by substituting in each bracket of the differential equation the term of the solution which will yield a certain power of x and equating to zero the coefficient of this power of x in the resulting series.

Example 2. *Find the complete solution in series of the differential equation*

$$2x^2(x-1)\frac{d^2y}{dx^2} + 3x(3x-1)\frac{dy}{dx} + (6x+1)y = 0.$$

We rewrite the equation grouping terms of equal weight, and we have

$$\left\{2x^3\frac{d^2y}{dx^2} + 9x^2\frac{dy}{dx} + 6xy\right\} - \left\{2x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - y\right\} = 0.$$

If we substitute the term $a_n x^{\varrho+n}$ in the right-hand bracket we obtain a multiple of $x^{\varrho+n}$. In the left-hand bracket we must substitute $a_{n-1}x^{\varrho+n-1}$ to obtain a multiple of $x^{\varrho+n}$.

From the right-hand bracket we obtain

$$\begin{aligned} &-a_n x^{\varrho+n}\{2(\varrho+n)(\varrho+n-1) + 3(\varrho+n) - 1\} \\ &= -a_n x^{\varrho+n}\{2(\varrho+n)^2 + (\varrho+n) - 1\} \\ &= -a_n x^{\varrho+n}(2\varrho+2n-1)(\varrho+n+1). \end{aligned}$$

From the left-hand bracket we obtain

$$\begin{aligned} &a_{n-1}x^{\varrho+n}\{2(\varrho+n-1)(\varrho+n-2) + 9(\varrho+n-1) + 6\} \\ &= a_{n-1}x^{\varrho+n}\{2(\varrho+n)^2 + 3(\varrho+n) + 1\} \\ &= a_{n-1}x^{\varrho+n}(2\varrho+2n+1)(\varrho+n+1). \end{aligned}$$

Hence, since the coefficient of $x^{\varrho+n}$ in the resulting series must be zero, we have

$$a_{n-1}(2\varrho+2n+1)(\varrho+n+1) - a_n(2\varrho+2n-1)(\varrho+n+1) = 0,$$

$$a_n = \frac{2\varrho+2n+1}{2\varrho+2n-1}a_{n-1}.$$

This equation holds for $n = 1, 2, 3$, etc., and hence we have a recurrence relation between the coefficients.

Since there is no term below a_0 in the series, putting $n = 0$ in the recurrence relation gives

$$-a_0(2\varrho-1)(\varrho+1) = 0,$$

and this is the indicial equation, giving $\varrho = \dfrac{1}{2}$ or -1.

If $\varrho = \frac{1}{2}$ we have the solution y_1 and the recurrence relations are

$$\begin{aligned} a_n &= \frac{n+1}{n} a_{n-1}, \\ a_1 &= 2a_0, \\ a_2 &= \frac{3}{2} a_1 = 3a_0, \\ a_3 &= \frac{4}{3} a_2 = 4a_0, \\ &\cdots\cdots \\ a_n &= (n+1)a_0, \end{aligned}$$

$$y_1 = a_0 x^{1/2}\{1 + 2x + 3x^2 + 4x^3 + \ldots + (n+1)x^n + \ldots\}.$$

If $\varrho = -1$ we have the solution y_2 and the recurrence relations are

$$\begin{aligned} a_n &= \frac{2n-1}{2n-3} a_{n-1}, \\ a_1 &= -1a_0, \\ a_2 &= \frac{3}{1} a_1 = -3a_0, \\ a_3 &= \frac{5}{3} a_2 = -5a_0, \\ &\cdots\cdots \\ a_n &= -(2n-1)a_0, \end{aligned}$$

$$y_2 = a_0 x^{-1}\{1 - x - 3x^2 - 5x^3 - \ldots - (2n-1)x^n + \ldots\}.$$

The complete solution is then

$$y = Ay_1 + By_2,$$

that is, absorbing the constant a_0,

$$\begin{aligned} y &= Ax^{1/2}(1 + 2x + 3x^2 + \ldots) + Bx^{-1}(1 - x - 3x^2 - 5x^3 \ldots) \\ &= A \sum_{n=0}^{\infty} (n+1)x^{n+1/2} - B \sum_{n=0}^{\infty} (2n-1)x^{n-1}. \end{aligned}$$

13.4 Gamma Functions

A convenient extension of the factorial notation is provided by gamma functions.

If n is a positive integer we write

$$\begin{aligned} \Gamma(n+1) &= n! \\ &= n(n-1)(n-2)\ldots 1. \end{aligned}$$

Thus $\Gamma(n+1) = n\Gamma(n)$.

When n is not an integer we define $\Gamma(a)$ for $0 < a \leqslant 1$ by the integral

$$\Gamma(a) = \int_0^\infty e^{-t} t^{a-1} dt,$$

$$\Gamma(a+1) = a\Gamma(a),$$

$$\Gamma(a+2) = (a+1)a\Gamma(a), \text{ etc.},$$

and hence the value of $\Gamma(n)$ can be found.

Values of $\Gamma(a)$ are tabulated for different values of a, and from the integral it can be shown that $\Gamma(1) = 1$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Thus
$$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi},$$
$$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\cdot\frac{3}{2}\sqrt{\pi},$$
$$\Gamma\left(\frac{7}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\sqrt{\pi},$$
$$\Gamma\left(n+\frac{1}{2}\right) = \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{(2n-1)}{2}\sqrt{\pi}.$$

Similarly,
$$\Gamma\left(6\frac{1}{4}\right) = 5\frac{1}{4}\times 4\frac{1}{4}\times 3\frac{1}{4}\times 2\frac{1}{4}\times 1\frac{1}{4}\times\frac{1}{4}\times\Gamma\left(\frac{1}{4}\right).$$
$$\Gamma\left(\frac{1}{2}\right) = -\frac{1}{2}\Gamma\left(-\frac{1}{2}\right)$$
$$= +\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(-\frac{3}{2}\right)$$
$$= -\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(-\frac{5}{2}\right),$$
$$\Gamma\left(-\frac{5}{2}\right) = -\frac{8}{15}\sqrt{\pi}.$$

Thus the gamma function is defined for any real number which is not a negative integer or zero.

Example 3. *Find the complete solution of the differential equation*
$$2x(x-1)\frac{d^2y}{dx^2} + (4x-1)\frac{dy}{dx} - 24y = 0.$$

Arranging the terms according to their weights we have
$$\left(2x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} - 24y\right) - \left(2x\frac{d^2y}{dx^2} + \frac{dy}{dx}\right) = 0.$$

Let $y = a_0x^\rho + a_1x^{\rho+1} + \ldots + a_nx^{\rho+n} + \ldots$.

When the series is substituted for y in the differential equation the coefficient of $x^{\rho+n}$ is obtained by substituting $a_nx^{\rho+n}$ in the first bracket and $a_{n+1}x^{\rho+n+1}$ in the second bracket.

From the first bracket we obtain
$$a_n\{2(\rho+n)(\rho+n-1) + 4(\rho+n) - 24\}x^{\rho+n}$$
$$= a_n\{2(\rho+n)^2 + 2(\rho+n) - 24\}x^{\rho+n}$$
$$= 2a_n(\rho+n+4)(\rho+n-3)x^{\rho+n}.$$

From the second bracket we obtain
$$-a_{n+1}\{2(\rho+n+1)\ (\rho+n\ \ + (\rho+n+1)\}x^{\rho+n}$$
$$= -2a_{n+1}(\rho+n+1)(\rho+n+\tfrac{1}{2})x^{\rho+n}.$$

Hence we have the recurrence relation

$$a_{n+1} = \frac{(\varrho + n + 4)(\varrho + n - 3)}{(\varrho + n + 1)(\varrho + n + \frac{1}{2})} a_n.$$

When $a_0 x^\rho$ is substituted in the second bracket we obtain

$$-a_0 x^{\rho - 1} \varrho(2\varrho - 1),$$

hence the indicial equation is

$$\varrho(2\varrho - 1) = 0,$$

and

$$\varrho = 0 \text{ or } \tfrac{1}{2}.$$

When $\varrho = 0$ we have

$$a_{n+1} = \frac{(n + 4)(n - 3)}{(n + 1)(n + \frac{1}{2})} a_n,$$

$$a_1 = \frac{4(-3)}{1(\frac{1}{2})} a_0,$$

$$a_2 = \frac{5(-2)}{2(\frac{3}{2})} a_1 = \frac{5.4}{1.2} \cdot \frac{3.2}{\frac{1}{2}.\frac{3}{2}} a_0,$$

$$a_3 = \frac{6(-1)}{3(\frac{5}{2})} a_2 = -\frac{4.5.6}{1.2.3} \cdot \frac{3.2.1}{\frac{1}{2}.\frac{3}{2}.\frac{5}{2}},$$

$$a_4 = 0.$$

It is clear that all the coefficients after a_3 are zero and hence we have the solution

$$y_1 = a_0(1 - 24x + 80x^2 - 64x^3).$$

When $\varrho = \frac{1}{2}$ we have

$$a_{n+1} = \frac{(n + \frac{9}{2})(n - \frac{5}{2})}{(n + \frac{3}{2})(n + 1)} a_n,$$

$$a_1 = \frac{\frac{9}{2}(-\frac{5}{2})}{\frac{3}{2}.1} a_0,$$

$$a_2 = \frac{\frac{11}{2}.(-\frac{3}{2})}{\frac{5}{2}.2} a_1 = \frac{\frac{9}{2}.\frac{11}{2}}{\frac{3}{2}.\frac{5}{2}} \cdot \frac{(-\frac{5}{2})(-\frac{3}{2})}{1.2} a_0,$$

$$a_3 = \frac{\frac{9}{2}.\frac{11}{2}.\frac{13}{2}(-\frac{5}{2})(-\frac{3}{2})(-\frac{1}{2})}{\frac{3}{2}.\frac{5}{2}.\frac{7}{2}.1.2.3} a_0,$$

$$a_4 = \frac{\frac{9}{2}.\frac{11}{2}.\frac{13}{2}.\frac{15}{2}(-\frac{5}{2})(-\frac{3}{2})(-\frac{1}{2})(\frac{1}{2})}{\frac{3}{2}.\frac{5}{2}.\frac{7}{2}.\frac{9}{2}.1.2.3.4} a_0.$$

Hence, it may be seen that

$$a_n = \frac{\Gamma(n + \frac{9}{2})}{\Gamma(\frac{9}{2})} \cdot \frac{\Gamma(n - \frac{5}{2})}{\Gamma(-\frac{5}{2})} \cdot \frac{\Gamma(\frac{3}{2})}{\Gamma(n + \frac{3}{2})} \cdot \frac{1}{n!} a_0.$$

Absorbing the gamma functions independent of n into the first term we have

$$y_2 = b_0 \sum_{n=0}^{\infty} \frac{\Gamma(n + \frac{9}{2})\,\Gamma(n - \frac{5}{2})}{\Gamma(n + \frac{3}{2})\,\Gamma(n + 1)} x^{n+1/2}.$$

The complete solution is then

$$y = y_1 + y_2.$$

13.5 Convergence of Solutions

The existence of convergent solutions in series of linear differential equations is proved in works on differential equations. The range of values of the variable for which any particular series solution is convergent may be found by applying some test of convergence, such as the ratio test, to the series.

If $$y = a_0x^\rho + a_1x^{\rho+1} + a_2x^{\rho+2} + \dots$$
be the solution, the ratio of the $(n + 1)$th term to the nth term is
$$\frac{a_{n+1}x^{\rho+n+1}}{a_nx^{\rho+n}} = \frac{a_{n+1}}{a_n}x.$$
The series is convergent if
$$\lim_{n\to\infty}\left|\frac{a_{n+1}x}{a_n}\right| < 1,$$
that is if $$|x| < \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right|.$$
For Example 2 we found the recurrence relation
$$\frac{a_{n-1}}{a_n} = \frac{2\varrho + 2n - 1}{2\varrho + 2n + 1}.$$
Therefore $$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{a_{n-1}}{a_n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{1+\frac{\varrho}{n}-\frac{1}{2n}}{1+\frac{\varrho}{n}+\frac{1}{2n}}\right|$$
$$= 1,$$
and hence the series is convergent if $|x| < 1$, that is $-1 < x < 1$.

For Example 3 we found
$$\frac{a_n}{a_{n+1}} = \frac{(\varrho + n + 1)(\varrho + n + \frac{1}{2})}{(\varrho + n + 4)(\varrho + n - 3)},$$
and $$\lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = 1,$$
and the solution y_2 is convergent for $|x| < 1$, that is $-1 < x < 1$.

13.6 Relation between Solutions

Let y_1 and y_2 be two distinct solutions of the differential equation $\frac{d^2y}{dx^2} + Q\frac{dy}{dx} + Ry = 0$, where Q and R are functions of x. This is a

general form for the second order linear differential equation when the expression multiplying $\dfrac{d^2y}{dx^2}$ has been cleared by division.

Let $y_2 = uy_1$, where u is a function of x.

$$\frac{dy_2}{dx} = y_1\frac{du}{dx} + u\frac{dy_1}{dx},$$

$$\frac{d^2y_2}{dx^2} = y_1\frac{d^2u}{dx^2} + 2\frac{dy_1}{dx}\frac{du}{dx} + u\frac{d^2y_1}{dx^2}.$$

$$\frac{d^2y_2}{dx^2} + Q\frac{dy_2}{dx} + Ry_2 = y_1\frac{d^2u}{dx^2} + 2\frac{dy_1}{dx}\frac{du}{dx} + Qy_1\frac{du}{dx} + u\left(\frac{d^2y_1}{dx^2} + Q\frac{dy_1}{dx} + Ry_1\right).$$

Hence, since y_1 and y_2 are both solutions of the differential equation, we have

$$y_1\frac{d^2u}{dx^2} + \left(2\frac{dy_1}{dx} + Qy_1\right)\frac{du}{dx} = 0.$$

This is a first-order differential equation for $\dfrac{du}{dx}$ and is solved by multiplying by the integrating factor $y_1e^{\int Qdx}$ to make the expression a perfect differential.

We have $\quad y_1^2e^{\int Qdx}\dfrac{d^2u}{dx^2} + \left(2y_1\dfrac{dy_1}{dx} + Qy_1^2\right)e^{\int Qdx}\dfrac{du}{dx} = 0,$

and on integrating, $y_1^2e^{\int Qdx}\dfrac{du}{dx} = c$ (a constant).

Therefore
$$\frac{du}{dx} = \frac{c}{y_1^2}e^{-\int Qdx},$$

$$u = c\int\frac{1}{y_1^2}e^{-\int Qdx}dx,$$

$$y_2 = cy_1\int\frac{1}{y_1^2}e^{-\int Qdx}dx.$$

Thus if one solution y_1 has been found a second solution y_2 can be found by integration.

Example 4. *Verify that* $y_1 = (x - 1)$ *is a solution of the differential equation* $x(x - 1)\dfrac{d^2y}{dx^2} + (x - 1)\dfrac{dy}{dx} - y = 0$, *and find a second solution.*

We have $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} - \dfrac{1}{x(x-1)}y = 0$ and it is easily seen that y_1 is a solution.

$$Q = \frac{1}{x},$$

$$\int Q\,dx = \log x,$$

$$e^{-\int Q dx} = \frac{1}{x},$$

$$\frac{1}{y_1^2}e^{-\int Q dx} = \frac{1}{x(x-1)^2},$$

$$= \frac{1}{x} - \frac{1}{x-1} + \frac{1}{(x-1)^2},$$

$$\int \frac{1}{y_1^2}e^{-\int Q dx}dx = \log\frac{x}{x-1} - \frac{1}{x-1},$$

$$y_1\int \frac{1}{y_1^2}e^{-\int Q dx}dx = (x-1)\log\frac{x}{x-1} - 1,$$

$$y_2 = c\left\{(x-1)\log\frac{x}{x-1} - 1\right\}.$$

EXERCISES 13 (*a*)

1. Find a solution in ascending powers of x of the differential equation $2x(x-1)\dfrac{dy}{dx} - (2x-1)y = 0$, and state the range of values of x for which the series converges.

2. Express in series of ascending powers of x the general solution of

$$x^2\frac{d^2y}{dx^2} + (x + x^2)\frac{dy}{dx} + (x-9)y = 0,$$

and show that one series terminates. (L.U., Pt. II)

3. Obtain the general solution, in series of ascending powers of x, of the equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 6y = 0. \qquad \text{(L.U., Pt. II)}$$

4. Show that the equation $4x\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} + y = 0$ has a solution in series of the form $\sum\limits_{n=0}^{\infty} a_n x^n$ where $2n(2n+1)a_n = -a_{n-1} (n > 1)$.

   Obtain a second solution in series and state the general solution. (L.U., Pt. II)

5. Obtain the general solution in series of the differential equation

$$2x(1-x)\frac{d^2y}{dx^2} + (1-6x)\frac{dy}{dx} - 2y = 0.$$

By applying the ratio test show that the series obtained are convergent if $-1 < x < 1$. (L.U., Pt. II)

6. Obtain the general solution in series of the differential equation

$$4x\frac{d^2y}{dx^2} + (4x + 3)\frac{dy}{dx} + y = 0.$$

Give the ranges of values of x for which each series is valid.

7. Find the general solution of the differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} - 7x\frac{dy}{dx} - 9y = 0$$

in series of ascending powers of x. By applying the ratio test, or otherwise, prove that the series converges for $-1 < x < 1$.
(L.U., Pt. II)

8. Find a complete solution in series of the differential equation

$$x^2\frac{d^2y}{dx^2} + (x^2 - 2)y = 0.$$

9. Find a complete solution in series of the differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - \left(x^2 + \frac{1}{4}\right)y = 0.$$

10. Find a complete solution in series of the differential equation

$$2x(x - 1)\frac{d^2y}{dx^2} - (2x + 1)\frac{dy}{dx} - 16y = 0.$$

11. Find a complete solution in series of the differential equation

$$x\frac{d^2y}{dx^2} + (b - x)\frac{dy}{dx} - ay = 0,$$

where b is not an integer.

12. Verify that $y = (x - 1)^{-2}$ is a solution of the differential equation

$$x(x - 1)^2\frac{d^2y}{dx^2} + (x - 1)\frac{dy}{dx} + (2 - 6x)y = 0,$$

and find a second solution.

13.7 Legendre Polynomials

The differential equation

$$(x^2 - 1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - n(n + 1)y = 0$$

is known as Legendre's equation. We shall show that when n is an integer this equation has a solution which is a polynomial of degree n in x.

Let $$y_1 = (x^2 - 1)^n,$$

then $$\frac{dy_1}{dx} = 2nx(x^2 - 1)^{n-1},$$

$$(x^2 - 1)\frac{dy_1}{dx} - 2nxy_1 = 0.$$

We shall differentiate this last equation $n + 1$ times using Leibnitz's theorem for the differentiation of a product. Writing D^r for $\frac{d^r}{dx^r}$, we have

$$D^{n+1}\{(x^2 - 1)Dy_1\} = (x^2 - 1)D^{n+2}y_1 + (n + 1).2xD^{n+1}y_1 + \frac{(n + 1)n}{2}.2D^n y_1.$$

$$D^{n+1}\{-2nxy_1\} = -2nxD^{n+1}y_1 - 2n(n + 1)D^n y_1.$$

Therefore

$$(x^2 - 1)D^{n+2}y_1 + 2xD^{n+1}y_1 - n(n + 1)D^n y_1 = 0.$$

Writing $y = D^n y_1$, we have

$$(x^2 - 1)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - n(n + 1)y = 0.$$

Thus $y = \frac{d^n}{dx^n}(x^2 - 1)^n$ is a solution of Legendre's equation and it is evident that y is a polynomial of degree n.

We write $$P_n(x) = \frac{1}{2^n(n)!}\frac{d^n}{dx^n}(x^2 - 1)^n,$$

and $P_n(x)$ is called the Legendre polynomial of the nth degree. The multiplying factor $\frac{1}{2^n(n)!}$ is introduced to ensure that for all values of n, $P_n(1) = 1$.

The polynomials for $n = 0, 1, 2, 3, 4 \ldots$ are easily found.

$$P_0(x) = 1,$$

$$P_1(x) = x,$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1),$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x),$$

$$P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3),$$

$$P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x).$$

To obtain a general expression for the polynomial $P_n(x)$ we write down the binomial expansion of $(x^2 - 1)^n$ and differentiate n times.

We have

$$(x^2 - 1)^n = x^{2n} - nx^{2n-2} + \frac{n(n-1)}{2}x^{2n-4} - \frac{n(n-1)(n-2)}{2.3}x^{2n-6} + \dots.$$

Now if $r < n$, $D^n x^r = 0$,

$$\text{if } r \geqslant n, \qquad D^n x^r = r(r-1)(r-2)\dots(r-n+1)x^{r-n},$$

$$= \frac{(r)!}{(r-n)!}x^{r-n}.$$

$$D^n x^{2n} = \frac{(2n)!}{(n)!}x^n,$$

$$D^n x^{2n-2} = \frac{(2n-2)!}{(n-2)!}x^{n-2},$$

$$D^n x^{2n-4} = \frac{(2n-4)!}{(n-4)!}x^{n-4},$$

$$\frac{1}{2^n(n)!}D^n(x^2-1)^n = \frac{1}{2^n(n)!}\left\{\frac{(2n)!}{(n)!}x^n - \frac{n(2n-2)!}{(n-2)!}x^{n-2} + \frac{n(n-1)(2n-4)!}{2(n-4)!}x^{n-4}\dots\right\}$$

$$= \frac{(2n)!}{2^n(n)!\,(n)!}\left\{x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4}\dots\right\}$$

that is

$$P_n(x) = \sum_r \frac{(-1)^r(2n-2r)!\,x^{n-2r}}{2^n(r)!\,(n-r)!\,(n-2r)!},$$

where $r = 0, 1, 2, \dots$, the last term corresponding to $r = \frac{1}{2}n$ if n is even and $r = \frac{1}{2}(n-1)$ if n is odd.

Values of $P_n(x)$ for $n = 2$ to $n = 12$ are tabulated for different values of x in the British Association Mathematical Tables, Part-Volume A, 1946.

13.8 Alternative Form for Legendre Polynomials

An alternative form for Legendre polynomials is obtained by writing $x = 1 - 2z$.

Then

$$(x^2-1)^n = 2^{2n}z^n(z-1)^n,$$

$$\frac{d}{dx} = -\frac{1}{2}\frac{d}{dz},$$

$$\left(\frac{d}{dx}\right)^n = (-1)^n\frac{1}{2^n}\left(\frac{d}{dz}\right)^n,$$

$$\begin{aligned}
P_n(x) &= \frac{1}{2^n(n)!}\left(\frac{d}{dx}\right)^n(x^2-1)^n,\\
&= \frac{1}{(n)!}\left(\frac{d}{dz}\right)^n z^n(1-z)^n,\\
&= \frac{1}{(n)!}\left(\frac{d}{dz}\right)^n\Big\{z^n - nz^{n+1} + \frac{n(n-1)}{2!}z^{n+2}\\
&\qquad - \frac{n(n-1)(n-2)}{3!}z^{n+3} + \dots\Big\}\\
&= \frac{1}{(n)!}\Big\{(n)! - n\frac{(n+1)!}{(1)!}z + \frac{n(n-1)}{2!}\cdot\frac{(n+2)!}{2!}z^2\\
&\qquad - \frac{n(n-1)(n-2)}{3!}\cdot\frac{(n+3)!}{3!}z^3 + \dots\Big\}\\
&= 1 - n(n+1)z + \frac{n(n-1)}{2!}\cdot\frac{(n+1)(n+2)}{2!}z^2\\
&\qquad - \frac{n(n-1)(n-2)}{(3)!}\cdot\frac{(n+1)(n+2)(n+3)}{3!}z^3\dots\\
&= \sum_{r=0}^{n}\frac{(n+r)!}{(n-r)!}\frac{(-z)^r}{(r!)^2}\\
&= \sum_{r=0}^{n}\frac{(n+r)!}{(n-r)!}\frac{(\frac{1}{2}x-\frac{1}{2})^r}{(r!)^2}.
\end{aligned}$$

13.9 Recurrence Formulae

The following recurrence formulae exist between Legendre polynomials of degree $n+1$, n and $n-1$ and their derivatives (denoted by primes).

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \tag{1}$$

$$P'_{n+1}(x) - xP'_n(x) - (n+1)P_n(x) = 0 \tag{2}$$

$$xP'_n(x) - P'_{n-1}(x) - nP_n(x) = 0 \tag{3}$$

$$P'_{n+1}(x) - (2n+1)P_n(x) - P'_{n-1}(x) = 0 \tag{4}$$

$$(x^2-1)P'_n(x) - nxP_n(x) + nP_{n-1}(x) = 0 \tag{5}$$

These formulae are proved by manipulation of the fundamental formula for $P_n(x)$. For example, we have

$$D^{n+2}(x^2-1)^{n+1} = D^{n+2}(x^2-1)^n.(x^2-1),$$

and hence using Leibnitz's theorem

$$D^{n+2}(x^2-1)^{n+1} = (x^2-1)D^{n+2}(x^2-1)^n + 2(n+2)xD^{n+1}(x^2-1)^n + (n+2)(n+1)D^n(x^2-1)^n,$$

$$\frac{1}{2^n(n)!}D^{n+2}(x^2-1)^{n+1} = (x^2-1)P''_n(x) + 2(n+2)xP'_n(x) + (n+2)(n+1)P_n(x).$$

The left-hand side is $2(n+1)P'_{n+1}(x)$ and since

$$(x^2-1)P''_n(x) + 2xP'_n(x) - n(n+1)P_n(x) = 0,$$

we have

$$2(n+1)P'_{n+1}(x) = 2(n+1)xP'_n(x) + 2(n+1)^2P_n(x),$$
$$P'_{n+1}(x) = xP'_n(x) + (n+1)P_n(x).$$

This is the formula (2). The other formulae are proved by similar methods and their proof is left as an exercise for the student.

13.10 Expression for Potential at a Point

If a source of attraction is situated at a point A distant a from a point O (Fig. 347) and P is a point distant r from O with the angle $POA = \theta$, the potential at P is proportional to $1/(PA)$.

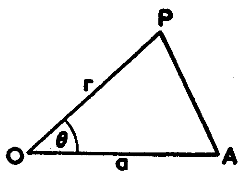

Fig. 347

$$\frac{1}{PA} = (a^2 - 2ar\cos\theta + r^2)^{-1/2},$$

$$= \frac{1}{a}(1 - 2h\cos\theta + h^2)^{-1/2}, \text{ where } h = r/a.$$

This quantity may be expanded as a power series in h for $|h| < 1$ and we have

$$(1 - 2h\cos\theta + h^2)^{-1/2} = 1 + \cos\theta . h + \frac{1}{2}(3\cos^2\theta - 1)h^2 + \ldots$$

$$= P_0(\cos\theta) + hP_1(\cos\theta) + h^2P_2(\cos\theta) + \ldots.$$

We shall show that the coefficient of h^n in this expansion is $P_n(\cos\theta)$.

Let $y = (1 - 2h\cos\theta + h^2)^{-1/2}$,

then
$$(1 - 2h\cos\theta + h^2)\frac{dy}{dh} + (h - \cos\theta)y = 0,$$

$$\left(h^2\frac{dy}{dh} + hy\right) - \cos\theta\left(2h\frac{dy}{dh} + y\right) + \frac{dy}{dh} = 0.$$

Let
$$y = a_0 + a_1h + a_2h^2 + \ldots + a_nh^n + \ldots.$$

The coefficient of h^n when y is substituted in the differential equation is

$$a_{n-1}(n - 1 + 1) - \cos\theta\, a_n(2n+1) + a_{n+1}(n+1).$$

Since this quantity is zero, the successive coefficients of the expansion are obtained from the recurrence formula

$$(n+1)a_{n+1} - (2n+1)\cos\theta\, a_n + na_{n-1} = 0,$$

and
$$a_0 = P_0(\cos\theta), \quad a_1 = P_1(\cos\theta).$$

Therefore since the coefficients are obtained by the recurrence formula (1) for Legendre polynomials we have

$$a_n = P_n(\cos\theta),$$

and
$$(1 - 2h\cos\theta + h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(\cos\theta).$$

If $\theta = 0$, we have $\cos\theta = 1$ and

$$\begin{aligned}(1 - 2h + h^2)^{-1/2} &= (1-h)^{-1}\\ &= 1 + h + h^2 + \ldots,\end{aligned}$$

and hence $P_n(1) = 1$.

If $\theta = \pi$, we have

$$\begin{aligned}(1 + 2h + h^2)^{-1/2} &= (1+h)^{-1}\\ &= 1 - h + h^2 - h^3 \ldots,\end{aligned}$$

and hence $P_n(-1) = (-1)^n$.

13.11 Integral Properties of Legendre Polynomials

We shall now show that if $P_n(x)$ and $P_m(x)$ are Legendre polynomials

$$\begin{aligned}\int_{-1}^{1} P_n(x)P_m(x)dx &= 0, \text{ if } m \neq n,\\ &= \frac{1}{n+\frac{1}{2}}, \text{ if } m = n.\end{aligned}$$

Legendre's differential equation may be written as

$$\frac{d}{dx}(x^2-1)P'_n(x) - n(n+1)P_n(x) = 0.$$

Therefore

$$\begin{aligned}
&n(n+1)\int_{-1}^{1} P_n(x)P_m(x)dx\\
&= \int_{-1}^{1} P_m(x)\,.\frac{d}{dx}(x^2-1)P'_n(x)\,.dx\\
&= \Big[(x^2-1)P'_n(x)\,.P_m(x)\Big]_{-1}^{1} - \int_{-1}^{1}(x^2-1)P'_n(x)P'_m(x)dx\\
&= -\int_{-1}^{1}(x^2-1)P'_m(x)P'_n(x)dx\\
&= -\Big[(x^2-1)P'_m(x)P_n(x)\Big]_{-1}^{1} + \int_{-1}^{1} P_n(x)\frac{d}{dx}(x^2-1)P'_m(x)dx\\
&= m(m+1)\int_{-1}^{1} P_n(x)P_m(x)dx
\end{aligned}$$

Hence if $m \neq n$ we must have $\int_{-1}^{1} P_n(x)P_m(x)dx = 0$. If $m = n$, we have from the above

$$n(n+1)\int_{-1}^{1} P_n(x)P_n(x)dx = -\int_{-1}^{1} (x^2 - 1)P'_n(x)P'_n(x)dx.$$

Now, from the recurrence formula (5)

$$(x^2 - 1)P'_n(x) = nxP_n(x) - nP_{n-1}(x),$$

therefore

$$(n+1)\int_{-1}^{1} P_n^2(x)dx = \int_{-1}^{1} P_{n-1}(x)P'_n(x)dx - \int_{-1}^{1} xP_n(x)P'_n(x)dx$$

$$= \int_{-1}^{1} P_{n-1}(x)P'_n(x)dx - \left[\frac{1}{2}xP_n^2(x)\right]_{-1}^{1} + \frac{1}{2}\int_{-1}^{1} P_n^2(x)dx.$$

$$\left(n + \frac{1}{2}\right)\int_{-1}^{1} P_n^2(x)dx = \int_{-1}^{1} P_{n-1}(x)P'_n(x)dx - 1.$$

From recurrence formula (2)

$$P'_n(x) = xP'_{n-1}(x) + nP_{n-1}(x).$$

Therefore

$$\left(n + \frac{1}{2}\right)\int_{-1}^{1} P^2_n(x)dx = -1 + \int_{-1}^{1} xP_{n-1}(x)P'_{n-1}(x)dx$$

$$+ n\int_{-1}^{1} P^2_{n-1}(x)dx$$

$$= -1 + \left[\frac{1}{2}xP^2_{n-1}(x)\right]_{-1}^{1} - \frac{1}{2}\int_{-1}^{1} P^2_{n-1}(x)dx$$

$$+ n\int_{-1}^{1} P^2_{n-1}(x)dx$$

$$= -1 + 1 + \left(n - \frac{1}{2}\right)\int_{-1}^{1} P^2_{n-1}(x)dx$$

$$= \left(n - \frac{1}{2}\right)\int_{-1}^{1} P^2_{n-1}(x)dx.$$

Now $\int_{-1}^{1} P_0^2(x)dx = 2$,

therefore writing $n = 1$ in the above formula

$$\int_{-1}^{1} P_1^2(x)dx = \frac{2}{3} = \frac{1}{1 + \frac{1}{2}},$$

$$\int_{-1}^{1} P_2^2(x)dx = \frac{3}{5} \times \frac{2}{3} = \frac{1}{2 + \frac{1}{2}},$$

$$\int_{-1}^{1} P_n^2(x)dx = \frac{1}{n + \frac{1}{2}}.$$

13.12 Expansions in Terms of Legendre Polynomials

If $f(x)$ be a polynomial of degree n in x we may express $f(x)$ in terms of Legendre polynomials $P_0(x)$ to $P_n(x)$.

We write $\quad f(x) = a_0P_0(x) + a_1P_1(x) + \ldots + a_nP_n(x),$

where $a_0, a_1, \ldots, a_n$ are constants. Equating coefficients of x^n on both sides of the equation gives a_n, and the other constants $a_{n-1}, a_{n-2}, \ldots, a_0$ are evaluated successively, there being $n + 1$ equations to determine the $n + 1$ constants.

Then
$$\int_{-1}^{1} f(x)P_r(x)dx = a_r\int_{-1}^{1} P_r^2(x)dx,$$
for $0 \leqslant r \leqslant n$, the integrals of products of $P_r(x)$ with the other polynomials being zero.

Therefore
$$\int_{-1}^{1} f(x)P_r(x)dx = \frac{a_r}{r+\frac{1}{2}},$$
$$a_r = \left(r+\frac{1}{2}\right)\int_{-1}^{1} f(x)P_r(x)dx.$$

Thus a polynomial of degree n can be uniquely represented as a series of the Legendre polynomials $P_0(x)$ to $P_n(x)$. Now any function of x, $f(x)$, which has a Maclaurin expansion in powers of x can be approximated to by taking n terms of the expansion. The approximation may be made as close as we desire by taking a large number of terms, and these terms may be expressed in terms of Legendre polynomials. Thus an infinite series of Legendre polynomials can represent the function, and we have
$$f(x) = \sum_{r=0}^{\infty} a_rP_r(x)$$
where
$$a_r = \left(r+\frac{1}{2}\right)\int_{-1}^{1} f(x)P_r(x)dx.$$
This expansion is analogous to a Fourier series expansion in terms of sines and cosines of multiples of x.

13.13 Complete Solution of Legendre's Equation

The second solution of Legendre's equation is denoted by $Q_n(x)$. As this solution is usually required for values of x greater than unity we shall obtain the solution in a series of *descending* powers of x, valid when $|x| > 1$.

Let $\quad y = a_0x^\rho + a_1x^{\rho-1} + \ldots + a_mx^{\rho-m} + \ldots$

be a solution of the equation
$$\left\{x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - n(n+1)y\right\} - \left\{\frac{d^2y}{dx^2}\right\} = 0.$$

The *highest* powers of x when y is substituted in the equation is obtained by substituting a_0x^ρ in the left-hand bracket. Thus the indicial equation is

$$\rho(\rho - 1) + 2\rho - n(n + 1) = 0,$$

$$(\rho - n)(\rho + n + 1) = 0,$$

and
$$\rho = n \text{ or } -n - 1.$$

Substituting $a_m x^{\rho-m}$ in the left-hand bracket and $a_{m-2}x^{\rho-m+2}$ in the right-hand bracket we have

$$a_m\{(\rho - m)(\rho - m - 1) + 2(\rho - m) - n(n + 1\}$$
$$- a_{m-2}(\rho - m + 2)(\rho - m + 1) = 0,$$

$$a_m = a_{m-2}\frac{(\rho - m + 2)(\rho - m + 1)}{(\rho - m - n)(\rho - m + n + 1)}.$$

If $\rho = n$ we have

$$a_m = a_{m-2}\frac{(n - m + 2)(n - m + 1)}{(-m)(2n + 1 - m)},$$

$$a_2 = -a_0\frac{n(n - 1)}{2.(2n - 1)},$$

$$a_4 = -a_2\frac{(n - 2)(n - 3)}{4(2n - 3)}.$$

The series terminates and with the appropriate value of a_0 this gives the solution

$$P_n(x) = \frac{(2n)!}{2^n(n)!(n)!}\left\{x^n - \frac{n(n - 1)}{2(2n - 1)}x^{n-2} + \dots\right\}.$$

If $\rho = -n - 1$,

$$a_m = a_{m-2}\frac{(n + m - 1)(n + m)}{(2n + m + 1)m},$$

$$a_2 = a_0\frac{(n + 1)(n + 2)}{2(2n + 3)},$$

$$a_4 = a_2\frac{(n + 3)(n + 4)}{4(2n + 5)},$$

$$y = a_0x^{-n-1}\left\{1 + \frac{(n + 1)(n + 2)}{2(2n + 3)}x^{-2}\right.$$
$$\left. + \frac{(n + 1)(n + 2)(n + 3)(n + 4)}{2.4(2n + 3)(2n + 5)}x^{-4} + \dots\right\}.$$

The second solution $Q_n(x)$ is defined by this series with a particular value of a_0.

We have

$$Q_n(x) = \frac{\Gamma(\frac{1}{2})\Gamma(n+1)}{2^{n+1}\Gamma(n+\frac{3}{2})}x^{-n-1}\left\{1 + \frac{(n+1)(n+2)}{2(2n+3)}x^{-2} + \ldots\right\}$$

$$= \frac{1}{2}\sum_{r=0}^{\infty}\frac{\Gamma(\frac{1}{2}n+\frac{1}{2}+r)\Gamma(\frac{1}{2}n+1+r)}{\Gamma(n+\frac{3}{2}+r)(r)!}x^{-n-1-2r}.$$

The ratio of the $(r+1)$th term to the rth

$$= \frac{(\frac{1}{2}n+\frac{1}{2}+r)(\frac{1}{2}n+1+r)}{(n+\frac{3}{2}+r)(r+1)}\cdot\frac{1}{x^2},$$

the limit of this ratio as r tends to infinity is $\frac{1}{x^2}$ and hence the series is convergent if $|x| > 1$.

The complete solution of Legendre's equation is then

$$AP_n(x) + BQ_n(x).$$

It should be noted that the functions $P_n(x)$ and $Q_n(x)$ may be defined by these series if n is not an integer, but in this case $P_n(x)$ is also an infinite series. It can be shown that the functions $Q_n(x)$ satisfy the five recurrence formulae given in § 13.9.

13.14 Associated Legendre Functions

Legendre's differential equation is

$$(x^2-1)y'' + 2xy' - n(n+1)y = 0.$$

Differentiating this equation m times, where $m \leqslant n$, by Leibnitz's theorem and writing D^r for $\frac{d^r}{dx^r}$, we have

$$(x^2-1)D^{m+2}y + m.2xD^{m+1}y + m(m-1)D^m y$$
$$+ 2xD^{m+1}y + 2mD^m y - n(n+1)D^m y = 0.$$

Hence writing $u = D^m y$ we have

$$(x^2-1)\frac{d^2u}{dx^2} + (m+1)2x\frac{du}{dx} - (n-m)(n+1+m)u = 0.$$

Now let $u = (x^2-1)^{-m/2}v$.

$$\frac{du}{dx} = (x^2-1)^{-m/2}\frac{dv}{dx} - mx(x^2-1)^{-m/2-1}v.$$

$$\frac{d^2u}{dx^2} = (x^2-1)^{-m/2}\frac{d^2v}{dx^2} - 2mx(x^2-1)^{-m/2-1}\frac{dv}{dx}$$
$$+ m(m+2)x^2(x^2-1)^{-m/2-2}v - m(x^2-1)^{-m/2-1}v.$$

We then have

$$(x^2-1)^{1-m/2}\frac{d^2v}{dx^2} - 2mx(x^2-1)^{-m/2}\frac{dv}{dx} + m(m+2)x^2(x^2-1)^{-m/2-1}v$$
$$- m(x^2-1)^{-m/2}v + 2(m+1)x(x^2-1)^{-m/2}\frac{dv}{dx}$$
$$- 2m(m+1)x^2(x^2-1)^{-m/2-1}v$$
$$- (n-m)(n+1+m)(x^2-1)^{-m/2}v = 0,$$

that is $$(x^2-1)\frac{d^2v}{dx^2} + 2x\frac{dv}{dx} - \left\{n(n+1) + \frac{m^2}{x^2-1}\right\}v = 0. \qquad (1)$$

This equation is known as the Associated Legendre equation. Its solutions are called Associated Legendre functions and denoted by the symbols $P_n{}^m(x)$ and $Q_n{}^m(x)$. From the method by which the differential equation (1) is formed we have that $v = (x^2-1)^{m/2}D^m y$, and hence

$$P_n{}^m(x) = (x^2-1)^{m/2}D^m P_n(x),$$
$$Q_n{}^m(x) = (x^2-1)^{m/2}D^m Q_n(x).$$

EXERCISES 13 (*b*)

1. Show that for $|x| < 1$ a second solution of Legendre's equation is

$$y = \sum_{r=0}^{\infty} \frac{\Gamma(-\tfrac{1}{2}n + \tfrac{1}{2} + r)\Gamma(\tfrac{1}{2}n + 1 + r)}{(\tfrac{3}{2} + r)(r)!}x^{2r+1} \text{ if } n \text{ is even,}$$

$$y = \sum_{r=0}^{\infty} \frac{\Gamma(-\tfrac{1}{2}n + r)\Gamma(\tfrac{1}{2}n + \tfrac{1}{2} + r)}{\Gamma(\tfrac{1}{2} + r)(r)!}x^{2r}, \text{ if } n \text{ is odd.}$$

2. Prove that $$\int_{-1}^{1} \frac{P_n(x)dx}{(1-2xh+h^2)^{1/2}} = \frac{h^n}{n+\tfrac{1}{2}}.$$

3. Prove that if $|h| < 1$,

$$h(1-2xh+h^2)^{-3/2} = \sum_{r=1}^{\infty} P'_r(x).h^r.$$

Hence prove that $$P'_n(1) = \frac{n(n+1)}{2},$$

$$P'_n(-1) = (-1)^{n-1}\, n\frac{(n+1)}{2}.$$

4. Prove that $$\int_{-1}^{1} P''_n(x)dx = \frac{1}{2}n(n+1)(1-\cos n\pi),$$

$$\int_{-1}^{1} xP'_n(x)dx = 1 + \cos n\pi.$$

5. Show that for $r < n$,

$$\left\{\frac{d^r}{dx^r}P_n(x)\right\}_{x=1} = \frac{(n+r)!}{(n-r)!\,2^r(r)!}.$$

6. Show that $y = P_n(\cos\theta)$ is a solution of the differential equation

$$\frac{d^2y}{d\theta^2} + \cot\theta\frac{dy}{d\theta} + n(n+1)y = 0.$$

Show that

$$P_n(\cos\theta) = 1 - n(n+1)\sin^2\frac{1}{2}\theta + \frac{1}{4}(n-1)n(n+1)(n+2)\sin^4\frac{1}{2}\theta + \dots.$$

7. Prove that if $P_n(x)$ is the Legendre polynomial of degree n,

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0.$$

8. Prove that for $n < m$, and $m + n$ even

$$\int_{-1}^{1} P'_n(x)P'_m(x)dx = m(m+1).$$

9. Prove that $\displaystyle\int_{-1}^{1} xP_n(x)P_{n+1}(x)dx = \frac{2(n+1)}{(2n+1)(2n+3)}.$

10. Show that

$$Q_0(x) = \frac{1}{2}\log\frac{x+1}{x-1},$$

$$Q_1(x) = \frac{1}{2}x\log\frac{x+1}{x-1} - 1.$$

11. From the values of $Q_0(x)$ and $Q_1(x)$ given in Exercise 10 with the aid of the recurrence formulae prove that

$$Q_2(x) = \frac{1}{4}(3x^2-1)\log\frac{x+1}{x-1} - \frac{3}{2}x,$$

$$Q_3(x) = \frac{1}{4}(5x^3-3x)\log\frac{x+1}{x-1} - \frac{5}{2}x^2 + \frac{2}{3}.$$

12. Show that if $u = \frac{1}{2}P_n(x)\log\frac{x+1}{x-1}$,

$$(x^2-1)\frac{d^2u}{dx^2} + 2x\frac{du}{dx} - n(n+1)u = -2P'_n(x).$$

Hence show that if $y = \frac{1}{2}P_n(x)\log\frac{x+1}{x-1} + v(x)$ is a solution of Legendre's equation of degree n

$$(x^2-1)\frac{d^2v}{dx^2} + 2x\frac{dv}{dx} - n(n+1)v = 2P'_n(x).$$

Show that this equation has a solution which is a polynomial of degree $n-1$ in x.

13. Show that for $|x|>1$ a second solution of Legendre's equation is

$$y = P_n(x)\int_x^{\infty}\frac{dx}{(1-x^2)[P_n(x)]^2}.$$

Verify the formula

$$Q_n(x) = P_n(x)\int_x^\infty \frac{dx}{(1-x^2)[P_n(x)]^2}$$

for $n = 1$.

14. Show that if $m \leqslant n$ the solution of the differential equation

$$(x^2 - 1)\frac{d^2y}{dx^2} + 2(m+1)x\frac{dy}{dx} - (n-m)(n+1+m)y = 0$$

is $y = AD^mP_n(x) + BD^mQ_n(x)$, where A and B are constants.

15. Use the recurrence formulae to show that

$$(n+1)\{P_{n+1}(x)Q_n(x) - Q_{n+1}(x)P_n(x)\}$$
$$= n\{P_n(x)Q_{n-1}(x) - Q_n(x)P_{n-1}(x)\}.$$

Hence show that $P_n(x)Q_{n-1}(x) - Q_n(x)P_{n-1}(x) = \dfrac{1}{n}$.

13.15 Bessel's Differential Equation

The differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0$$

is known as Bessel's equation of order n.

To obtain a solution in series of this equation we group the terms according to weight and write

$$\left(x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - n^2y\right) + (x^2y) = 0.$$

Setting $y = a_0x^\varrho + a_1x^{\varrho+1} + \ldots$, the lowest power of x is obtained by substituting a_0x^ϱ in the left-hand bracket, giving the indicial equation

$$a_0\{\varrho(\varrho-1) + \varrho - n^2\}x^\varrho = 0,$$

that is

$$\varrho^2 - n^2 = 0,$$

$$\varrho = \pm n.$$

Substituting $a_rx^{\varrho+r}$ in the left-hand bracket and $a_{r-2}x^{\varrho+r-2}$ in the right-hand bracket we have

$$\{(\varrho+r)^2 - n^2\}a_r + a_{r-2} = 0,$$

$$a_r = -\frac{a_{r-2}}{(\varrho+r-n)(\varrho+r+n)}.$$

If $\varrho = n$ we have

$$a_r = -\frac{a_{r-2}}{r(2n + r)},$$

$$a_2 = -\frac{a_0}{2^2(n + 1)},$$

$$a_4 = -\frac{a_2}{2^2.2(n + 2)},$$

$$a_6 = -\frac{a_4}{2^2.3(n + 3)},$$

$$a_{2r} = \frac{(-1)^r a_0}{2^{2r}(r)!\,(n + 1)(n + 2)\ldots(n + r)},$$

$$= a_0\frac{\Gamma(n + 1)(-1)^r}{2^{2r}(r)!\Gamma(n + r + 1)}.$$

The ratio of successive terms tends to zero as n tends to infinity, and hence the series is convergent for all finite values of x.

Therefore we have a solution

$$y = a_0\left\{x^n - \frac{x^{n+2}}{2^2.1.(n + 1)} + \frac{x^{n+4}}{2^4.2!(n + 1)(n + 2)} - \ldots\right\}.$$

Writing $a_0 = \dfrac{1}{2^n\Gamma(n + 1)}$ we have the standard solution,

$$J_n(x) = \frac{1}{\Gamma(n + 1)}\left\{(x/2)^n - \frac{(x/2)^{n+2}}{1.(n + 1)} + \frac{(x/2)^{n+4}}{1.2.(n + 1)(n + 2)} - \ldots\right\}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r(x/2)^{n+2r}}{(r)!\Gamma(n + r + 1)}.$$

$J_n(x)$ is called the Bessel function of the first kind of order n, and is given by this formula for integer and fractional values of n. Values of $J_n(x)$ for different values of x and n have been extensively tabulated; for example, in British Association Mathematical Tables, Vol. 5 (1937), values of $J_0(x)$ and $J_1(x)$ are given to ten decimal places.

If we put $\varrho = -n$ in the recurrence relations, we have

$$a_r = -\frac{a_{r-2}}{r(-2n + r)}.$$

If n is not an integer we can find a set of values of a_2, a_4, etc., in terms of a_0 as before, but *if n is an integer* the recurrence relations break down when $r = 2n$ and the second solution of the differential equation must be found by other methods.

If n is not an integer we have

$$a_{2r} = a_0 \frac{(-1)^r \Gamma(-n+1)}{2^{2r}(r)!\,\Gamma(-n+r+1)},$$

and we define the second solution of the differential equation as

$$J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{-n+2r}}{(r)!\,\Gamma(-n+r+1)}.$$

Thus if n is not an integer the complete solution of the differential equation is

$$y = AJ_n(x) + BJ_{-n}(x),$$

where A and B are constants.

Example 5. *Find the complete solution of Bessel's differential equation when* $n = \frac{1}{2}$.

We have

$$J_{1/2}(x) = \frac{1}{\Gamma(1+\frac{1}{2})}\left\{(x/2)^{1/2} - \frac{(x/2)^{2+(1/2)}}{1.\frac{3}{2}} + \frac{(x/2)^{4+(1/2)}}{1.2.\frac{3}{2}.\frac{5}{2}} - \frac{(x/2)^{6+(1/2)}}{1.2.3.\frac{3}{2}.\frac{5}{2}.\frac{7}{2}}\cdots\right\}$$

$$= \frac{(x/2)^{1/2}}{\frac{1}{2}\Gamma(\frac{1}{2})}\left\{1 - \frac{(x/2)^2}{1.\frac{3}{2}} + \frac{(x/2)^4}{1.2.\frac{3}{2}.\frac{5}{2}} - \frac{(x/2)^6}{1.2.3.\frac{3}{2}.\frac{5}{2}.\frac{7}{2}} + \cdots\right\}$$

$$= \sqrt{(2x/\pi)}\left\{1 - \frac{x^2}{1.2.3} + \frac{x^4}{1.2.3.4.5} - \frac{x^6}{1.2.3.4.5.6.7} + \cdots\right\}$$

$$= \sqrt{(2/\pi x)}\left\{x - \frac{x^3}{(3)!} + \frac{x^5}{(5)!} - \frac{x^7}{(7)!} + \cdots\right\}$$

$$= \sqrt{(2/\pi x)}\sin x.$$

$$J_{-(1/2)}(x) = \frac{1}{\Gamma(1-\frac{1}{2})}\left\{(x/2)^{-(1/2)} - \frac{(x/2)^{2-(1/2)}}{1.\frac{1}{2}} + \frac{(x/2)^{4-(1/2)}}{1.2.\frac{1}{2}.\frac{3}{2}}\cdots\right\}$$

$$= \frac{(x/2)^{-(1/2)}}{\sqrt{\pi}}\left\{1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} + \cdots\right\}$$

$$= \sqrt{(2/\pi x)}\cos x.$$

Hence a complete solution is

$$y = x^{-(1/2)}(A\cos x + B\sin x).$$

13.16 Bessel Functions of Order Zero

If $n = 0$ we have one solution of Bessel's equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = 0,$$

namely
$$J_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{2r}}{(r)!\,(r)!}$$

$$= 1 - \frac{x^2}{2.2} + \frac{x^4}{2.2.4.4} - \frac{x^6}{2.2.4.4.6.6} + \cdots.$$

The second solution is defined as $Y_0(x)$, where

$$Y_0(x) = \frac{2}{\pi}\left(\log\frac{x}{2} + \gamma\right)J_0(x) - V_0(x),$$

$$V_0(x) = \frac{2}{\pi}\sum_{r=1}^{\infty}\frac{(-1)^r(x/2)^{2r}}{(r)!\,(r)!}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{r}\right).$$

The number γ is known as Euler's constant and its value is 0·5772 approximately.

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} - \log n\right).$$

We shall show that $Y_0(x)$ is in fact a solution of Bessel's equation.

Let $\quad y = \frac{2}{\pi}\left(\log\frac{1}{2}x + \gamma\right)J_0(x) - V_0(x).$

Then $\quad \frac{dy}{dx} = \frac{2}{\pi}\left(\log\frac{1}{2}x + \gamma\right)J'_0(x) + \frac{2}{\pi x}J_0(x) - V'_0(x),$

$$\frac{d^2y}{dx^2} = \frac{2}{\pi}\left(\log\frac{1}{2}x + \gamma\right)J''_0(x) + \frac{4}{\pi x}J'_0(x) - \frac{2}{\pi x^2}J_0(x) - V''_0(x).$$

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + y = \frac{2}{\pi}\left(\log\frac{1}{2}x + \gamma\right)\left\{J''_0(x) + \frac{1}{x}J'_0(x) + J_0(x)\right\}$$
$$+ \frac{4}{\pi x}J'_0(x) - \left\{V''_0(x) + \frac{1}{x}V'_0(x) + V_0(x)\right\}.$$

Hence, if y is to be a solution of Bessel's equation we must have

$$\frac{d^2V_0}{dx^2} + \frac{1}{x}\frac{dV_0}{dx} + V_0 = \frac{4}{\pi x}J'_0(x)$$
$$= \frac{2}{\pi}\sum_{r=1}^{\infty}\frac{(-1)^r(x/2)^{2r-2}}{(r)!\,(r-1)!}.$$

Let $$V_0 = \frac{2}{\pi}\sum_{r=1}^{\infty} b_r(x/2)^{2r}.$$

$$\frac{1}{x}\frac{dV_0}{dx} = \frac{2}{\pi}\sum_{r=1}^{\infty}\frac{1}{2}rb_r(x/2)^{2r-2},$$

$$\frac{d^2V_0}{dx^2} = \frac{2}{\pi}\sum_{r=1}^{\infty} r(r-1/2)b_r(x/2)^{2r-2},$$

$$\frac{d^2V_0}{dx^2} + \frac{1}{x}\frac{dV_0}{dx} + V_0 = \frac{2}{\pi}\sum_{r=1}^{\infty}(r^2b_r + b_{r-1})(x/2)^{2r-2}.$$

Hence we have

$$r^2b_r + b_{r-1} = \frac{(-1)^r}{(r)!\,(r-1)!}, \text{ for } r = 1, 2, \ldots.$$

Since $b_0 = 0$ we have

$$b_1 = -1,$$

$$4b_2 + b_1 = \frac{1}{2}, \qquad b_2 = \frac{1}{(2!)^2}\left(1 + \frac{1}{2}\right),$$

$$9b_3 + b_2 = -\frac{1}{2!3!}, \qquad b_3 = \frac{-1}{(3!)^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right),$$

$$16b_4 + b_3 = \frac{1}{3!4!}, \qquad b_4 = \frac{1}{(4!)^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}\right),$$

$$b_r = \frac{(-1)^r}{(r!)^2}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{r}\right),$$

$$Y_0(x) = \frac{2}{\pi}\sum_{r=1}^{\infty}\frac{(-1)^r(x/2)^{2r}}{(r)!\,(r)!}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{r}\right).$$

Thus $Y_0(x)$ is a solution of Bessel's equation of order zero. Tables of $Y_0(x)$ are given in British Association Mathematical Tables, Vol. 6 (1937). The complete solution of Bessel's equation is then

$$y = AJ_0(x) + BY_0(x),$$

where A and B are constants. Both $J_0(x)$ and $Y_0(x)$ are oscillatory functions and their graphs are as shown in Fig. 348.

It is evident that as x tends to zero $Y_0(x)$ tends to $-\infty$ and for small values of x is approximately $\frac{2}{\pi}\left(\log \frac{1}{2}x + \gamma\right)$

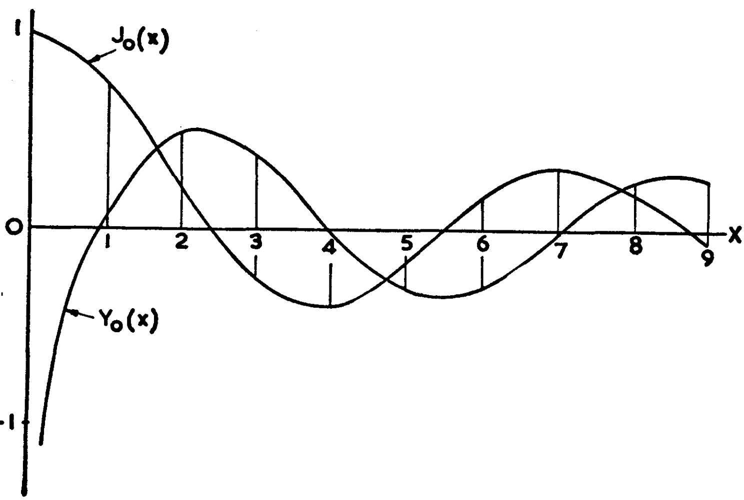

Fig. 348

13.17 Bessel Functions of Integer Order

If the parameter n in Bessel's equation is an integer we have the solution

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{n+2r}}{(r)!\,(n+r)!}.$$

The second solution is denoted by $Y_n(x)$ and is the rather complicated expression

$$\begin{aligned} Y_n(x) = {} & \frac{2}{\pi}\left(\log \frac{1}{2}x + \gamma\right) J_n(x) \\ & -\frac{1}{\pi}\sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{2r+n}}{(r)!\,(n+r)!}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{r}\right) \\ & -\frac{1}{\pi}\sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{2r+n}}{(r)!\,(n+r)!}\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n+r}\right) \\ & -\frac{1}{\pi}\sum_{r=0}^{n-1} \frac{(n-r-1)!}{(r)!}(x/2)^{2r-n}. \end{aligned}$$

This expression may be obtained as in the previous section by substituting in the differential equation

$$y = \frac{2}{\pi}\left(\log \frac{1}{2}x + \gamma\right) J_n(x) - V_n(x),$$

and it will be found that $V_n(x)$ must satisfy the differential equation

$$\frac{d^2 V_n}{dx^2} + \frac{1}{x}\frac{dV_n}{dx} + \left(1 - \frac{n^2}{x^2}\right) V_n = \frac{4}{\pi x} J'_n(x).$$

This equation can only be satisfied if an expression for $V_n(x)$ is taken to be of the form

$$(x/2)^{-n}\{c_0 + c_1(x/2)^2 + c_2(x/2)^4 + \ldots\},$$

and by substitution the expression for $V_n(x)$ can be deduced.

The recurrence relations for the coefficients c_r are

$$\begin{aligned} r(r-n)c_r + c_{r-1} &= 0, \text{ for } 0 < r < n \\ &= \frac{2(-1)^{n+r}}{\pi}\,\frac{(2r-n)}{(r)!\,(r-n)!}, \text{ for } r > n. \end{aligned}$$

With $c_0 = \dfrac{1}{\pi}(n-1)!$, these recurrence relations lead to the above expression for $Y_n(x)$.

The function $Y_n(x)$ is defined for non-integer values of n as

$$Y_n(x) = \frac{J_n(x)\cos n\pi - J_{-n}(x)}{\sin n\pi}.$$

It is evident that this is a solution of Bessel's equation. It can be shown that the limiting value of this quantity as n tends to some integer value

is identical with the expression we have found for $Y_n(x)$ when n is an integer.

Thus we may write the complete solution of Bessel's equation in the form

$$y = AJ_n(x) + BY_n(x),$$

for all values of n.

13.18 Recurrence Formulae for Bessel Functions

The following recurrence formulae exist between Bessel functions of order $n + 1$, n, $n - 1$, and their derivatives.

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \tag{1}$$

$$J_{n-1}(x) - J_{n+1}(x) = 2J'_n(x) \tag{2}$$

$$\frac{n}{x} J_n(x) - J'_n(x) = J_{n+1}(x) \tag{3}$$

$$\frac{n}{x} J_n(x) + J'_n(x) = J_{n-1}(x) \tag{4}$$

We shall prove the formulae (3) and (4); formulae (1) and (2) follow directly.

$$\frac{J_n(x)}{x^n} = 2^{-n} \sum_{r=0}^{\infty} \frac{(-1)^r (\frac{1}{2}x)^{2r}}{(r)!\Gamma(r + n + 1)},$$

$$\frac{d}{dx}\left\{\frac{J_n(x)}{x^n}\right\} = 2^{-n} \sum_{r=1}^{\infty} \frac{(-1)^r r (\frac{1}{2}x)^{2r-1}}{(r)!\Gamma(r + n + 1)}$$

$$= -2^{-n} \sum_{r=0}^{\infty} \frac{(-1)^r (\frac{1}{2}x)^{2r+1}}{(r)!\Gamma(r + n + 2)}$$

$$= -x^{-n} J_{n+1}(x).$$

Hence $$\frac{J'_n(x)}{x^n} - \frac{nJ_n(x)}{x^{n+1}} = -\frac{1}{x^n} J_{n+1}(x),$$

and formulae (3) follows.

$$x^n J_n(x) = 2^n \sum_{r=0}^{\infty} \frac{(-1)^r (\frac{1}{2}x)^{2n+2r}}{(r)!\Gamma(n + r + 1)},$$

$$\frac{d}{dx}\{x^n J_n(x)\} = 2^n \sum_{r=0}^{\infty} \frac{(-1)^r (n + r)(\frac{1}{2}x)^{2n+2r-1}}{(r)!\Gamma(n + r + 1)}$$

$$= x^n \sum_{r=0}^{\infty} \frac{(-1)^r (\frac{1}{2}x)^{n-1+2r}}{(r)!\Gamma(n + r)}$$

$$= x^n J_{n-1}(x).$$

Hence $x^n J'_n(x) + nx^{n-1} J_n(x) = x^n J_{n-1}(x)$, and (4) follows.

A useful result follows if we put $n = 0$ in (3), namely

$$J'_0(x) = -J_1(x).$$

It can be shown that the functions $Y_n(x)$ satisfy the same recurrence formulae as the functions $J_n(x)$.

Example 6. *Express $J_{5/2}(x)$ in finite form.*

We have (Example 5)

$$\begin{aligned}
J_{1/2}(x) &= \sqrt{(2/\pi x)} \sin x, \\
J_{n+1}(x) &= -x^n \frac{d}{dx}\{x^{-n} J_n(x)\}, \\
J_{3/2}(x) &= -x^{1/2} \frac{d}{dx}\{x^{-1/2} J_{1/2}(x)\} \\
&= -x^{1/2} \frac{d}{dx}\{\sqrt{(2/\pi)} x^{-1} \sin x\} \\
&= -\sqrt{(2/\pi)} x^{1/2} (x^{-1} \cos x - x^{-2} \sin x). \\
J_{5/2}(x) &= -x^{3/2} \frac{d}{dx}\{x^{-3/2} J_{3/2}(x)\} \\
&= \sqrt{(2/\pi)} x^{3/2} \frac{d}{dx}(x^{-2} \cos x - x^{-3} \sin x) \\
&= \sqrt{(2/\pi)} x^{3/2} (-2x^{-3} \cos x - x^{-3} \cos x - x^{-2} \sin x + 3x^{-4} \sin x) \\
&= \sqrt{(2/\pi)} x^{3/2} \left\{\left(\frac{3}{x^4} - \frac{1}{x^2}\right) \sin x - \frac{3}{x^3} \cos x\right\}.
\end{aligned}$$

13.19 Bessel's Integral

When n is an integer,

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta.$$

If $n = 0$,

$$\begin{aligned}
\frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)\,d\theta &= \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r)!} \frac{x^{2r}}{\pi} \int_0^\pi \sin^{2r}\theta\,d\theta \\
&= \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r)!} x^{2r} \cdot \frac{(2r)!}{2^{2r}(r)!(r)!} \\
&= J_0(x).
\end{aligned}$$

Assume that

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta.$$

Now

$$\frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta).(n - x\cos\theta)\,d\theta = \frac{1}{\pi}\Big[\sin(n\theta - x\sin\theta)\Big]_0^\pi = 0,$$

therefore

$$\frac{n}{x} J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\cos\theta\,d\theta.$$

Also
$$J'_n(x) = \frac{1}{\pi}\int_0^\pi \sin(n\theta - x\sin\theta)\sin\theta d\theta,$$
therefore
$$\frac{n}{x}J_n(x) - J'_n(x) = \frac{1}{\pi}\int_0^\pi \cos\{(n+1)\theta - x\sin\theta\}d\theta$$
$$= J_{n+1}(x).$$

Therefore, on the assumption that the formula is true for $J_n(x)$ we have proved it true for $J_{n+1}(x)$, and hence, since it is true for $n = 0$, the formula is established.

Since the function in the integrand is less or equal to unity, it is clear that $|J_n(x)| \leqslant 1$, and it can be proved that this is so for all values of n. The functions $Y_n(x)$ tend to $-\infty$ as x tends to zero because of the factor $\log x$, but if x is greater than a certain value $Y_n(x)$ also lies between -1 and 1.

13.20 Asymptotic Values

The series defining the Bessel functions $J_n(x)$ and $Y_n(x)$ converge very slowly when x is large. Approximate values of these functions may however be obtained for large values of x.

In the differential equation
$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0,$$
let $y = e^{ix}u$. Substituting we have
$$\left(x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} - n^2u\right) + i\left(2x^2\frac{du}{dx} + xu\right) = 0.$$
Let $u = x^\rho(a_0 + a_1x^{-1} + a_2x^{-2} + \ldots)$.

The highest power of x when u is substituted in the differential equation is
$$ia_0(2\rho + 1)x^{\rho+1}.$$
We must therefore take $\rho = -\dfrac{1}{2}$, and we obtain the relations between the coefficients
$$\{(r + \tfrac{1}{2})^2 - n^2\}a_r + i\{-2(r+1)\}a_{r+1} = 0,$$
$$a_{r+1} = i\frac{4n^2 - (2r+1)^2}{2^3(r+1)}a_r.$$
The resulting series is not convergent but may nevertheless be used for computing values of the solutions.

As a first approximation we have a solution $a_0e^{ix}x^{-(1/2)}$, and similarly a solution $b_0e^{-ix}x^{-(1/2)}$.

$J_n(x)$ and $Y_n(x)$ are identified with combinations of these approximate solutions by writing

$$J_n(x) \sim \frac{\cos\left(x - \frac{\pi}{4} - n\frac{\pi}{2}\right)}{\sqrt{(\pi x/2)}},$$

$$Y_n(x) \sim \frac{\sin\left(x - \frac{\pi}{4} - n\frac{\pi}{2}\right)}{\sqrt{(\pi x/2)}}.$$

It will be seen that if $n = \frac{1}{2}$, these give the accurate solutions

$$J_{1/2}(x) = \frac{\sin x}{\sqrt{(\pi x/2)}}, \quad Y_{1/2}(x) = \frac{-\cos x}{\sqrt{(\pi x/2)}}.$$

Closer approximation may be obtained by taking further terms of the series of descending powers of x.

13.21 Hankel Functions

For some purposes it is convenient to take the fundamental solutions of Bessel's equation in a different form.

Hankel functions, which are sometimes called Bessel functions of the third kind, are defined as

$$H_n^{(1)}(x) = J_n(x) + iY_n(x),$$
$$H_n^{(2)}(x) = J_n(x) - iY_n(x).$$

These functions are complex quantities and are evidently solutions of Bessel's equation. It is also clear that the functions $H_n^{(1)}(x)$ and $H_n^{(2)}(x)$ satisfy the recurrence formulae of § 13.18.

13.22 Modified Bessel Functions

If we substitute ix for x in Bessel's equation it becomes

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \left(1 + \frac{n^2}{x^2}\right)y = 0. \tag{1}$$

This equation, therefore, has the solution

$$y = AJ_n(ix) + BY_n(ix).$$

Writing
$$I_n(x) = (i)^{-n}J_n(ix)$$
$$= \sum_{r=0}^{\infty} \frac{(\frac{1}{2}x)^{n+2r}}{(r)!\,\Gamma(n+r+1)},$$

it is clear that $I_n(x)$ is a real solution of the equation (1). $I_n(x)$ is called a *modified Bessel function* of the first kind.

The modified Bessel function of the second kind is denoted by $K_n(x)$ and defined by the equation

$$K_n(x) = \frac{\frac{1}{2}\pi}{\sin n\pi}\{I_{-n}(x) - I_n(x)\},$$

or by the limit of this quantity if n is an integer.

The complete solution of the equation (1) is therefore

$$y = AI_n(x) + BK_n(x).$$

The functions $I_n(x)$ and $K_n(x)$ have been extensively tabulated.

13.23 Transformations of Bessel's Equation

If we make the substitution $y = x^n.u$ in Bessel's equation, we have

$$\frac{dy}{dx} = x^n\left(\frac{du}{dx} + \frac{n}{x}u\right),$$

$$\frac{d^2y}{dx^2} = x^n\left\{\frac{d^2u}{dx^2} + \frac{2n}{x}\frac{du}{dx} + \frac{n(n-1)u}{x^2}\right\},$$

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = x^n\left\{\frac{d^2u}{dx^2} + \frac{(2n+1)}{x}\frac{du}{dx} + u\right\}.$$

Hence the differential equation

$$\frac{d^2u}{dx^2} + \frac{(2n+1)}{x}\frac{du}{dx} + u = 0 \tag{1}$$

has the solution $u = x^{-n}\{AJ_n(x) + BY_n(x)\}$.

Similarly, the equation

$$\frac{d^2u}{dx^2} + \frac{(1-2n)}{x}\frac{du}{dx} + u = 0 \tag{2}$$

has the solution $u = x^n\{AJ_n(x) + BY_n(x)\}$.

If x is replaced by ax in Bessel's equation, we have

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(a^2 - \frac{n^2}{x^2}\right)y = 0, \tag{3}$$

of which the solution is $y = AJ_n(ax) + BY_n(ax)$.

If x is replaced by ax in equation (2) above, we have

$$\frac{d^2u}{dx^2} + \frac{1-2n}{x}\frac{du}{dx} + a^2u = 0.$$

Thus the complete solution of the differential equation

$$\frac{d^2u}{dx^2} + \frac{m}{x}\frac{du}{dx} + a^2u = 0 \tag{4}$$

is $$u = x^{(1-m)/2}\{AJ_{(1-m)/2}(ax) + BY_{(1-m)/2}(ax)\}.$$

13.24 Vibration of a Tapering Beam

In § 6.9 we found the equation of vibration of a uniform beam as

$$EI\frac{\partial^4 y}{\partial x^4} = -\frac{m}{g}\frac{\partial^2 y}{\partial t^2}.$$

If the beam is not of uniform cross-section this becomes

$$\frac{\partial^2}{\partial x^2}EI\frac{\partial^2 y}{\partial x^2} = -\frac{m}{g}\frac{\partial^2 y}{\partial t^2}.$$

Here E is Young's modulus, I is the moment of inertia of the section about the neutral axis, and m the mass per unit length. Also y is the deflexion at time t of a point distant x from one end.

If the beam tapers to a point, at distance x from the point I will be proportional to x^4 and m to x^2. Let $I = ax^4$, $m = bx^2$.

Then
$$\frac{Eag}{b}\frac{\partial^2}{\partial x^2}x^4\frac{\partial^2 y}{\partial x^2} = -x^2\frac{\partial^2 y}{\partial t^2}.$$

Let $y = u \cos \omega t$, where u depends on x alone and is therefore the maximum deflexion.

Then
$$\frac{\partial^2 y}{\partial t^2} = -\omega^2 u \cos \omega t,$$

and we have
$$\frac{\partial^2}{\partial x^2}x^4\frac{\partial^2 u}{\partial x^2} = \frac{b\omega^2}{Eag}x^2 u.$$

Let $k^4 = \dfrac{b\omega^2}{Eag}$. Since u is a function of x only we may write the differential coefficients as ordinary coefficients and we have the differential equation

$$\frac{d^2}{dx^2}\left(x^4\frac{d^2 u}{dx^2}\right) - k^4x^2u = 0,$$

that is
$$x^2\frac{d^4u}{dx^4} + 8x\frac{d^3u}{dx^3} + 12\frac{d^2u}{dx^2} - k^4u = 0. \qquad (1)$$

This equation may be written in the form

$$\left(x\frac{d^2}{dx^2} + 3\frac{d}{dx} - k^2\right)\left(x\frac{d^2}{dx^2} + 3\frac{d}{dx} + k^2\right)u = 0.$$

This may be verified by differentiating out this expression, and it is clear that the order of the brackets can be changed. Hence if u is a solution of the equation (1) we have

either
$$x\frac{d^2u}{dx^2} + 3\frac{du}{dx} + k^2u = 0, \qquad (2)$$

or
$$x\frac{d^2u}{dx^2} + 3\frac{du}{dx} - k^2u = 0. \qquad (3)$$

To solve the equation (2) let $t = 2kx^{1/2}$.

Then
$$\frac{dt}{dx} = \frac{2k^2}{t},$$
$$\frac{du}{dx} = \frac{2k^2}{t}\frac{du}{dt},$$
$$\frac{d^2u}{dx^2} = \frac{2k^2}{t}\frac{d}{dt}\frac{2k^2}{t}\frac{du}{dt}$$
$$= \frac{4k^4}{t^2}\frac{d^2u}{dt^2} - \frac{4k^4}{t^3}\frac{du}{dt}.$$

Substituting in (2) and dividing by k^2 we have
$$\frac{d^2u}{dt^2} + \frac{5}{t}\frac{du}{dt} + u = 0.$$

Putting $u = \frac{v}{t^2}$, this becomes the Bessel equation,
$$\frac{d^2v}{dt^2} + \frac{1}{t}\frac{dv}{dt} + \left(1 - \frac{4}{t^2}\right)v = 0.$$

Hence
$$u = t^{-2}\{AJ_2(t) + BY_2(t)\}$$
$$= x^{-1}\{AJ_2(2kx^{1/2}) + BY_2(2kx^{1/2})\}.$$

The equation (3) is identical with (2) if k is replaced by ik. Thus the solution involves modified Bessel functions of argument $2kx^{1/2}$ and we have
$$u = x^{-1}\{CI_2(2kx^{1/2}) + DK_2(2kx^{1/2})\}.$$

Thus the complete solution is
$$y = \cos \omega t . x^{-1}\{AJ_2(2kx^{1/2}) + BY_2(2kx^{1/2}) + CI_2(2kx^{1/2}) + DK_2(2kx^{1/2})\}.$$

EXERCISES 13 (c)

1. Show that
$$e^{(1/2)x(t-1/t)} = \left\{1 + (\tfrac{1}{2}x)t + \frac{(\frac{1}{2}x)^2t^2}{2!} + \dots\right\}\left\{1 - (\tfrac{1}{2}x)\frac{1}{t} + \frac{(\frac{1}{2}x)^2}{2!}\frac{1}{t^2} - \dots\right\}$$
$$= J_0(x) + J_1(x)\left(t - \frac{1}{t}\right) + J_2(x)\left(t^2 + \frac{1}{t^2}\right) + \dots.$$

2. Show that $y = J_0(x)\int \frac{dx}{x\{J_0(x)\}^2}$ is a solution of Bessel's equation of zero order.

3. Given that $J_0(\frac{1}{2}) = 0{\cdot}9385$, $J_1(\frac{1}{2}) = 0{\cdot}2423$, find the values of $J'_1(\frac{1}{2})$, $J_2(\frac{1}{2})$, $J'_3(\frac{1}{2})$.

4. Given that $J_0(1) = 0{\cdot}765$, $J_1(1) = 0{\cdot}440$, $Y_0(1) = 0{\cdot}088$, find the solution of the differential equation $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + xy = 0$, such that $y = 2$ and $\dfrac{dy}{dx} = 1$ when $x = 1$.

5. Given that $J_{1/2}(x) = \sqrt{(2/\pi x)} \sin x$, use the recurrence formulae to find the values of $J_{-1/2}(x)$ and $J_{-3/2}(x)$.

6. Prove that
$$I_{n-1}(x) - I_{n+1}(x) = \frac{2n}{x} I_n(x),$$
$$I_{n-1}(x) + I_{n+1}(x) = 2I'_n(x).$$

7. Prove that
$$J_3(x) = -\left\{\frac{d^3}{dx^3} + \frac{3}{x}\frac{d^2}{dx^2} - \frac{3}{x^2}\frac{d}{dx}\right\} J_0(x).$$

8. Write down the differential equation one of whose solutions is $J_0(x) + J_2(x)$, and find the other solution.

9. Prove that
$$x\{J_n(x)Y'_n(x) - Y_n(x)J'_n(x)\} = \text{constant} = \frac{2}{\pi}.$$

10. Prove that
$$J_{n+1}(x)Y_n(x) - Y_{n+1}(x)J_n(x) = \frac{2}{\pi x}.$$

11. Show that the solution y of Bessel's equation of order n, which is such that $y = \lambda$ and $\dfrac{dy}{dx} = \mu$ when $x = a$, is
$$y = \tfrac{1}{2}\pi a\{\lambda Y'_n(a) - \mu Y_n(a)\}J_n(x) - \tfrac{1}{2}\pi a\{\lambda J'_n(a) - \mu J_n(a)\}Y_n(x).$$

12. Prove that
$$\int_0^{\frac{1}{2}\pi} J_0(x\cos\theta)\cos\theta\, d\theta = \frac{\sin x}{x}.$$

13. Prove that
$$\int x^{n+1}J_n(x)dx = x^{n+1}J_{n+1}(x),$$
$$\int x^{1-n}J_n(x)dx = -x^{1-n}J_{n-1}(x).$$

14. Prove that
$$\int \frac{1}{x} J_1^2(x)dx = -\frac{1}{2}\{J_0^2(x) + J_1^2(x)\}.$$

15. Find the values of $I_{1/2}(\frac{1}{2}\pi)$ and $K_{1/2}(\frac{1}{2}\pi)$.

16. Write down the complete solution of the differential equation
$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + 4xy = 0.$$

17. Solve the differential equation
$$\frac{d^2y}{dx^2} - \frac{3}{x}\frac{dy}{dx} + 4y = 0.$$

18. Solve the differential equation

$$\frac{d^2y}{dx^2} - \frac{2}{x}\frac{dy}{dx} + 9y = 0.$$

19. Show that the complete solution of the differential equation

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + (a^2x^2 + 1)y = 0 \text{ is } y = x^{-1}\{AJ_0(ax) + BY_0(ax)\}.$$

20. Show that the substitution $x = t^{1/2}$ reduces the differential equation $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} + 4\left(x^2 - \dfrac{1}{x^2}\right)y = 0$ to Bessel's equation, and find its solution.

21. Solve the equation $\dfrac{d^2y}{dx^2} + \dfrac{7}{x}\dfrac{dy}{dx} - \left(1 + \dfrac{7}{x^2}\right)y = 0.$

22. Solve the differential equation $\dfrac{d^2y}{dx^2} + xy = 0$ by substituting $t^2 = \dfrac{4}{9}x^3$.

13.25 The Wave Equation

Consider a wave which is propagated along the x-axis with velocity c, so that at any point x there is at time t a disturbance measured by V. V may be, for example the height of a wave in a liquid, and its magnitude depends on the values of the two variables x and t, so that we may write

$$V = f(x, t).$$

When $t = 0$, $V = f(x, 0)$.

If the wave moves along the x-axis without change of shape, the value of V at x at time t, will be the same as that at x_1 at time $t = 0$ where $x_1 = x - ct$ (Fig. 349).

That is

$$f(x, t) = f(x_1, 0)$$
$$= f(x - ct, 0).$$

Therefore V is a function of $x - ct$ and we may write

$$V = \phi(x - ct).$$

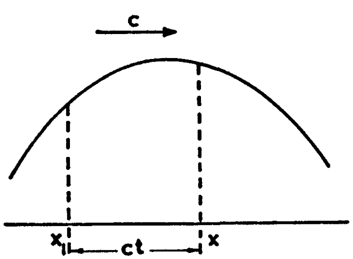

Fig. 349

It follows that V must be a solution of the partial differential equation,

$$\frac{\partial^2 V}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0.$$

This is the one-dimensional form of the wave equation.

If the wave is a harmonic wave which at time $t = 0$ has the form $V = a \cos nx$, then at time t it must be of the form,

$$V = a \cos n(x - ct).$$

For a given value of t, V has the same value at $x = 0$ and $x = 2\pi/n$, and the length $2\pi/n$ is called the *wavelength* λ.

Thus $$V = a\cos\frac{2\pi}{\lambda}(x - ct).$$

Also, for a given value of x, V has the same value when $ct = 0$ and $ct = \lambda$. The time λ/c is called the *period* of the wave and c/λ its *frequency*.

Now consider the motion in three dimensions of a plane wave, that is, one in which V has a constant value at all points of any plane which is perpendicular to the direction of propagation of the wave.

In Cartesian coordinates, if l, m, n be the direction cosines of the direction of propagation, the equation of any such plane is

$$lx + my + nz = p,$$

and p is the distance of the plane from the origin. If the disturbance on this plane at time t was at time $t = 0$ on the plane $lx + my + nz = p_1$, then $p - p_1$ is the distance between the planes and $p - p_1 = ct$. Thus it follows that V is a function of $lx + my + nz - ct$, and we may write

$$V = \phi(lx + my + nz - ct).$$

Therefore

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = (l^2 + m^2 + n^2)\phi''(lx + my + nz - ct),$$

$$\frac{\partial^2 V}{\partial t^2} = c^2\phi''(lx + my + nz - ct),$$

where $\phi''(\lambda)$ is the second derivative of $\phi(\lambda)$, λ being any variable. Therefore, since $l^2 + m^2 + n^2 = 1$, we have

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0.$$

This is the three-dimensional form of the wave equation. The equation is linear, and if V_1 and V_2 be two solutions, then $AV_1 + BV_2$ is also a solution, A and B being any constants.

The most general solution of the wave equation is

$$V = f(lx + my + nz - ct) + g(lx + my + nz + ct),$$

where f and g are arbitrary functions.

The wave equation applies to waves other than plane waves. Thus we may have waves with spherical symmetry due to propagation in all directions from a point source. From the transformation of the wave equation into spherical polar coordinates (§ 13.28) we shall see

that if V is independent of the angles θ and ϕ, the wave equation becomes

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0,$$

that is

$$\frac{\partial^2}{\partial r^2}(rV) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(rV) = 0.$$

Thus $rV = f(r - ct) + g(r + ct)$, where f and g are arbitrary functions. The motion is therefore similar to that of a plane wave but the magnitude diminishes as the distance increases.

13.26 Stationary Waves

If we add two harmonic waves of the same amplitude and wave length moving along the x-axis in opposite directions with the same velocity c, we have the solution of the wave equation,

$$\frac{\partial^2 V}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0,$$

$$V = a \cos\frac{2\pi}{\lambda}(x - ct) + a\cos\frac{2\pi}{\lambda}(x + ct)$$

$$= 2a\cos\frac{2\pi}{\lambda}x\cos\frac{2\pi}{\lambda}ct.$$

This value of V represents what is called a *stationary wave.* It is clear from the expression for V that $V = 0$ for $x = \frac{1}{4}\lambda, \frac{3}{4}\lambda, \ldots$, for all values of t and $V = 2a\cos\frac{2\pi x}{\lambda}$ represents the maximum amplitude for any value of x. The general picture is therefore that of a succession of waves each similar to a mode of vibration of a taut string (§ 6.7) separated by the zeros of V which are called the nodes.

In what follows we shall deal with solutions of the wave equation which represent stationary waves possessing axial or spherical symmetry. These solutions can be applied to many types of electrical and mechanical vibrations.

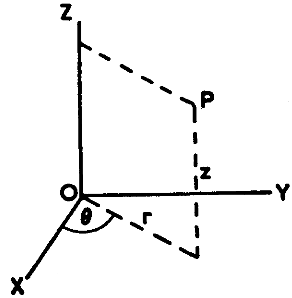

Fig. 350

13.27 Cylindrical Coordinates

Cylindrical coordinates are obtained from Cartesian coordinates (x, y, z) by writing $x = r\cos\theta, y = r\sin\theta$, and describing the point by the coordinates (r, θ, z) (Fig. 350).

To express the wave equation in cylindrical coordinates, we have

$$\frac{\partial V}{\partial r} = \cos\theta\frac{\partial V}{\partial x} + \sin\theta\frac{\partial V}{\partial y},$$

$$\frac{\partial^2 V}{\partial r^2} = \cos^2\theta\frac{\partial^2 V}{\partial x^2} + 2\cos\theta\sin\theta\frac{\partial^2 V}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 V}{\partial y^2},$$

$$\frac{\partial V}{\partial \theta} = -r\sin\theta\frac{\partial V}{\partial x} + r\cos\theta\frac{\partial V}{\partial y},$$

$$\frac{\partial^2 V}{\partial \theta^2} = r^2\sin^2\theta\,\frac{\partial^2 V}{\partial x^2} - 2\,r^2\cos\theta\sin\theta\frac{\partial^2 V}{\partial x\partial y} + r^2\cos^2\theta\,\frac{\partial^2 V}{\partial y^2}$$
$$- r\cos\theta\frac{\partial V}{\partial x} - r\sin\theta\frac{\partial V}{\partial y}.$$

Therefore $$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}.$$

The wave equation then becomes

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0.$$

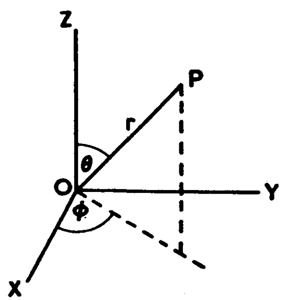

Fig. 351

13.28 Spherical Polar Coordinates

Spherical polar coordinates (r, θ, ϕ) are connected with Cartesian coordinates (x, y, z) by the relations (Fig. 351),

$$x = r\sin\theta\cos\phi,$$
$$y = r\sin\theta\sin\phi,$$
$$z = r\cos\theta.$$

To transform the wave equation into these coordinates we have

$$\frac{\partial V}{\partial r} = \sin\theta\,\cos\phi\frac{\partial V}{\partial x} + \sin\theta\,\sin\phi\frac{\partial V}{\partial y} + \cos\theta\frac{\partial V}{\partial z},$$

$$\frac{\partial^2 V}{\partial r^2} = \sin^2\theta\cos^2\phi\frac{\partial^2 V}{\partial x^2} + \sin^2\theta\sin^2\phi\frac{\partial^2 V}{\partial y^2} + \cos^2\theta\frac{\partial^2 V}{\partial z^2}$$
$$+ 2\sin\theta\cos\theta\,\sin\phi\frac{\partial^2 V}{\partial y\partial z} + 2\sin\theta\cos\theta\,\cos\phi\frac{\partial^2 V}{\partial z\partial x}$$
$$+ 2\sin^2\theta\sin\phi\,\cos\phi\frac{\partial^2 V}{\partial x\partial y}.$$

$$\frac{1}{r}\frac{\partial V}{\partial \theta} = \cos\theta\,\cos\phi\frac{\partial V}{\partial x} + \cos\theta\,\sin\phi\frac{\partial V}{\partial y} - \sin\theta\frac{\partial V}{\partial z},$$

$$\frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = \cos^2\theta\cos^2\phi\frac{\partial^2 V}{\partial x^2} + \cos^2\theta\sin^2\phi\frac{\partial^2 V}{\partial y^2} + \sin^2\theta\frac{\partial^2 V}{\partial z^2}$$
$$- 2\sin\theta\cos\theta\sin\phi\frac{\partial^2 V}{\partial y\partial z} - 2\sin\theta\cos\theta\cos\phi\frac{\partial^2 V}{\partial z\partial x}$$
$$+ 2\cos^2\theta\sin\phi\cos\phi\frac{\partial^2 V}{\partial x\partial y} - \frac{1}{r}\sin\theta\cos\phi\frac{\partial V}{\partial x}$$
$$- \frac{1}{r}\sin\theta\sin\phi\frac{\partial V}{\partial y} - \frac{1}{r}\cos\theta\frac{\partial V}{\partial z}.$$

$$\frac{1}{r}\frac{\partial V}{\partial \phi} = -\sin\theta\sin\phi\frac{\partial V}{\partial x} + \sin\theta\cos\phi\frac{\partial V}{\partial y},$$

$$\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} = \sin^2\theta\sin^2\phi\frac{\partial^2 V}{\partial x^2} + \sin^2\theta\cos^2\phi\frac{\partial^2 V}{\partial y^2} - 2\sin^2\theta\sin\phi\cos\phi\frac{\partial^2 V}{\partial x\partial y}$$
$$- \frac{1}{r}\sin\theta\cos\phi\frac{\partial V}{\partial x} - \frac{1}{r}\sin\theta\sin\phi\frac{\partial V}{\partial y}.$$

Hence

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{\cos\theta}{r^2\sin\theta}\frac{\partial V}{\partial \theta} = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}.$$

The wave equation is therefore

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\sin\theta\frac{\partial V}{\partial \theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0.$$

13.29 Separation of Variables

Solutions of the wave equation are obtained by finding differential equations satisfied by each of the variables separately.

Thus for the one-dimensional equation,

$$\frac{\partial^2 V}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0,$$

we assume that V is the product of a function of x and a function of t, that is,

$$V = X.T.$$

Then
$$\frac{\partial^2 V}{\partial x^2} = T\frac{d^2X}{dx^2} = \frac{V}{X}\frac{d^2X}{dx^2},$$

$$\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = \frac{X}{c^2}\frac{d^2T}{dt^2} = \frac{V}{c^2T}\frac{d^2T}{dt^2},$$

therefore
$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{c^2T}\frac{d^2T}{dt^2}.$$

The left-hand side of this equation is independent of t and the right-hand side is independent of x, therefore each side must be equal to a constant $-m^2$ (say).

Then $$\frac{d^2X}{dt^2} = -m^2X, \quad X = a\cos(mx + \varepsilon),$$

$$\frac{d^2T}{dt^2} = -m^2c^2T, \quad T = b\cos(mct + \eta).$$

Thus typical solutions are

$$\begin{aligned} V_1 &= A\cos mx \cos mct, \\ V_2 &= B\cos mx \sin mct, \\ V_3 &= C\sin mx \cos mct, \\ V_4 &= D\sin mx \sin mct, \end{aligned}$$

and the sum of these solutions is also a solution.

Taking different values of m, usually integer values, a solution may be obtained which is the sum of any number of terms of this type.

The constant used in the separation of the variables might have been taken as positive, $= +m^2$ (say). This would lead to exponential solutions of the type

$$V = A\cosh mx \cosh mct.$$

The same method may be applied to the three-dimensional form of the wave equation,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0.$$

Writing $$V = X.Y.Z.T,$$

where each quantity is a function of one variable only, we have

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} - \frac{1}{c^2T}\frac{d^2T}{dt^2} = 0.$$

Then each of these quantities must be a constant and we write

$$\frac{1}{X}\frac{d^2X}{dx^2} = -m_1^2,$$

$$\frac{1}{Y}\frac{d^2Y}{dy^2} = -m_2^2,$$

$$\frac{1}{Z}\frac{d^2Z}{dz^2} = -m_3^2,$$

$$\frac{1}{c^2T}\frac{d^2T}{dt^2} = -n^2,$$

where $n^2 = m_1^2 + m_2^2 + m_3^2$.

We therefore have an infinite number of solutions of the type

$$V = A\cos m_1x \cos m_2y \cos m_3z \cos nct.$$

13.30 Solution in Cylindrical Coordinates

We now obtain a solution by applying the method of separation of variables to the wave equation,

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0.$$

Let $V = R.\Theta.Z.T$ be a solution, where the functions R, Θ, Z and T are functions of r, θ, z and t respectively.

We have

$$\frac{1}{R}\left\{\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right\} + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} - \frac{1}{c^2T}\frac{d^2T}{dt^2} = 0.$$

Let
$$\frac{1}{Z}\frac{d^2Z}{dz^2} = -n^2,$$
$$\frac{1}{\Theta}\frac{d^2\Theta}{d\theta^2} = -m^2,$$
$$\frac{1}{T}\frac{d^2T}{dt^2} = -c^2p^2,$$

then
$$\frac{1}{R}\left\{\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right\} - \frac{m^2}{r^2} = n^2 - p^2 = -a^2 \text{ (say)},$$

that is
$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + R\left(a^2 - \frac{m^2}{r^2}\right) = 0.$$

This last equation is Bessel's equation of order m with argument ar. therefore

$$R = AJ_m(ar) + BY_m(ar).$$

We thus have an infinite number of solutions of the type

$$V = J_m(ar) \cos m\theta \cos nz \cos cpt,$$

where $a^2 = -n^2 + p^2$. Other solutions may take the Bessel functions of the second kind and may take sines instead of cosines.

A solution which has axial symmetry and is independent of θ is found by putting $m = 0$, and we have

$$V = J_0(ar) \cos nz \cos cpt,$$

where $a^2 = -n^2 + p^2$.

If the wave is also independent of z we have

$$V = J_0(pr) \cos cpt.$$

13.31 Vibration of a Membrane

Consider the vibration of a thin membrane stretched tightly over a circular ring of unit radius. We shall suppose the tension to be uniformly equal to T over the area of the circle, so that a small rectangular

element of sides δx and δy (Fig. 352) is subjected to forces $T\delta x$ and $T\delta y$ perpendicular to its edges.

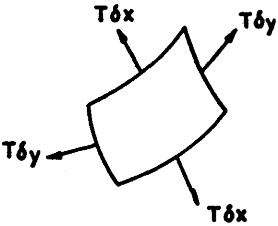

Fig. 352

We have seen (§ 6.7) that if y is the displacement of an element δx of a taut string, the restoring force is $(T\delta x)\dfrac{d^2y}{dx^2}$, where T is the tension in the string.

Here we shall take z to be the displacement of an element of the membrane. Then the restoring force due to the tension $T\delta y$ is $T\delta y\delta x\dfrac{\partial^2 z}{\partial x^2}$, and that due to the tension $T\delta x$ is $T\delta y\delta x\dfrac{\partial^2 z}{\partial y^2}$. If ϱ be the surface density of the membrane the effective force on the element is

$$\varrho\frac{\delta y\delta x}{g}\frac{\partial^2 z}{\partial t^2},$$

and we have
$$T\delta y\delta x\left(\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}\right)=\varrho\frac{\delta y\delta x}{g}\frac{\partial^2 z}{\partial t^2},$$

that is
$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2 z}{\partial t^2},$$

where $c^2 = \dfrac{Tg}{\varrho}$.

Thus the equation of motion to be solved is the two-dimensional form of the wave equation, with the boundary condition $z = 0$ on the bounding circle.

To obtain a solution with symmetry about the centre of the circle we find the solution in cylindrical coordinates which is independent of θ (§ 13.30) and we write

$$z = J_0(pr)\cos cpt.$$

Since $z = 0$ when $r = 1$, we must have $J_0(p) = 0$. The least root of the equation $J_0(p) = 0$ is $p = 2{\cdot}405$, and hence the fundamental mode of oscillation is

$$z = J_0(2{\cdot}405r).$$

This represents the maximum displacement corresponding to any value of r. We can obtain other modes of vibration by considering the further zeros of $J_0(p)$, the next in order of magnitude being $p = 5{\cdot}52$, and the corresponding mode

$$z = J_0(5{\cdot}52r).$$

The frequency of the fundamental mode is

$$\frac{cp}{2\pi}=\frac{2{\cdot}405c}{2\pi},$$

and higher modes have correspondingly greater frequencies.

13.32 Solution in Spherical Polar Coordinates

Let $V = R.\Theta.\Phi.T$ be a solution of the wave equation

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial V}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2} - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = 0,$$

where R, Θ, Φ and T are functions of r, θ, ϕ and t respectively.

We have

$$\frac{1}{R}\left\{\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right\} + \frac{1}{r^2\sin\theta.\Theta}\frac{d}{d\theta}\sin\theta\frac{d\Theta}{d\theta} + \frac{1}{r^2\sin^2\theta.\Phi}\frac{d^2\Phi}{d\phi^2} - \frac{1}{c^2T}\frac{d^2T}{dt^2} = 0.$$

Let

$$\frac{1}{c^2T}\frac{d^2T}{dt^2} = -p^2,$$

$$\frac{1}{\Phi}\frac{d^2\Phi}{d\phi^2} = -m^2,$$

$$\frac{1}{\sin\theta.\Theta}\frac{d}{d\theta}\sin\theta\frac{d\Theta}{d\theta} - \frac{m^2}{\sin^2\theta} = -n(n+1),$$

then

$$\frac{1}{R}\left\{\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right\} = \frac{n(n+1)}{r^2} - p^2.$$

We have, therefore, $T = \cos cpt$ or $\sin cpt$,

$\Phi = \cos m\phi$ or $\sin m\phi$.

The equation for Θ is

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d\Theta}{d\theta} + \left\{n(n+1) - \frac{m^2}{\sin^2\theta}\right\}\Theta = 0.$$

Writing $x = \cos\theta$, we have

$$\frac{1}{\sin\theta}\frac{d}{d\theta} = -\frac{d}{dx},$$

$$\frac{d}{dx}(1-x^2)\frac{d\Theta}{dx} + \left\{n(n+1) + \frac{m^2}{x^2-1}\right\}\Theta = 0,$$

$$(x^2-1)\frac{d^2\Theta}{dx^2} + 2x\frac{d\Theta}{dx} - \left\{n(n+1) + \frac{m^2}{x^2-1}\right\}\Theta = 0.$$

This is the Associated Legendre equation, § 13.14, of which the solution is

$$\Theta = AP_n^m(x) + BQ_n^m(x),$$
$$= AP_n^m(\cos\theta) + BQ_n^m(\cos\theta).$$

The equation for R is

$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left\{p^2 - \frac{n(n+1)}{r^2}\right\}R = 0.$$

The substitution $R = r^{1/2}y(r)$, reduces this equation to Bessel's equation.

We have
$$\frac{dR}{dr} = r^{-(1/2)}\frac{dy}{dr} - \frac{1}{2}r^{-(3/2)}y,$$
$$\frac{d^2R}{dr^2} = r^{-(12)}\frac{d^2y}{dr^2} - r^{-(3/2)}\frac{dy}{dr} + \frac{3}{4}r^{-(5/2)}y,$$
$$\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + \left\{p^2 - \frac{n(n+1)}{r^2}\right\}R$$
$$= r^{-(1/2)}\left[\frac{d^2y}{dr^2} + \frac{1}{r}\frac{dy}{dr} + \left\{p^2 - \frac{(n+\frac{1}{2})^2}{r^2}\right\}y\right] = 0,$$
$$y = AJ_{n+1/2}(pr) + BY_{n+1/2}(pr).$$

Hence
$$R = r^{-(1/2)}\{AJ_{n+1/2}(pr) + BY_{n+1/2}(pr)\}.$$

Hence we have solutions of the wave equation of the form
$$V = r^{-(1/2)}J_{n+1/2}(pr)\,.\,P_n{}^m(\cos\theta)\,.\cos m\phi\,.\cos cpt.$$

A solution which has spherical symmetry and is independent of θ and ϕ is found by putting $m = n = 0$, and we have
$$V = r^{-(1/2)}J_{1/2}(pr)\cos cpt,$$
$$= \frac{1}{r}\sin pr\cos cpt.$$

A solution which has axial symmetry and is independent of ϕ is found by putting $m = 0$ and we have
$$V = r^{-(1/2)}J_{n+1/2}(pr)P_n(\cos\theta)\cos cpt.$$

13.33 Laplace's Equation

Laplace's equation in Cartesian coordinates is
$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

This equation has many applications which include a fluid in motion, gravitational or electrostatic potential, thermal equilibrium in solids and stresses in solids.

Its solutions may be considered as solutions of the wave equation independent of the time and may be obtained by the separation of the variables. Thus we have solutions obtained by separating x, y and z of the type
$$V = \cos lx\,.\cos my\,.\cos nz,$$
where $l^2 + m^2 + n^2 = 0$. This implies that one or two of the quantities l, m, n must be imaginary and the corresponding cosine replaced by a hyperbolic function.

In cylindrical coordinates the equation is

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = 0,$$

and we have the solutions obtained in § 13.30 with $p = 0$,

$$V = J_m(nr) \cos m\theta \cosh nz.$$

In spherical polar coordinates Laplace's equation is

$$\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta} \sin\theta\frac{\partial V}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0.$$

Writing $V = R.\Theta.\Phi$ and separating the variables as in § 13.32, we have

$$\frac{1}{R}\left\{\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right\} = \frac{n(n+1)}{r^2},$$

that is
$$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} - n(n+1)R = 0.$$

It is easily verified that a solution of this equation is $R = r^n$ or $R = r^{-n-1}$. The other equations are as in § 13.32 and we have solutions of the type

$$V = r^n P_n{}^m(\cos\theta) \cos m\phi,$$

or
$$V = r^{-n-1} P_n{}^m(\cos\theta) \cos m\phi.$$

13.34 The Equation of Telegraphy

The equation of telegraphy is

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{c^2}\left\{\frac{\partial^2 V}{\partial t^2} + k\frac{\partial V}{\partial t}\right\},$$

and represents the flow of a wave along the x-axis with damping proportional to the velocity of vibration.

Writing $V = X.T$, and separating the variables we have

$$\frac{d^2X}{dx^2} = -m^2X,$$

$$\frac{d^2T}{dt^2} + k\frac{dT}{dt} = -m^2c^2T.$$

The second equation is that of damped simple harmonic motion and has the solution

$$T = e^{-(1/2)kt} \cos(pt + \varepsilon),$$

where $p^2 = m^2c^2 - \frac{1}{4}k^2$. Thus we have solutions of the type

$$V = e^{-(1/2)kt} \cos mx \cos pt.$$

13.35 The Heat Flow Equation

The heat flow equation applies to the conduction of heat in bodies. Let v be the temperature at any point of a body. Then v will, in general, be a function of the coordinates $x, y, z,$ of the point and of the time t. At any instant there will be surfaces in the body along which the temperature is constant and these isothermal surfaces have equations $v(x, y, z) =$ constant.

The direction of flow of heat at any point is normal to the isothermal surface through the point in the direction in which v decreases. The direction ratios of the normal are $\dfrac{\partial v}{\partial x} : \dfrac{\partial v}{\partial y} : \dfrac{\partial v}{\partial z}$, and if K be the thermal conductivity of the substance the flow per unit area across the surface has components f_x, f_y, f_z parallel to the axes of coordinates, where

$$f_x = -K\frac{\partial v}{\partial x}, f_y = -K\frac{\partial v}{\partial y}, f_z = -K\frac{\partial v}{\partial z}.$$

The flow is therefore a vector with these components, and is called the *current vector*. The flow across a small area about a point is therefore the product of the area and the current vector at the point.

Consider an element of volume which is a rectangular parallelepiped with edges $2\delta x$, $2\delta y$, $2\delta z$, parallel to the axes of coordinates, and with centre at the point $P(x, y, z)$ (Fig. 353). The flow into the element parallel to the x-axis is

$$4\delta y\delta z\left(f_x - \frac{\partial f_x}{\partial x}\delta x\right),$$

and the flow out of the element through the opposite face is

$$4\delta y\delta z\left(f_x + \frac{\partial f_x}{\partial x}\delta x\right).$$

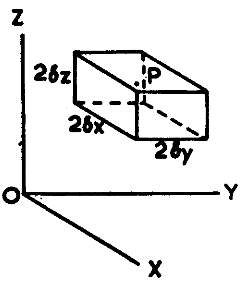

Fig. 353

Thus the increase of heat in the element due to the flow through these two faces is

$$-8\delta x\delta y\delta z\frac{\partial f_x}{\partial x}.$$

By considering the flow parallel to the axes of y and z we find increases of heat

$$-8\delta x\delta y\delta z\frac{\partial f_y}{\partial y},$$

and

$$-8\delta x\delta y\delta z\frac{\partial f_z}{\partial z}, \text{ respectively.}$$

Hence, the total gain of heat by the element is

$$-8\delta x\delta y\delta z\left(\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}\right).$$

If ϱ be the density and c the specific heat of the substance, the rate at which the element is gaining heat is

$$(8\delta x\delta y\delta z\varrho)c\frac{\partial v}{\partial t}.$$

Therefore
$$\varrho c\frac{\partial v}{\partial t}+\frac{\partial f_x}{\partial x}+\frac{\partial f_y}{\partial y}+\frac{\partial f_z}{\partial z}=0,$$

that is
$$K\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)=\varrho c\frac{\partial v}{\partial t},$$

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=\frac{1}{h^2}\frac{\partial v}{\partial t}, \tag{1}$$

where $h^2 = K/(\varrho c)$ is called the *thermometric conductivity* of the substance.

If the temperature does not vary with the time we have

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}=0, \tag{2}$$

so that v satisfies Laplace's equation.

Solutions of the equations (1) and (2) can be found by separation of the variables as in § 13.29. Thus, if the flow of heat is parallel to the x-axis throughout the body, we have

$$\frac{\partial^2 v}{\partial x^2}=\frac{1}{h^2}\frac{\partial v}{\partial t}.$$

Let $v = X.T$, where X depends on x only, and T on t only.

We have
$$\frac{1}{X}\frac{d^2X}{dx^2}=\frac{1}{h^2T}\frac{dT}{dt}=\text{constant},$$
$$=-n^2 \text{ (say)}.$$

We then have solutions
$$X = A\cos(nx+\varepsilon),$$
$$T = Be^{-n^2h^2t},$$
$$v = Ce^{-n^2h^2t}\cos(nx+\varepsilon).$$

The sum of any number of such solutions with different values of n 's, of course, itself a solution of the equation.

The expression $\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial x^2}$ may be expressed in cylindrical and spherical polar coordinates as in § 13.27 and § 13.28.

Thus in cylindrical coordinates (1) becomes

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}+\frac{\partial^2 v}{\partial z^2}=\frac{1}{h^2}\frac{\partial v}{\partial t}. \tag{3}$$

In spherical polar coordinates (1) becomes

$$\frac{\partial^2 v}{\partial r^2}+\frac{2}{r}\frac{\partial v}{\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\sin\theta\frac{\partial v}{\partial \theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v}{\partial \phi^2}=\frac{1}{h^2}\frac{\partial v}{\partial t}. \tag{4}$$

13.36 Boundary Conditions

Problems of conductivity of heat involve finding solutions of the heat flow equation which satisfy certain boundary conditions.

A bounding surface of a body may be maintained at zero temperature, in which case the solution found must be zero on this surface for all values of t.

If a bounding surface is maintained at a constant temperature other than zero it is often necessary to express this constant temperature as a Fourier series and to compare the Fourier series with the value of the solution obtained on the boundary.

A bounding surface may also be insulated so that there is no flow of heat across it. In this case it is necessary that the current vector derived from the solution obtained for v shall be zero on the surface.

Example 7. *A large sheet of metal of thickness a and thermometric conductivity h^2 has initially a uniform temperature of T degrees centigrade. If its bounding surfaces are kept at a constant temperature of zero degrees centigrade, find the temperature distribution at time t.*

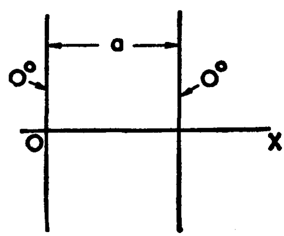

Fig. 354

Taking the x-axis normal to the plate (Fig. 354) we have for the flow of heat,

$$\frac{\partial^2 v}{\partial x^2}=\frac{1}{h^2}\frac{\partial v}{\partial t}$$

and we have found solutions of this equation of the type

$$v = Ce^{-n^2h^2t}\sin(nx+\varepsilon).$$

Since $v=0$ when $x=0$, we have $\varepsilon=0$, and since $v=0$ when $x=a$ we have $na=k\pi$, where k is an integer.

The flow across the surfaces is

$$-K\frac{\partial v}{\partial x} = -nKCe^{-n^2h^2t}\cos\frac{k\pi x}{a}.$$

This must have the same value with opposite sign when $x=0$ and when $x=a$, therefore k must be an odd integer, and hence $n=(2r+1)\pi/a$, where r is an integer.

Therefore $\quad v = Ce^{-(2r+1)^2\pi^2h^2t/a^2}\sin\{(2r+1)\pi x/a\}.$

This cannot give a uniform temperature T when $t = 0$ unless we take a combination of solutions of this type and write

$$v = \sum_{r=0}^{\infty} C_r e^{-(2r+1)^2\pi^2h^2t/a^2} \sin\{(2r+1)\pi x/a\},$$

so that when $t = 0$ we have

$$(v)_{t=0} = \sum_{r=0}^{\infty} C_r \sin\{(2r+1)\pi x/a\}.$$

We can express the constant T as a Fourier series of sines of odd multiples of θ.

Thus if $$T = \sum_{r=0}^{\infty} b_r \sin(2r+1)\theta,$$

$$b_r = \frac{4}{\pi}\int_0^{\frac{1}{2}\pi} T\sin(2r+1)\theta d\theta$$

$$= \frac{4T}{\pi}\cdot\frac{1}{(2r+1)}.$$

Thus $$T = \frac{4T}{\pi}\sum_{r=0}^{\infty}\frac{\sin(2r+1)\theta}{(2r+1)}, \text{ for } 0 < \theta < \pi.$$

$$= \frac{4T}{\pi}\sum_{r=0}^{\infty}\frac{\sin\{(2r+1)\pi x/a\}}{(2r+1)}, \text{ for } 0 < x < a.$$

Therefore the solution obtained for v satisfies the condition

$$(v)_{t=0} = T,$$

$$\text{if } C_r = \frac{4T}{\pi}\frac{1}{(2r+1)}.$$

Hence, we have the solution

$$v = \frac{4T}{\pi}\sum_{r=0}^{\infty} e^{-(2r+1)\pi^2h^2t/a^2}\frac{\sin\{(2r+1)\pi x/a\}}{(2r+1)}.$$

Example 8. *A square thin metal plate of side a has two adjacent sides kept at a temperature of zero degrees and one side kept at a temperature of T degrees, the remaining side being insulated. Find the temperature distribution over the plate in the steady state.*

Let the edges of the plate be the lines $x = 0$, $x = a$, $y = 0$, $y = a$ (Fig. 355), and let the edge $y = 0$ be insulated and $x = a$ be at a temperature T. The equation to be solved is, since there is no flow of heat,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

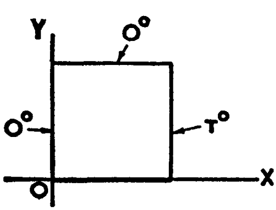

Fig. 355

Solutions of the type

$$v = C\sinh nx\cos ny$$

will satisfy the equation and the boundary condition at $x = 0$. They will also satisfy the boundary condition for $y = 0$ since $\frac{\partial v}{\partial y}$ must be zero on this edge.

To satisfy the boundary condition on the edge $y = a$ we must have $\cos na = 0$,

that is $$na = (2r + 1)\frac{1}{2}\pi, \text{ where } r \text{ is an integer.}$$

To satisfy the remaining boundary condition we write

$$v = \sum_{r=0}^{\infty} C_r \sinh\{(2r + 1)\pi x/(2a)\} \cos\{(2r + 1)\pi y/(2a)\},$$

and we have

$$(v)_{x=a} = \sum_{r=0}^{\infty} C_r \sinh\{(2r + 1)\pi/2\} \cos\{(2r + 1)\pi y/(2a)\}.$$

We can express T as a Fourier series,

$$T = \sum_{r=0}^{\infty} a_r \cos(2r + 1)\theta,$$

where $$a_r = \frac{4T}{\pi}\int_0^{\frac{1}{2}\pi} \cos(2r + 1)\theta d\theta$$
$$= \frac{4T}{\pi}\frac{(-1)^r}{(2r + 1)}.$$

Thus $$T = \frac{4T}{\pi}\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r + 1)}\cos(2r + 1)\theta, \quad 0 < \theta < \frac{1}{2}\pi$$
$$= \frac{4T}{\pi}\sum_{r=0}^{\infty}\frac{(-1)^r}{(2r + 1)}\cos\{(2r + 1)\pi y/(2a)\}, \quad 0 < y < a.$$

Hence $$C_r = \frac{4T}{\pi}\frac{(-1)^r}{(2r + 1)\sinh(r + \frac{1}{2})\pi},$$
$$v = \frac{4T}{\pi}\sum_{r=0}^{\infty}\frac{(-1)^r \sinh\{(2r + 1)\pi x/(2a)\}\cos\{(2r + 1)\pi y/(2a)\}}{(2r + 1)\sinh(r + \frac{1}{2})\pi}.$$

Example 9. *A thin semicircular plate of radius a has its bounding diameter kept at temperature zero and its circumference at temperature T. Find the temperature distribution in the steady state.*

Taking the bounding diameter as the x-axis with the origin at the centre (Fig. 356), the equation to be satisfied is

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0,$$

or, in polar coordinates (r, θ),

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2} = 0.$$

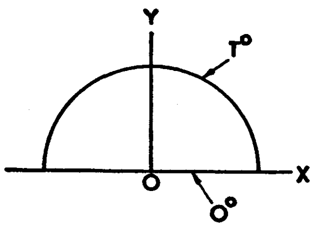

Fig. 356

Separating the variables by writing $v = R.\Theta$, we have

$$\frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} = 0.$$

Since $\dfrac{1}{\theta}\dfrac{d^2\Theta}{d\theta^2} = \text{constant} = -n^2$ (say),

we have $$\Theta = \sin(n\theta + \varepsilon),$$

and $$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{n^2}{r^2}R = 0$$

This last equation is satisfied by r^m, where $m = \pm n$, and hence we have solutions

$$v = (Ar^n + Br^{-n}) \sin (n\theta + \varepsilon).$$

We may take $B = 0$, since the solution must be finite at the origin, and if $\varepsilon = 0$ the boundary condition is satisfied along the diameter if n is an integer. Also, for symmetry in the two quadrants n must be an odd integer. Therefore we have

$$v = \sum_{m=0}^{\infty} C_m r^{2m+1} \sin (2m + 1)\theta.$$

On the circumference we have

$$(v)_a = T = \sum_{m=0}^{\infty} C_m a^{2m+1} \sin (2m + 1)\theta.$$

Using the Fourier series for T obtained in Example 7, we have

$$T = \sum_{m=0}^{\infty} \frac{4T}{\pi} \frac{\sin (2m + 1)\theta}{(2m + 1)}, \text{ for } 0 < \theta < \pi,$$

and hence $$v = \frac{4T}{\pi} \sum_{m=0}^{\infty} \frac{(r/a)^{2m+1}}{(2m + 1)} \sin (2m + 1)\theta.$$

EXERCISES 13 (*d*)

1. Find a solution of the equation $\dfrac{\partial^2 V}{\partial x^2} = \dfrac{1}{c^2} \dfrac{\partial^2 V}{\partial t^2}$ which is zero when $x = 0$ and when $t = 0$ and is never greater than unity.

2. Find a solution of the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0,$$

which is zero when x is infinite and when y is zero.

3. Assuming that

$$u = \frac{1}{r} F(r) \cos (\omega t + \alpha)$$

is a solution of the partial differential equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

where ω, α and c are constants, and $F(r)$, is a function of r only, obtain the ordinary differential equation satisfied by $F(r)$ and give the general solution for $F(r)$.

Given that, for all values of t, (i) u is finite at $r = 0$, (ii) $\dfrac{\partial u}{\partial r} = 0$ at $r = a$ and that u is not identically zero, prove that $(\omega a)/c = \beta$ must satisfy the equation $\beta = \tan \beta$. (L.U., Pt. II)

4. Show that $u = Ae^{mx} \cos (\omega t + mx) + Be^{-mx} \cos (\omega t - mx)$ is a solution of $\dfrac{\partial^2 u}{\partial x^2} = 2\dfrac{\partial u}{\partial t}$, where A, B, m and ω are constants, provided that $m^2 = \omega$.

Find values of the constants, given the conditions (i) $m > 0$, (ii) u remains finite as $x \to \infty$, (iii) $u = \cos t$ when $x = 0$.

(L.U., Pt. II)

5. Show that $V = J_0(nr) \cos nz$ is a solution of Laplace's equation in cylindrical coordinates which is independent of θ and finite at the origin.

6. Show that $r^n P_n (\cos \theta)$ is a solution of Laplace's equation in terms of spherical polar coordinates (r, θ, ϕ), which is independent of ϕ.

7. Find a solution of the two-dimensional wave equation $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} - \dfrac{1}{c^2}\dfrac{\partial^2 V}{\partial t^2} = 0$, such that $V = 0$ along the lines $x = 0$, $y = 0$, $x = a$, $y = a$.

8. Show that a solution of the wave equation in two dimensions which is symmetrical about the origin and finite there and is zero on a circle of radius a is

$$J_0\left(\frac{\alpha r}{a}\right) \cos \frac{c\alpha t}{a},$$

where α is a zero of $J_0(x)$.

9. Show that a solution of the wave equation which has spherical symmetry about the origin and is finite there is of the form

$$Ar^{-1} \sin pr \cos (cpt + \varepsilon).$$

10. A square metal plate of side a has edges represented by the lines $x = a$ and $y = a$ insulated. The edge $x = 0$ is kept at a temperature of zero degrees and the edge $y = 0$ at a temperature of T degrees. Show that the temperature distribution in the steady state is given by

$$v = \sum_{r=0}^{\infty} C_r \cosh \{(2r + 1)\pi(a - y)/(2a)\} \sin \{(2r + 1)\pi x/(2a)\},$$

and that $C_r = \dfrac{4T \operatorname{sech} (2r+1) \pi/2}{\pi(2r + 1)}$.

11. A thin metal strip is bounded by the lines $x = 0$, $y = 0$ and $y = a$ and is of infinite length. If the temperature is T degrees along the edge $x = 0$ and is zero along the other edges and at infinity, show that the temperature distribution in the steady state is given by

$$v = \sum_{r=0}^{\infty} C_r e^{-(2r+1)\pi x/a} \sin \{(2r + 1)\pi y/a\},$$

where $$C_r = \frac{4T}{\pi(2r + 1)}.$$

12. A large sheet of metal of thickness a has initially a uniform temperature of T degrees. The lower face is kept at a temperature of zero degrees and the other face is insulated. Show that the temperature after time t at distance x from the lower face is given by

$$v = \sum_{r=0}^{\infty} C_r e^{-h^2(2r+1)^2\pi^2 t/(4a^2)} \sin \{(2r + 1)\pi x/(2a)\},$$

where $C_r = \dfrac{4T}{\pi(2r + 1)}$, and h^2 is the thermometric conductivity of the metal.

13. Show that if $f(x) = 0$ for $0 < x < \frac{1}{2}a$, and $f(x) = T$ for $\frac{1}{2}a < x < a$,

$$f(x) = \frac{1}{2}T - \frac{2T}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r}{(2r + 1)} \cos (2r + 1)\pi x/a,$$

for $0 < x < a$.

Show that

$$v = \frac{1}{2}T - \frac{2T}{\pi} \sum_{r=0}^{\infty} \frac{(-1)^r \cosh \{(2r + 1)\pi(a - y)/a\} \cos \{(2r + 1)\pi x/a\}}{(2r + 1) \cosh (2r + 1)\pi},$$

gives the steady state temperature distribution on a square plate with sides $x = 0$, $y = 0$, $x = a$, $y = a$, insulated along three sides and with the side $y = 0$ kept at temperature zero for $0 < x < \frac{1}{2}a$, and at temperature T for $\frac{1}{2}a < x < a$.

ANSWERS TO EXERCISES

EXERCISES 1 (*a*)

1. 8529, ft.lb./sec.
2. (*a*) 1 ft./sec., 39·1 ft. (*b*) 2·5 ft./sec., 97·7 ft.
4. 0·91.
5. (*a*) 11·39 h.p., (*b*) 52·1 lb.wt.
6. 625 lb., 12 ft./sec.
7. 67·5.
8. 1·226 ft./sec.^{2}.
9. 40·1 per cent.
10. 0·56 ft./sec., 173·6 ft. tons, 59 tons.
11. 1·10, 1·34, 1·46 ft./sec., 1 ft.ton.
12. 1195 ft./sec.
13. 22 h.p., 193 lb.wt.
14. 1172 lb.wt., 1/4, 85 h.p.

EXERCISES 1 (*b*)

1. 0·65 miles.
2. 80 ft./sec., 1440 ft., 85·3 ft./sec.
4. $(u^2 - ga + ga^3/x^2)^{1/2}$.
5. 1416 ft./sec.
6. $\dfrac{1}{k}\log\{U + (U^2 - V^2)^{1/2}\}/V$, $\dfrac{1}{2}mV^2 \sinh 2kt$.
7. $\{2ghR/(R + h)\}^{1/2}$.
8. 84 miles, 340 sec.

EXERCISES 1 (*c*)

5. 10 ft./sec., 0·71 sec.
7. 1·36 miles.
9. 520 ft.
11. $\dfrac{1}{k}\log(1 + kv_0/g)$.
12. $x = v\cos\alpha(1 - e^{-kt})/k$, $y = (kv\sin\alpha + g)(1 - e^{-kt})/k^2 - gt/k$.
15. $k^3UV/(\mu - k^2V^2)$, $\{\mu k^2/(\mu - k^2V^2)\}^{1/2}$.

EXERCISES 1 (*d*)

1. 13·5 ft./sec.
2. 7 lb.wt.
3. $gT\{1 - (2P/Q)\log(1 + Q/2P)\}$.
4. $W(1 + v^2/gl)$.
6. 567 ft./sec., 1843 ft.
7. 427 ft./sec., 1541 ft.
8. $v = v_0(1 - c^2)/(1 + c^2)$, $c = (1 - rt/m_0)^{u/v_0}$.

EXERCISES 1 (*e*)

1. 2·67 ft., 4·44 ft.
2. 36° 52′.
3. $120\sqrt{6}/\pi$.
4. 88·2 r.p.m.
5. 160·7 r.p.m., 2·36 in.
6. 625 ft.lb., 38 h.p.
7. 440 r.p.m.

EXERCISES 1 (*f*)

1. $\frac{1}{2}v_0(v_0^2 + 4gc)^{1/2}/g$.
3. $V\sec\psi/(\sec\psi + \tan\psi)^k$.
7. $\sqrt{7a\omega}$.
8. $a(1 + \omega^2/\lambda^2)$, $a\omega^2/\lambda^2$, $2\pi/\lambda$.
11. 687 days, 7·482, 7·449 × 10^9 ft.

EXERCISES 2 (*a*)

1. $\pi/8$ sec., 8 ft./sec.
6. $\lambda^2 = 1 + 2k,\ C = \frac{1}{2}b$.

EXERCISES 2 (*b*)

1. 7, 5·50 sec.
3. $2m/s,\ \pi A^2 k(ag)^{-1/2}$ per cycle.
4. $\frac{3}{4}(2)^{2/3}$.
5. $3nx = (4an - u)e^{-nt} + (u - an)e^{-4nt}$.
8. $\frac{1}{5}(8 \sin 4t - 4 \cos 4t)$.
9. $a(k^2p^4 + \lambda^2)^{1/2}\{m^2p^4 + (k^2 - 2\lambda m)p^2 + \lambda^2\}^{-(1/2)}$.

EXERCISES 2 (*c*)

1. $CR^2 = 4L,\ EC(1 + 2t/CR)e^{-2t/CR},\ -(Et/L)\,e^{-2t/CR}$.
5. No term $t \cos t/(LC)^{1/2}$.
6. $q = EC\pi\{(\lambda + \mu\beta)e^{\alpha t_1} - (\lambda + \mu\alpha)e^{\beta t_1}\}/(\alpha - \beta)(\lambda^2 + \mu^2)$,
   $\lambda = (1 - LC\pi^2),\ \mu = RC\pi,\ t_1 = t - 2T,\ \alpha\beta = 1/LC,\ \alpha + \beta = -R/L$.
7. $Ae^{-\alpha t} \cosh (\beta t + \varepsilon)$ or $Ae^{-\alpha t} \cos (\beta t + \varepsilon) + (E/R) \cos \phi \sin (pt + \phi)$,
   $\alpha = -R/2L,\ \beta = \pm \{(R/2L)^2 - (1/CL)\}^{1/2},\ \tan \phi = (1 - LCp^2)/RCp$.
8. 0·0001, 3·2.
9. $(E/R) \cos \phi \sin (\omega t + \phi),\ \tan \phi = (1 - LC\omega^2)/RC\omega$.
10. $CR^2 = 4L$.
11. $E\{1 - e^{-\omega t}(\cos 2\omega t + \frac{1}{2} \sin 2\omega t)\},\ \frac{5}{2}CE\omega e^{-\omega t} \sin 2\omega t$.

EXERCISES 2 (*d*)

1. $i_1 = (E/R)(1 - e^{-Rt/L}),\ i_2 = (E/R)e^{-t/CR}$.
2. $$\frac{E\{4R^2(L^2C^2\omega^4 + 3R^2C^2\omega^2 + 1 - LC\omega^2)^2 + (R^2C - L)^2(1 - LC\omega^2)^2\omega^2\}^{1/2}}{(3R^2C\omega + L\omega)^2 + 4R^2(1 - LC\omega^2)^2},$$
   $\tan^{-1}\{(R^2C - L)(1 - LC\omega^2)\omega/2R(L^2C^2\omega^4 + 3R^2C^2\omega^2 + 1 - LC\omega^2)\}$.
3. $E(R^2 + L^2\omega^2)^{1/2}(9R^2 + 4L^2\omega^2)^{-(1/2)} \cos (\omega t + \beta)$,
   $\cot \beta = (3R^2 + 2L^2\omega^2)/RL\omega$.

EXERCISES 3 (*a*)

1. $1{\cdot}625M,\ 1{\cdot}531M$.
2. 1·22, 1·99 ft.
3. $3{\cdot}1M$.
4. $3{\cdot}33a^4$.
5. $93Ma^2/70,\ M(93a^2 + 32h^2)/140$.
6. $0{\cdot}46Ma^2,\ 1{\cdot}21Ma^2$.
7. $62ma^2/3$.
8. $3{\cdot}5Ma^2,\ 2{\cdot}25Ma^2$.
9. $0{\cdot}475Ma^2,\ 0{\cdot}86Ma^2$.
10. 3·90, 2·74 in.
11. 2·2 in.
12. 13·31 in.
13. 3·22, 4·37, 4·18 in.
14. 6 in., 5440 in.^{4}.
15. $\frac{1}{6}sa^4 \sin \theta \cos^3 \theta,\ \frac{1}{2}sa^4 \sin^3 \theta \cos \theta,\ \sqrt{3}sa^4/32,\ 3\sqrt{3}sa^4/32$.

EXERCISES 3 (*b*)

1. $Ma^2/(2\pi),\ (9\pi - 32)Ma^2/(18\pi^2)$.
2. $b^2(4a^2 + \pi ab + 2b^2)/3$.
3. 240π in.^{4}.
5. 105·4, 23·4, −27·7 in.^{4}.
6. 68·8, 13·9, −18·6 in.^{4}.

EXERCISES 3 (c)

1. (i) $ab(a^2 + \frac{1}{3}b^2)$, (ii) $8a^3b^3/3(a^2 + b^2)$.
2. $\pi a^4/16 \pm a^4/8$.
3. $22\frac{1}{2}°$, $(5/3 \pm \sqrt{2})a^4$.
4. 119·6 in.^{4}.
5. 17° 1′, (64·4 ± 49·5) in.^{4}.
6. 32·4 in.^{4}.

EXERCISES 3 (d)

1. 3·40, 2·14 in.
2. 2·12 in.
4. 29° 52′, 119° 52′, (4·17 ± 2·32) in.^{4}.
5. 6014·25 ft.
7. 2240, 2224, 20 lb.ft.^{2}.
8. 3120, 3088, 64 lb.ft.^{2}.

EXERCISES 4 (a)

1. 16 lb.ft.^{2}.
2. $3g/(4a)$.
3. 68·4 r.p.m.
4. $\{g/2a)\}^{1/2}$.
6. 23·2 rad./sec.
7. 4·375 lb.ft.^{2}, 35 lb.wt.
8. 673 r.p.m.
9. 37,011 ft.lb., 59 ft.lb., 57 sec.
11. 6·26 sec.
12. 5350 lb.

EXERCISES 4 (b)

1. 7·63 ft./sec., 326 pdl.
2. $3\pi^2 \times 10^4$ ft.pdl., 150π ft.pdl., 2 min.
3. $\{ag(\pi - 2)/3\}^{1/2}$.
4. $\frac{1}{3}g$.
5. 3·04 rad./sec.
6. $4g/71$, 0·02 ft.cwt.
7. $(P/K)^{1/2}$
8. $\dfrac{a(G-P)I_2 + bQI_1}{a^2I_2 + b^2I_1}, \dfrac{(G-P)b^2 - Qab}{a^2I_2 + b^2I_1}, \dfrac{\pi N^2 a}{b} \cdot \dfrac{a^2I_2 + b^2I_1}{(G-P)b^2 - Qab}$.
9. 19·6 ft.lb., 0·89.
10. $Wr^2\{gT^2/(4\pi a) - 1\}$.
11. $3g > 4ak^2$, $\theta = ae^{-kt}(\cos pt + \dfrac{k}{p}\sin pt)$, $p^2 + k^2 = 3g/4a$.

EXERCISES 4 (c)

2. 3200 ft.lb.
3. 200 r.p.m.
4. $\cos^{-1}(11/12)$.
5. $1/\sqrt{3}$ lb.sec.
6. 762 r.p.m., 98,000 ft.lb.
7. 1733 ft.sec.
8. 41·175 ft.lb., 6·02 lb.sec.
9. 125 r.p.m. in B's direction, 2·36 sec.
10. 1·83 r.p.s
11. 17/9.
12. $(I_1r_2^2 + I_2r_1^2)\omega_1/\{I_1r_2^2 + (I + I_2)r_1^2\}$.
13. $R_1\omega_1I_1I_2(1 + \mu^2)^{1/2}/\{\mu P(I_1R_2^2 + I_2R_1^2)\}$.

EXERCISES 4 (d)

1. 2·2 sec.
2. 1° 45′.
3. 5·62 rad./sec., 2·235 sec.
4. $1{\cdot}78a$, $2{\cdot}33a^{1/2}$.
5. $2{\cdot}49r^{1/2}$.
6. $0{\cdot}42a$, $1{\cdot}08a$ from m.
7. $2\pi(2l/3g)^{1/2}$, $2\pi(l/g)^{1/2}$.
8. 29·5.
10. $2\pi\{(b^2 + 3a^2)/2ag\}^{1/2}$.
11. Circle of radius 14·4 cm.
12. $2\pi\{(a^2 + 2ab + 4b^2)/3gb\}^{1/2}$.
13. $(8g/5a)^{1/2}$.
14. $\frac{1}{4}mg$.

EXERCISES 4 (*e*)

1. 3·18 in.
2. $4\pi^2 I, \dfrac{I}{2g}\left(\dfrac{2a\pi^2}{180}\right)^2$.
3. $\dfrac{1}{2\pi}\left\{\dfrac{GJ}{a}\left(\dfrac{1}{I_1}+\dfrac{1}{I_2}\right)\right\}^{1/2}$.
4. $2\pi\{I_1 I_2/gk(I_1+I_2)\}^{1/2}$.
6. $2\pi(l/2g)^{1/2}$.
7. $2\pi(l/3g)^{1/2}$.
9. 1·2 in.
10. 6700 lb.ft.^{2}.

EXERCISES 5 (*a*)

1. 0·165, 1·23 sec.
2. cylinder, 0·03 sec.
3. $\frac{2}{3}g, \frac{1}{3}Mg$.
4. 77·5 sec., 0·1 ft./sec.
5. $2a^2 g\sin\alpha/(2a^2+k_1^2+k_2^2)$; $\frac{1}{2}(k_2^2-k_1^2)mg\sin\alpha/(2a^2+k_1^2+k_2^2)$.
7. 2·96 sec.
8. $4P/3M$.
9. $2T\cos\alpha/3M$.
11. $\frac{3}{4}mg(3\cos\theta-2)\sin\theta, \frac{1}{4}mg(1-3\cos\theta)^2$, 0·35.
13. $\frac{1}{4}mg(12\sin\theta-1-9\sin^2\theta)$; $\sin^{-1}\frac{1}{3}$.
14. $\cos^{-1}(10/17), \{10g(a+b)/17\}^{1/2}$.
16. $m(1+k^2/r^2)\alpha$.
17. $4R\mu r^2 g/(Mr^2+2mk^2)$.

EXERCISES 5 (*b*)

1. $\{3g(\cos\alpha-\cos\beta)/2a\}^{1/2}$.
3. $(15g/13a)^{1/2}, 149Mg/104$.
4. $a\omega^2/6g$.
5. $Mg(7-6\cos\alpha)$.
6. 1745 ft. tons, 4·65 per cent.
7. $\cos\theta\{18ga(1-\cos\theta)/(9\sin^2\theta+7)\}^{1/2}$.
8. $3mg-7m(a\Omega^2+g)/25$.

EXERCISES 5 (*c*)

7. $\{2a\omega(\mu\cos\alpha-\sin\alpha)+3u\mu\cos\alpha\}/\{a(5\mu\cos\alpha-2\sin\alpha)\}$.

EXERCISES 5 (*d*)

1. $m(\frac{1}{3}a^2\omega^2\pm au)$; $m(\frac{1}{3}a^2\omega+u^2/\omega)$.
2. $m(a+\frac{3}{2}b)b\omega$.
3. $\frac{1}{3}Mr\dot{\omega}, \frac{3}{8}Mr\dot{\omega}$.
4. $a\dot{\theta}$.
5. 32·2 ft./sec., 797 ft.lb.
6. $(g\cos\alpha/2a)^{1/2}$.

EXERCISES 5 (*e*)

3. $w/\sqrt{3}$.
4. (*a*) $15g/16a, 83w/128$; (*b*) $15g/56a, 403w/448$.
5. 0·2.
6. $(3g\sin\theta/a)^{1/2}$.
7. $mg\sin\alpha.(l-a)(l-3a)/4l$; $mg\sin\alpha\, a(l-a)^2/4l$.
8. $\sqrt{185}mg\sin\alpha/22, mga\sin\alpha/22$.

EXERCISES 5 (*f*)

2. $3\omega/4$.
3. $(M+6m)\omega/(M+3m), M(M+4m)a\omega/(M+3m)$.
4. 1/13.
6. $6u/7l$.
7. $\frac{1}{2}Mma^2u^2/\{Ma^2+m(a^2+3x^2)\}$.

EXERCISES 5 (g)

1. $mMV/2(M+3m)$, $mMV/(M+3m)$.
2. $m(3gl)^{1/2}/7$.
4. $\omega = 3Jb/\{2m(4a^2+b^2)\}$.
5. J/m, $5J/8am$, $37J^2/64m$.
6. $\omega = 3v/8a$, $\dot{x} = 5v/8$, $\dot{y} = 3v/8$.
8. (a) 41·175 ft.lb. (b) 6·02 lb.sec.
9. $5V/6$.
10. $3I/5Ma$.
11. $8v/7$.

EXERCISES 5 (h)

1. $\frac{1}{3}Ml\omega$.
2. $5l/11$, $ml\omega/6$, $ml\omega/3$, $11ml\omega/6$.
4. $4ma\omega/5$.
5. $\frac{1}{3}a$ from side, $M(ag/12)^{1/2}$.

EXERCISES 6 (a)

1. $2\pi(l^2/3ag)^{1/2}$.
2. $2\pi(83a/120g)^{1/2}$.
5. $2\pi(b/3g)^{1/2}$.
8. $17{\cdot}05a$.
10. 6·33 ft./sec., 4·05 sec.

EXERCISES 6 (b)

1. (i) $a+b$, (ii) $ab/(a+b)$.
3. $Wl^3/(192EI)$.
4. $\theta = \{12al\pi^2n^2/(3gl - 8\pi^2n^2l^2)\}\sin 2\pi nt$.
6. $(\tan nl - nl)/n$, $EIn^2 = mg$.
7. $\sec\alpha(\tan nl - nl)/n$.

EXERCISES 6 (c)

2. 49·0, 23·3 per sec.
3. 264 r.p.m.
4. 6240 r.p.m.
6. 2750 r.p.m.

EXERCISES 6 (d)

1. $\frac{1}{3}\pi^2 + 4\Sigma\cos n\pi\left(\frac{1}{n^2}\cos nx - \frac{1}{n}\sin nx\right)$.
2. $-\frac{1}{2}\Sigma\cos n\pi\cos(2n-1)x/(2n-1)$.
3. $\{(2\sinh\pi)/\pi\}\left\{\frac{1}{2} + \Sigma\frac{\cos n\pi}{1+n^2}(\cos nx - n\sin nx)\right\}$.
5. $\frac{1}{3}a + \frac{a\sqrt{3}}{2\pi}\left(\cos\theta - \frac{1}{2}\cos 2\theta + 0\right) + \frac{3a}{2\pi}\left(\sin\theta + \frac{1}{2}\sin 2\theta + 0\right)$.
6. $\frac{1}{\pi} + \frac{1}{2}\sin x - \frac{2}{\pi}\Sigma\frac{\cos 2nx}{4n^2-1}$.

EXERCISES 6 (e)

1. $b_{4m} = 0$, $b_{4m+2} = -\frac{4}{(4m+2)\pi}$, $b_{2m+1} = \frac{2}{(2m+1)\pi}$.
3. $\frac{8}{\pi}\Sigma\frac{\sin(2n-1)x}{(2n-1)^3}$.
5. $2a\Sigma\frac{\cos(2n-1)a\cos(2n-1)x}{(2n-1)^2a^2 - (\pi/2)^2}$.
6. $\frac{ca}{\pi} + \frac{2c}{\pi}\Sigma\sin na\cos nx/n$.
7. $\frac{2}{3}\Sigma\frac{1}{n}\sin\frac{n\pi}{3}\left(1 + 2\cos\frac{n\pi}{3}\right)\cos nx$.

8. $\cos p\pi/4p^2$.

9. $\dfrac{4}{\pi}\Sigma\dfrac{\cos(2n+1)x}{(2n+1)^2}$.

10. $-\dfrac{2}{\pi}+\dfrac{1}{2}\cos x+2\Sigma\dfrac{\cos m\pi}{4m^2-1}\cos 2mx$.

11. $-2\pi\Sigma\dfrac{\cos n\pi}{n}\sin nx-\dfrac{8}{\pi}\Sigma\dfrac{\sin(2n-1)x}{(2n-1)^3}$.

12. $\dfrac{8}{\pi}\Sigma\dfrac{\sin(2n+1)x}{(2n+1)^3}$, $\dfrac{\pi^2}{6}-\Sigma\dfrac{\cos 2nx}{n^2}$.

13. $\dfrac{6\sqrt{3}}{\pi}\left(\sin x-\dfrac{\sin 5x}{5^2}+\dfrac{\sin 7x}{7^2}-\dfrac{\sin 11x}{11^2}+\dfrac{\sin 13x}{13^2}\right)$,
$4\sqrt{6}(1+5^{-2}-7^{-2}-11^{-2}-13^{-2})$.

14. $\dfrac{\pi^2}{12}+\Sigma\dfrac{\cos n\pi}{n^2}\cos nx$; $\dfrac{\pi^2}{6}$.

16. $2{\cdot}27+0{\cdot}83\cos x-0{\cdot}54\cos 2x-0{\cdot}22\cos 3x$
$+1{\cdot}04\sin x+0{\cdot}41\sin 2x+0{\cdot}14\sin 3x$.

17. $4{\cdot}26+3{\cdot}62\cos x+0{\cdot}52\cos 2x+0{\cdot}005\cos 3x-1{\cdot}02\sin x+0{\cdot}01\sin 3x$.

EXERCISES 7 (*a*)

1. $\frac{1}{3}(\mathbf{a}+2\mathbf{b})$.
2. 9, $(i+4j+8k)/9$.
3. $\sqrt{(55)}$, $\cos^{-1}5/\sqrt{(110)}$, $\cos^{-1}6/\sqrt{(110)}$, $\cos^{-1}7/\sqrt{(110)}$.
4. 7, $(3/7, -2/7, -6/7)$.
5. $120°$.
7. $\sqrt{(17)}$; $2/\sqrt{(17)}$, $-2/\sqrt{(17)}$, $3/\sqrt{(17)}$.
8. $\frac{1}{4}(\mathbf{a}+\mathbf{b}+\mathbf{c})$.
9. $\frac{1}{3}(\mathbf{a}+\mathbf{b}+\mathbf{c})$.
10. $\frac{1}{3}h$, $\cos^{-1}\{(a+b\cos\gamma+c\cos\beta)/h\}$,
$h^2=a^2+b^2+c^2+2bc\cos\alpha+2ca\cos\beta+2ab\cos\gamma$.

EXERCISES 7 (*b*)

1. $2\sqrt{2}(t^2+\frac{1}{2})^{1/2}$, $2\sqrt{2}$, $\cos^{-1}\{t/(t^2+\frac{1}{2})^{1/2}\}$.
7. $-(3i+6j+2k)/7$.
8. $7\sqrt{(10)}$; $3/\sqrt{(10)}$, 0, $-1/\sqrt{(10)}$.
9. 40 ft.lb.
13. $22i+11k$, $-5i+15j+10k$, $27i-15j+k$.
15. $25\sqrt{(13/14)}$.
17. $5\sqrt{3}/2$, $(i-j-k)/\sqrt{3}$, $5/3$.
19. $2\pi a^2$, $\int_C r^2 d\theta$ perpendicular to $z=0$.
21. $(-2i+2j+3k)/3$, $-2/3$.

EXERCISES 8 (*a*)

2. $38g/(73a)$, $10g/(73a)$.
3. $2\pi(7a/9g)^{1/2}$, $4a/9$.
6. $4\ddot{\theta}+3\ddot{\phi}\cos(\phi-\theta)-3\dot{\phi}^2/\sin(\phi-\theta)=-(9g/4a)\sin\theta$,
$\ddot{\phi}+\ddot{\theta}\cos(\phi-\theta)+\dot{\theta}^2\sin(\phi-\theta)=-(g/2a)\sin\phi$.
7. $l\ddot{\theta}+a\ddot{\phi}\sin(\phi-\theta)+a\dot{\phi}^2\cos(\phi-\theta)=-g\sin\theta$,
$\frac{4}{3}a\ddot{\phi}+l\ddot{\theta}\sin(\phi-\theta)-l\dot{\theta}^2\cos(\phi-\theta)=g\cos\phi$, $3g/(4a)$.
11. $Tn_1n_3/(I_1n_1^2+I_2n_2^2+I_3n_3^2)$.
12. $5mg\sin\alpha\cos\alpha/\{7(M+m)-5m\cos^2\alpha\}$.
13. $(3M+m_1)g\sin\alpha/(3M+6m+m_1)$;
$(3M+2m+m_1)g\sin\alpha/(3M+6m+m_1)$.
14. $2a\cos\alpha/(1+3\cos^2\alpha)$.
16. $-Mga(\frac{4}{3}\cos\theta+2\cos\phi)$, $-28g\sin\alpha/(20a)$, $3g\sin\alpha/(20a)$.
20. $\ddot{\theta}=(\lambda a\omega^2\cos\alpha+g)\sin\alpha/(\lambda a)$, $\ddot{\phi}=0$, $\ddot{x}=g\cos\alpha$.

EXERCISES 8 (*b*)

2. 157 rad./sec., 8·2 rad./sec.
3. 0·475 sec.
4. 1165 rad./sec., 0·096 sec.
5. 1·43, 12·1 rad./sec., 3·5 ft.lb.
6. 135 ft.lb.
8. 1227 ft.lb. to the left.
9. 7450 ft.lb.

EXERCISES 8 (*c*)

1. $13\theta = 10a\cos(g/2)^{1/2}t + 3a\cos(g/3)^{1/2}t$,
   $13\phi = 15a\cos(g/2)^{1/2}t - 2a\cos(g/3)^{1/2}t$.
2. $\lambda^2 = 1 + k$, $C = \frac{1}{2}b$.
4. $2\pi(4a/g)^{1/2}$, $2\pi(2a/g)^{1/2}$.
6. a, $6a$.
7. $2\pi(4a/g)^{1/2}$, $2\pi(a/3g)^{1/2}$.
8. $2\pi(4a/g)^{1/2}$, $2\pi(a/3g)^{1/2}$.
9. $2\pi(4a/g)^{1/2}$, $2\pi(2a/3g)^{1/2}$.
10. $2\pi(6a/g)^{1/2}$, $2\pi(a/g)^{1/2}$.
11. $2\pi(Il/\mu)^{1/2}$, $2\pi(Il/2\mu)^{1/2}$.
15. $\theta + \phi$, $2\pi(a/2g)^{1/2}$, $5\theta - 2\phi$, $2\pi(5a/3g)^{1/2}$.
16. $2a/3$, $22a/3$.
17. $2\pi(11a/3g)^{1/2}$, $2\pi(2a/3g)^{1/2}$.

EXERCISES 9 (*a*)

1. $40/\sqrt{(3)}$, $40/\sqrt{(3)}$, 5 lb.
3. 5, 5, 25 cwt.; $30a$.
4. 4 ft., 7 cwt., 50 ft.cwt., 6 ft. cwt.
5. $22w$, $\tan^{-1} 1{\cdot}90$ to horizontal.
7. $bW/(2ab - a^2)^{1/2}$.
8. 575 lb.
9. aW/l.
10. 3 lb., 10·1 lb., 20° 10′.
11. 5, 3·88 ft.
12. 6 tons, 8 tons.
13. $W/4\sqrt{(47)}$, $25W/4R$.
15. $WR/2\sqrt{(6)}$.

EXERCISES 9 (*b*)

6. $1/\sqrt{(3)}$, 125, $25\sqrt{(3)}$ lb.
7. 8 ft.
8. 4·44 lb.
11. 6·94 lb.

EXERCISES 9 (*c*)

1. 1·18 in.
2. $\tan^{-1}(17/12)$, 7·65 lb., $\pi - \tan^{-1}(17/12)$.
3. 1·79 in.
4. 10·4 in., 5·5 in.
5. 6·07 in.
6. $16\pi(7 + 2\pi)/3$.
7. $1{\cdot}65\pi$.
8. 4·07 in.
10. 2·44, 2·06 in.
11. 3·16 in.
12. 2·20 in.

EXERCISES 9 (*d*)

1. 14·14 lb., 45° to AB.
2. 24·6, 6·4 lb.
3. $h_2 = 3h_1$, $h_3 = 6h_1$, $Wx/(10a)$, $Wx/(10h_1)$.
4. $Wr/\sqrt{(3)}$, $\pm Wr/n\sqrt{(3)}$.
5. ED 5, BE 4 tons, comp.; CD 4, CE 3, BC 5, AC 8 tons, tension.
6. AB − 1·52, AC 0·88, DC − 1·08, DB 0·63, BC − 1·30 tons.
7. 250, 212·1, 427 cwt.
8. AC 4·2, AE 14·3, ED 15·7, CD 11·8, BC 16·0, CE 3·5 tons.
9. 49·5, 22·7, −60·8 tons.
10. 4·62, 1·64, 1·11 tons.
11. $-3{\cdot}75W$, $3{\cdot}75W$, $0{\cdot}75\sqrt{(2)}W$, $-0{\cdot}75\sqrt{(2)}W$.
12. 16·05, 20·63, 11·69 tons.

EXERCISES 9 (e)

1. 0·16 ft. tons.
2. 13, 440 ft.lb.
3. 11·5 ft.lb.
4. $(R - \mu)/(r + \mu)$, $r(R - \mu)/R(r + \mu)$.
5. $Wb/4a$, $b/(4an)$.
6. 15 lb., 750 ft.lb.

EXERCISES 9 (f)

1. $a + Wa/(2\pi\lambda)$.
5. $W\sqrt{(3)}$.
9. $\frac{1}{2}W\sqrt{(5)}$, $\frac{1}{2}W$, $\frac{3}{2}W$.
11. $Wb(a^2 + ab + 2b^2)/(a^2 + b^2)$.
12. $Wa^2 \sin\theta/\{l^2 - 4a^2 \sin^2 \frac{1}{2}\theta\}^{1/2}$.
13. 74 per cent.
15. $\frac{3}{4}W\sqrt{(6)}$, $\frac{1}{4}W\sqrt{(6)}$.
16. $W/\sqrt{(14)}$, $W/4\sqrt{(14)}$.

EXERCISES 9 (g)

2. Unstable.
3. Unstable.
4. $\tan^{-1}\{\sin(\alpha - \beta)/(2 \sin\alpha \sin\beta)\}$.
5. Two stable, one unstable.
12. $\sin^{-1}(\sqrt{(3)}/6)$, unstable.
16. $\theta = 0$, unstable, $\theta = 60°$, stable.

EXERCISES 10 (a)

3. 390 lb., 0·45 ft.
5. $33\frac{1}{3}$ h.p.
6. $\tan^{-1} 4$, $\tan^{-1} 2$, 0; $\frac{1}{2}W$.
7. 15 ft., 30 ft., 33·9 tons.

EXERCISES 10 (b)

1. 76·3 ft., 28·9 ft.
4. 9·43 ft., 8·70 ft.
6. $W\sqrt{2}$.
7. 60°, 30°, $2 + 4/\sqrt{3}$.
8. 27·32 ft., 5·86 ft.
9. $l(2\sqrt{2} - \sqrt{5})$.
12. $l = ce^{a/c}$, 49° 36′.

EXERCISES 10 (c)

1. 14·6 tons/in.^{2}.
2. 501 tons, 2·58 tons/ft.
3. 0·115, 28° 40′.
4. 262·3 ft.
5. 29·5 ft.
6. 66·5 ft., 49·9 ft.
7. 67·2 tons, 19° 35′.
8. 1·62 ft., 2·53 tons, 1·88 tons.
9. 61·1 tons, 2·25 ft., 15·2 tons.
10. 120·36 ft., 1·8 in.
11. 5·58 tons/in.^{2}, 0·6 in.
12. 9918 lb., 10,050 lb.
14. 3·5 ft.

EXERCISES 11 (a)

1. 7 in. from one end, 81 in.lb.
3. $w(AB)^2$.
4. 49·8 ft.cwt., $7\frac{3}{8}$ ft. from A.
5. 18 ft.ton at 12 ft. from A.
6. $\frac{3}{8}Wl$, $\frac{1}{4}Wl$.
7. 9·24 ft.ton, 6·93 ft. from A.
8. 31·25 ton, 25 ton, 31·64 ft.ton.
9. One wheel $\frac{1}{4}a$ from centre.
11. $\frac{1}{8}W(l - 4a)$, $\frac{1}{4}W(l - 2a)^2/l$, $0·293l$.
12. $0·293l$ from end.

EXERCISES 11 (b)

1. 8 in.
2. 236 lb., 41·6 lb.
3. 3200 lb.
4. 2·35 tons.
5. 13·1 in.
6. 1415 lb./in.^{2}, 47·2 lb./in.^{2}.
7. (a) 142 lb., (b) 176 lb.
9. 3·4 tons/in.^{2}, 4·6 tons/in.^{2}.

EXERCISES 11 (c)

1. $wx(x^2 - l^2)(3x^2 - 17l^2)/(360\ EIl)$. 2. $wa^2/12$, wa.
4. $\frac{1}{5}W$, $\frac{4}{5}W$, $3Wl^3/(640EI)$.
8. $6W/5$, $-\frac{1}{5}W$, $11Wl^3/(240EI)$, $Wl^2/(60EI)$.
9. $(W + 2W')l^3/(48EI)$. 10. $5Wl^3/(96EI)$.
11. $11wa^4/(48EI)$. 12. $7Wa^3/(144EI)$.
13. $\frac{1}{2}Wa^2(2a+3b)/(a + b)^3$, $\frac{1}{2}Wab(a + 2b)/(a + b)^2$, $0{\cdot}076Wa^3/EI$.
14. $w(a + b)^3/(8ab)$. 15. $11wa/28$, $8wa/7$, $13wa/14$; $0{\cdot}107wa^2$.

EXERCISES 11 (d)

2. 166, 1285 lb., 10,800 lb.
10. $b = a(\sec nl - 1)$; $Wa \sec nl$; $\pi^2EI/(9l^2)$, $W = EIn^2$.

EXERCISES 12 (a)

1. $h/r < \{2s(1 - s)\}^{-1/2}$. 6. $s < 0{\cdot}28$ or $s > 0{\cdot}72$.
7. $(4W/3\pi)(l^2 - r^2)\theta/r$. 12. 0·8 ft.

EXERCISES 12 (b)

1. 708 ft./sec., 89 ft. lb./sec. 2. 92·5 gal./min., 2 ft.
3. 14·42 ft., 4·44 ft. 4. 5·06 ft./sec., 3·16 ft.
6. 475 gal. 7. 18·16 lb./in.^{2}.
9. 0·89 ft./sec., 116 gal./min. 11. 1612·5 ft.lb.
12. 0·18 cu. ft./sec. 13. 388 gal./min.
14. 1·77 cu. ft./sec. 15. 25·3 ft./sec., 7 min. 35 sec.
17. 6·6 min. 18. 187 r.p.m.
19. 492 r.p.m.

EXERCISES 12 (c)

1. $\psi = xy$. 2. $(-1{\cdot}5, 0)$, $(1{\cdot}5, 0)$.
3. (a) $Ur \sin (\theta - \alpha)$, (b) $m\theta/(2\pi)$, (c) $\mu \sin \theta/(2\pi r)$.
4. $5r \sin \theta\{6 - 5/(\pi r^2)\}$.

EXERCISES 12 (d)

1. 12,400 ft.lb. 2. 11·62 cu. ft., 42,000 ft.lb.
3. 12·1. 4. $2{\cdot}7 \times 10^5$ lb./in.^{2}.
5. 1057 ft./sec. 6. 315·5 ft./sec.
7. 315·5 ft./sec., 0·18, 0·28. 10. 6·83 lb./sec.
11. 14 cu. ft./sec. 12. 23 lb./sec.
13. 189 ft./sec. 14. 0·29 lb./sec.
15. 14·2 lb./in.^{2}, 0·074 lb./cu. ft.
16. −24·6° C., 6·75 lb./in.^{2}, 0·041 lb./cu. ft.
17. 1030 ft./sec. 18. 8·95 lb./in.^{2}, 0·51 lb./cu. ft.
19. 3·45 lb./in.^{2}, 0·0238 lb./cu. ft., 975 ft./sec.
20. 2·72 lb./in.^{2}, 0·0188 lb./cu. ft.

EXERCISES 13 (a)

1. $\Sigma \dfrac{\Gamma(n-\frac{1}{2})}{(n)!} x^{n+1/2}$, $-1 < x < 1$.

2. $A\Sigma \dfrac{(-1)^n (n+3)!}{(n)!(n+6)!} x^{n+3} + B\left(\dfrac{1}{x} - \dfrac{8}{x^2} + \dfrac{20}{x^3}\right)$.

3. $A(1 - 3x^2) + B\left(x - \dfrac{4}{(3)!}x^3 - \dfrac{1.4.6}{(5)!}x^5 - \dfrac{1.3.4.6.8}{(7)!}x^7 \ldots\right)$.

4. $Ax^{-1/2} \cos x^{1/2} + Bx^{-1/2} \sin x^{1/2}$.

5. $\Sigma \dfrac{\{(n)!\}^2 (4x)^n}{(2n)!}$, $x^{1/2} \Sigma \dfrac{(2n+1)!(x/4)^n}{(n)!(n)!}$.

6. $A\Sigma \dfrac{\Gamma(n+\frac{1}{4})(-x)^n}{\Gamma(n+\frac{3}{4})(n)!} + Bx^{1/4} \Sigma \dfrac{\Gamma(n+\frac{1}{2})(-x)^n}{(n)!(n)!}$, all finite x.

7. $A\Sigma \dfrac{\Gamma(n+\frac{3}{2})\Gamma(n+\frac{3}{2})}{(2n)!}(2x)^{2n} + B\Sigma \dfrac{(n+1)!(n+1)!}{(2n+1)!}(2x)^{2n+1}$.

8. $A\Sigma \dfrac{(-1)^n (\frac{1}{2}x)^{2n+2}}{\Gamma(n+\frac{5}{2})(n)!} + B\Sigma \dfrac{(-1)^n (\frac{1}{2}x)^{2n-1}}{\Gamma(n-\frac{1}{2}).(n)!}$.

9. $A\Sigma \dfrac{x^{2n-1/2}}{(2n)!} + B\Sigma \dfrac{x^{2n+1/2}}{(2n+1)!}$.

10. $A\left(1 - 16x + 48x^2 - \dfrac{256}{5}x^3 + \dfrac{128}{7}x^4\right) + B\Sigma \dfrac{\Gamma(n-\frac{7}{2})\Gamma(n+\frac{5}{2})}{\Gamma(n+\frac{3}{2})(n)!} x^{n+1/2}$.

11. $A\Sigma \dfrac{\Gamma(a+n)x^n}{\Gamma(b+n)(n)!} + B\Sigma \dfrac{\Gamma(a-b+1+n)}{\Gamma(2-b+n)} \dfrac{x^{n+1-b}}{(n)!}$.

12. $B(x-1)^2(4x+1)$.

EXERCISES 13 (c)

3. $-0{\cdot}2423$, $0{\cdot}0307$, $0{\cdot}0109$.

4. $2{\cdot}32 J_0(x) + 2{\cdot}59 Y_0(x)$.

5. $(\frac{1}{2}\pi x)^{-1/2} \cos x$, $-(\frac{1}{2}\pi x)^{-1/2}(\sin x + x^{-1} \cos x)$.

8. $xy'' + 3y' + xy = 0$, $Y_0(x) + Y_2(x)$.

15. $\dfrac{2}{\pi} \sinh \dfrac{1}{2}\pi$, $e^{-(1/2)\pi}$.

16. $AJ_0(2x) + BY_0(2x)$.

17. $x^2\{AJ_2(2x) + BY_2(2x)\}$.

18. $x^{3/2}\{AJ_{3/2}(3x) + BY_{3/2}(3x)\}$.

20. $AJ_1(x^2) + BY_1(x^2)$.

21. $x^{-2}\{AI_4(x) + BK_4(x)\}$.

22. $x^{1/2}\{AJ_{1/3}(\frac{2}{3}x^{3/2}) + BY_{1/3}(\frac{2}{3}x^{3/2})\}$.

EXERCISES 13 (d)

1. $\sin nx \sin cnt$.

2. $A \sin ny\, e^{-nx}$.

3. $F = A \cos(\omega r/c + \varepsilon)$.

4. $A = 0$, $B = m = \omega = 1$.

7. $\sin(n\pi x/a) \sin(m\pi y/a) \cos\{\pi c(m^2 + n^2)^{1/2} t/a\}$.

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